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Determinantal forms and recursive relations of the Delannoy two-functional sequence

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Abstract

In the paper, the authors establish closed forms for the Delannoy two-functional sequence and its difference in terms of the Hessenberg determinants, derive recursive relations for the Delannoy two-functional sequence and its difference, and deduce closed forms in terms of the Hessenberg determinants and recursive relations for the Delannoy one-functional sequence, the Delannoy numbers, and central Delannoy numbers.

Keywords: closed form; recursive relation; difference; Hessenberg determinant; Delannoy two-functional sequence; Delannoy one-functional sequence; Delannoy number; central Delannoy number.

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1. Introduction

A tridiagonal determinant is a determinant whose nonzero elements locate only on the diagonal and slots horizontally or vertically adjacent the diagonal. Technically speaking, a determinant $H = |h_{ij}|_{n \times n}$ is called a tridiagonal determinant if $h_{ij} = 0$ for all pairs (i, j) such that $|i - j| > 1$. For more information, please refer to the paper [11]. A determinant $H = |h_{ij}|_{n \times n}$ is called a lower (or an upper, respectively) Hessenberg determinant if $h_{ij} = 0$ for all pairs (i, j) such that $i + 1 < j$ (or $j + 1 < i$, respectively). For more information, please refer to the paper [18].

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In combinatorial number theory, the Delannoy number $D(m, n)$ for $n, m \geq 0$ can be regarded as the number of lattice paths from $(0, 0)$ to (m, n) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$. The Delannoy numbers $D(m, n)$ can be defined by means of the recurrence relation

$$D(m, n) = D(m - 1, n) + D(m - 1, n - 1) + D(m, n - 1)$$

with initial values $D(0, n) = D(m, 0) = D(0, 0) = 1$. The historic significance of these numbers $D(m, n)$ was explained in the paper [1]. The Delannoy numbers $D(m, n)$ can be computed by explicit formulas

$$D(m, n) = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} 2^k \quad \text{and} \quad D(m, n) = \sum_{\ell=0}^n \binom{n}{\ell} \binom{m+n-\ell}{n}$$

and can be generated by

$$\frac{1}{1 - x - y - xy} = \sum_{n,m \geq 0} D(m, n) x^m y^n.$$

For more information on the Delannoy numbers $D(m, n)$, please refer to [1, 19, 59, 63] and closely related references therein.

The numbers $D(k) = D(k, k)$ for $k \geq 0$ are known as central Delannoy numbers. In [26, Theorem 1.3], central Delannoy numbers $D(k)$ were represented by

$$D(k) = \frac{1}{\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{\sqrt{(t-3+2\sqrt{2})(3+2\sqrt{2}-t)}} \frac{1}{t^{k+1}} dt, \quad k \geq 0.$$

In [48, Section 4], the central Delannoy numbers $D(k)$ were generalized as

$$D_{a,b}(k) = \frac{1}{\pi} \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t^{k+1}} dt, \quad k \geq 0, \quad b > a > 0$$

and, from [26, Lemma 2.4], it is derived that the quantities $D_{a,b}(k)$ have the generating function

$$\frac{1}{\sqrt{(x+a)(x+b)}} = \sum_{k=0}^{\infty} D_{a,b}(k) x^k.$$

By virtue of conclusions in [24, Section 2.4] and [38, Remark 4.1], we can find that the quantities $D_{a,b}(k)$ for $k \geq 0$ can be computed by

$$D_{a,b}(k) = \frac{1}{G(a,b)} \frac{(-1)^k}{[2A(a,b)]^k} \sum_{\ell=0}^k (-1)^\ell 2^{2\ell} \frac{(2\ell-1)!!}{(2\ell)!!} \binom{\ell}{k-\ell} \left[\frac{A(a,b)}{H(a,b)} \right]^\ell$$

and

$$D_{a,b}(k) = \frac{1}{\sqrt{ab}} \frac{1}{b^k} \sum_{\ell=0}^k \frac{(2\ell-1)!!}{(2\ell)!!} \frac{[2(k-\ell)-1]!!}{[2(k-\ell)]!!} \left(\frac{b}{a} \right)^\ell,$$

where $A(a, b) = \frac{a+b}{2}$, $G(a, b) = \sqrt{ab}$, and $H(a, b) = \frac{2}{1/a+1/b}$ are respectively known as arithmetic, geometric, and harmonic means of $a, b > 0$. The quantities $D_{a,b}(k)$ can be further generalized as

$$D_{a,b;\lambda}(z) = \frac{\sin(\lambda\pi)}{\pi} \int_a^b \frac{1}{(t-a)^\lambda (b-t)^{1-\lambda}} \frac{1}{t^{z+1}} dt,$$

where $0 < a < b$, $\lambda \in (0, 1)$, and $z \in \mathbb{C}$. It is obvious that $D_{a,b;1/2}(k) = D_{a,b}(k)$. Theorems 2.1 and 2.2 in the paper [38] demonstrate that the quantities $D_{a,b;\lambda}(k)$ can be generated by

$$\frac{1}{(x+a)^\lambda (x+b)^{1-\lambda}} = \sum_{k=0}^{\infty} D_{a,b;\lambda}(k) x^k$$

and can be computed by the closed forms

$$D_{a,b;\lambda}(k) = \frac{1}{n!} \frac{1}{a^\lambda b^{1-\lambda}} \frac{1}{b^n} \sum_{\ell=0}^n \binom{n}{\ell} (\lambda)_\ell (1-\lambda)_{n-\ell} \left(\frac{b}{a}\right)^\ell$$

and

$$D_{a,b;\lambda}(k) = \frac{1}{a^\lambda b^{1-\lambda}} \frac{1}{b^n} \sum_{k=0}^n \left(\frac{b}{a}\right)^k \sum_{\ell=0}^k \langle \lambda \rangle_\ell \binom{k-1}{\ell-1} \frac{1}{\ell!} \left(1 - \frac{a}{b}\right)^\ell,$$

where

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1} (x+k), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

and

$$\langle x \rangle_n = \begin{cases} \prod_{k=0}^{n-1} (x-k), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

are known as the rising and falling factorials respectively. The ideas, significance, and reasonability of these generalizations $D_{a,b}(k)$ and $D_{a,b;\lambda}(z)$ come from the papers [25, 34, 37, 38, 39, 43, 52, 53, 54, 55] and closely related references therein.

Inspired by the identity

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k} t^k = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{n} (t-1)^k$$

in [8, eq. 3.17], the Delannoy numbers $D(m, n)$ were extended to the functional sequence

$$D(x; n) = \sum_{k=0}^n \binom{n}{k} \binom{x}{k} 2^k = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{n}$$

in [61]. Various arithmetic properties and congruence relations for the Delannoy one-functional sequence $D(x; n)$ have been studied in [9, 16, 17, 59, 60, 63].

In [58], the Delannoy one-functional sequence $D(x; n)$ was further generalized as

$$D(x, r; n) = \sum_{k=0}^n \binom{x+r+k}{k} \binom{x-r}{n-k},$$

the generating function

$$\frac{(1+t)^{x-r}}{(1-t)^{x+r+1}} = \sum_{n=0}^{\infty} D(x, r; n) t^n, \quad |t| < 1 \tag{1.1}$$

was derived, and a plenty of identities for the Delannoy two-functional sequence $D(x, r; n)$ were acquired.

In [7], the Delannoy two-functional sequence $D(x, r; n)$ was generalized again to the Delannoy two-functional polynomials

$$D(x, r; n; y) = \sum_{k=0}^n \binom{x+r+k}{k} \binom{x-r}{n-k} y^k$$

and, among other things, the generating function and recurrence formula for the Delannoy two-functional polynomials $D(x, r; n; y)$ were derived.

It is noted that $D(x, 0; n) = D(x; n)$ and $D(x, r; n; 1) = D(x, r; n)$.

In combinatorial number theory, it is significant to express concrete sequences or arrays of integer numbers or polynomials in terms of tridiagonal determinants or the Hessenberg determinants. In this respect, the Bernoulli numbers and polynomials [2, 6, 20, 27, 36, 40, 62], the Euler numbers and polynomials [29, 32, 64], (central) Delannoy numbers and polynomials [10, 19, 25, 26, 31, 47, 48, 49], the Horadam polynomials [41], (generalized) Fibonacci numbers and polynomials [13, 14, 15, 21, 25, 33, 46, 50], the Lucas polynomials [41, 46], and the like, have been represented via tridiagonal determinants or the Hessenberg determinants, and consequently many remarkable relations have been obtained. For more information in this area and direction, please refer to [12, 22, 23, 30, 35, 42, 44, 45, 51, 56, 57] and closely related references therein.

In this paper, we will present closed forms [3] for the Delannoy two-functional sequence $D(x, r; n)$ and its difference $D(x, r; n) - D(x, r; n - 1)$ in terms of the Hessenberg determinants, derive recursive relations for the Delannoy two-functional sequence $D(x, r; n)$ and its difference $D(x, r; n) - D(x, r; n - 1)$, and deduce closed forms in terms of the Hessenberg determinants and recursive relations for the Delannoy one-functional sequence $D(x; n)$, the Delannoy numbers $D(m, n)$, and central Delannoy numbers $D(k)$.

2. Determinantal forms of the Delannoy two-functional sequence and its difference

In this section, we will present closed forms for the Delannoy two-functional sequence $D(x, r; n)$ and for its difference $D(x, r; n) - D(x, r; n - 1)$ in terms of the Hessenberg determinants.

Theorem 2.1. *The Delannoy two-functional sequence $D(x, r; n)$ for $n \geq 0$ can be determinantly expressed by*

$$D(x, r; n) = \frac{(-1)^n}{n!} |P_{(n+1) \times 1}(x, r) \quad B_{(n+1) \times n}(x, r)|_{(n+1) \times (n+1)}, \tag{2.1}$$

where

$$P_{(n+1) \times 1}(x, r) = (\langle x - r \rangle_0 \quad \langle x - r \rangle_1 \quad \langle x - r \rangle_2 \quad \dots \quad \langle x - r \rangle_n)^T,$$

$$B_{(n+1) \times n}(x, r) = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle x + r + 1 \rangle_{i-j} \right)_{(n+1) \times n},$$

and T denotes the transpose of a matrix.

The difference $D(x, r; n) - D(x, r; n - 1)$ for $n \geq 1$ can be determinantly expressed by

$$D(x, r; n) - D(x, r; n - 1) = \frac{(-1)^n}{n!} |P_{(n+1) \times 1} \quad C_{(n+1) \times n}|_{(n+1) \times (n+1)}, \tag{2.2}$$

where

$$C_{(n+1) \times n}(x, r) = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle x + r \rangle_{i-j} \right)_{(n+1) \times n}.$$

Proof. Let $p_{x,r}(t) = (1 + t)^{x-r}$ and $q_{x,r}(t) = (1 - t)^{x+r+1}$. Then

$$p_{x,r}^{(k)}(t) = \langle x - r \rangle_k (1 + t)^{x-r-k} \rightarrow \langle x - r \rangle_k$$

and

$$q_{x,r}^{(k)}(t) = (-1)^k \langle x + r + 1 \rangle_k (1 - t)^{x-r-k} \rightarrow (-1)^k \langle x + r + 1 \rangle_k$$

as $t \rightarrow 0$.

The equation (1.1) implies that

$$D(x, r; n) = \frac{1}{n!} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left[\frac{(1 + t)^{x-r}}{(1 - t)^{x+r+1}} \right]. \tag{2.3}$$

Let $u(t)$ and $v(t) \neq 0$ be two n th differentiable functions for $n \in \mathbb{N}$. Exercise 5) in [4, p. 40] reads that the n th derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{d^n}{dx^n} \left[\frac{u(t)}{v(t)} \right] = (-1)^n \frac{|W_{(n+1) \times (n+1)}(t)|}{v^{n+1}(t)}, \tag{2.4}$$

where $|W_{(n+1) \times (n+1)}(t)|$ is the determinant of the $(n + 1) \times (n + 1)$ matrix

$$W_{(n+1) \times (n+1)}(t) = \begin{pmatrix} U_{(n+1) \times 1}(t) & V_{(n+1) \times n}(t) \end{pmatrix}_{(n+1) \times (n+1)},$$

the matrix $U_{(n+1) \times 1}(t)$ is an $(n + 1) \times 1$ matrix whose elements satisfy $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \leq k \leq n + 1$, and $V_{(n+1) \times n}(t)$ is an $(n + 1) \times n$ matrix whose elements meet $v_{i,j}(t) = \binom{i-1}{j-1} v^{(i-j)}(t)$ for $1 \leq i \leq n + 1$ and $1 \leq j \leq n$. The formula (2.4) is a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions.

Combining (2.3) and (2.4) gives

$$\begin{aligned} D(x, r; n) &= \frac{(-1)^n}{n!} \lim_{t \rightarrow 0} \frac{|U_{(n+1) \times 1}(t) \quad V_{(n+1) \times n}(t)|}{q_{x,r}^{n+1}(t)} \\ &= \frac{(-1)^n}{n!} \frac{|\lim_{t \rightarrow 0} U_{(n+1) \times 1}(t) \quad \lim_{t \rightarrow 0} V_{(n+1) \times n}(t)|}{\lim_{t \rightarrow 0} q_{x,r}^{n+1}(t)} \\ &= \frac{(-1)^n}{n!} |\lim_{t \rightarrow 0} U_{(n+1) \times 1}(t) \quad \lim_{t \rightarrow 0} V_{(n+1) \times n}(t)|, \end{aligned}$$

where

$$\begin{aligned} \lim_{t \rightarrow 0} U_{(n+1) \times 1}(t) &= \lim_{t \rightarrow 0} \begin{pmatrix} p_{x,r}^{(0)}(t) & p_{x,r}^{(1)}(t) & p_{x,r}^{(2)}(t) & \dots & p_{x,r}^{(n)}(t) \end{pmatrix}^T \\ &= (\langle x - r \rangle_0 \quad \langle x - r \rangle_1 \quad \langle x - r \rangle_2 \quad \dots \quad \langle x - r \rangle_n)^T \end{aligned}$$

and

$$\lim_{t \rightarrow 0} V_{(n+1) \times n}(t) = \left(\lim_{t \rightarrow 0} \binom{i-1}{j-1} q_{x,r}^{(i-j)}(t) \right)_{(n+1) \times n} = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle x + r + 1 \rangle_{i-j} \right)_{(n+1) \times n}.$$

The determinantal form (2.1) is thus proved.

The generating function in (1.1) can be rewritten as

$$\frac{(1+t)^{x-r}}{(1-t)^{x+r}} = (1-t) \sum_{n=0}^{\infty} D(x, r; n) t^n = D(x, r; 0) + \sum_{n=1}^{\infty} [D(x, r; n) - D(x, r; n-1)] t^n.$$

By the same arguments as in the derivation of the determinantal form (2.1), we can obtain the determinantal forms (2.2) immediately. The proof of Theorem 2.1 is complete. □

3. Recursive relations of the Delannoy two-functional sequence and its difference

In this section, with the aid of Theorem 2.1, we will derive recursive relations for the Delannoy two-functional sequence $D(x, r; n)$ and for its differences $D(x, r; n) - D(x, r; n - 1)$.

Theorem 3.1. *The Delannoy two-functional sequence $D(x, r; n)$ for $n \geq 1$ satisfies*

$$D(x, r; n) = \frac{\langle x - r \rangle_n}{n!} + (-1)^n \sum_{s=1}^n (-1)^s \frac{\langle x + r + 1 \rangle_{n-s+1}}{(n - s + 1)!} D(x, r; s - 1) \tag{3.1}$$

and the differences $D(x, r; n) - D(x, r; n - 1)$ for $n \geq 2$ meet

$$D(x, r; n) - D(x, r; n - 1) = \frac{\langle x - r \rangle_n}{n!} + (-1)^n \sum_{s=2}^n (-1)^s \frac{\langle x + r \rangle_{n-s+1}}{(n - s + 1)!} [D(x, r; s - 1) - D(x, r; s - 2)]. \tag{3.2}$$

Proof. Let $Q_0 = 1$ and

$$Q_n = \begin{vmatrix} e_{1,1} & e_{1,2} & 0 & \dots & 0 & 0 \\ e_{2,1} & e_{2,2} & e_{2,3} & \dots & 0 & 0 \\ e_{3,1} & e_{3,2} & e_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{n-2,1} & e_{n-2,2} & e_{n-2,3} & \dots & e_{n-2,n-1} & 0 \\ e_{n-1,1} & e_{n-1,2} & e_{n-1,3} & \dots & e_{n-1,n-1} & e_{n-1,n} \\ e_{n,1} & e_{n,2} & e_{n,3} & \dots & e_{n,n-1} & e_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. In [5, p. 222, Theorem], it was proved that the sequence Q_n for $n \geq 0$ satisfies $Q_1 = e_{1,1}$ and

$$Q_n = \sum_{s=1}^n (-1)^{n-s} e_{n,s} \left(\prod_{j=s}^{n-1} e_{j,j+1} \right) Q_{s-1} \tag{3.3}$$

for $n \geq 2$, where any empty product is understood to be 1 while any empty sum is understood to be 0.

Substituting $(-1)^{n-1}(n-1)!D(x, r; n-1)$ for Q_n , replacing $e_{k,1}$ for $1 \leq k \leq n$ by $\langle x-r \rangle_{k-1}$, switching $e_{i,j}$ for $1 \leq i \leq n$ and $2 \leq j \leq n$ into $\binom{i-1}{j-2}(-1)^{i-j+1}\langle x+r+1 \rangle_{i-j+1}$ in (3.3), and simplifying arrive at

$$\begin{aligned} & (-1)^{n-1}(n-1)!D(x, r; n-1) \\ &= (-1)^{n-1}\langle x-r \rangle_{n-1} + \sum_{s=2}^n (-1)^{n-s} \binom{n-1}{s-2} (-1)^{n-s+1} \langle x+r+1 \rangle_{n-s+1} (-1)^{s-2} (s-2)!D(x, r; s-2) \end{aligned}$$

which can be reformulated as

$$\begin{aligned} & (-1)^{n-1}(n-1)!D(x, r; n-1) \\ &= (-1)^{n-1}\langle x-r \rangle_{n-1} - \sum_{s=2}^n (-1)^s (s-2)! \binom{n-1}{s-2} \langle x+r+1 \rangle_{n-s+1} D(x, r; s-2). \end{aligned}$$

The recursive relation (3.1) is thus proved.

Similarly, substituting $(-1)^{n-1}(n-1)! [D(x, r; n-1) - D(x, r; n-2)]$ for Q_n , replacing $e_{k,1}$ for $1 \leq k \leq n$ by $\langle x-r \rangle_{k-1}$, switching $e_{i,j}$ for $1 \leq i \leq n$ and $2 \leq j \leq n$ into $\binom{i-1}{j-2}(-1)^{i-j+1}\langle x+r \rangle_{i-j+1}$ in (3.3), and straightforwardly simplifying lead to the recursive relation (3.2). The proof of Theorem 3.1 is complete. \square

4. Determinantal forms and recursive relations for the Delannoy one-functional sequence, the Delannoy numbers, and central Delannoy numbers

In this section, with the help of Theorems 2.1 and 3.1, we deduce closed forms in terms of the Hessenberg determinants and recursive relations for the Delannoy one-functional sequence $D(x; n)$, the Delannoy numbers $D(m, n)$, and central Delannoy numbers $D(k)$.

Since $D(x, 0; n) = D(x; n)$, when taking $r = 0$ in Theorems 2.1 and 3.1, we derive the following conclusions.

Theorem 4.1. *The Delannoy one-functional sequence $D(x; n)$ for $n \geq 0$ can be determinantly expressed by*

$$D(x; n) = \frac{(-1)^n}{n!} \left| \mathfrak{P}_{(n+1) \times 1}(x) \quad \mathfrak{B}_{(n+1) \times n}(x) \right|_{(n+1) \times (n+1)}, \tag{4.1}$$

where

$$\mathfrak{P}_{(n+1) \times 1}(x) = (\langle x \rangle_0 \quad \langle x \rangle_1 \quad \langle x \rangle_2 \quad \dots \quad \langle x \rangle_n)^T$$

and

$$\mathfrak{B}_{(n+1) \times n}(x) = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle x+1 \rangle_{i-j} \right)_{(n+1) \times n}.$$

The difference $D(x; n) - D(x; n - 1)$ for $n \geq 1$ can be determinantly expressed by

$$D(x; n) - D(x; n - 1) = \frac{(-1)^n}{n!} \left| \mathfrak{B}_{(n+1) \times 1}(x) \quad \mathfrak{C}_{(n+1) \times n}(x) \right|_{(n+1) \times (n+1)},$$

where

$$\mathfrak{C}_{(n+1) \times n}(x) = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle x \rangle_{i-j} \right)_{(n+1) \times n}.$$

Theorem 4.2. The Delannoy one-functional sequence $D(x; n)$ for $n \geq 1$ satisfies

$$D(x; n) = \frac{\langle x \rangle_n}{n!} + (-1)^n \sum_{s=1}^n (-1)^s \frac{\langle x+1 \rangle_{n-s+1}}{(n-s+1)!} D(x; s-1)$$

and the differences $D(x; n) - D(x; n - 1)$ for $n \geq 2$ meet

$$D(x; n) - D(x; n - 1) = \frac{\langle x \rangle_n}{n!} + (-1)^n \sum_{s=2}^n (-1)^s \frac{\langle x \rangle_{n-s+1}}{(n-s+1)!} [D(x; s-1) - D(x; s-2)].$$

When taking $x = m$ in Theorems 4.3 and 4.4, we derive the following conclusions.

Theorem 4.3. The Delannoy numbers $D(m; n)$ for $m, n \geq 0$ can be determinantly expressed by

$$D(m; n) = \frac{(-1)^n}{n!} \left| \mathcal{P}_{(n+1) \times 1}(m) \quad \mathcal{B}_{(n+1) \times n}(m) \right|_{(n+1) \times (n+1)}, \tag{4.2}$$

where

$$\mathcal{P}_{(n+1) \times 1}(m) = (\langle m \rangle_0 \quad \langle m \rangle_1 \quad \langle m \rangle_2 \quad \dots \quad \langle m \rangle_n)^T$$

and

$$\mathcal{B}_{(n+1) \times n}(m) = \left((-1)^{i-j} \binom{i-1}{j-1} \langle m+1 \rangle_{i-j} \right)_{(n+1) \times n}.$$

The differences $D(m; n) - D(m; n - 1)$ for $m \geq 0$ and $n \geq 1$ can be determinantly expressed by

$$D(m; n) - D(m; n - 1) = \frac{(-1)^n}{n!} \left| \mathcal{P}_{(n+1) \times 1}(m) \quad \mathcal{C}_{(n+1) \times n}(m) \right|_{(n+1) \times (n+1)},$$

where

$$\mathcal{C}_{(n+1) \times n}(m) = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle m \rangle_{i-j} \right)_{(n+1) \times n}.$$

Theorem 4.4. The Delannoy numbers $D(m; n)$ for $m \geq 0$ and $n \geq 1$ satisfy

$$D(m; n) = \binom{m}{n} + (-1)^{n-1} \sum_{s=0}^{n-1} (-1)^s \binom{m+1}{n-s} D(m; s) \tag{4.3}$$

and the differences $D(m; n) - D(m; n - 1)$ for $m \geq 0$ and $n \geq 2$ meet

$$D(m; n) - D(m; n - 1) = \binom{m}{n} + (-1)^n \sum_{s=2}^n (-1)^s \binom{m}{n-s+1} [D(m; s-1) - D(m; s-2)].$$

When taking $m = n = k$ in Theorem 4.3, we derive the following conclusions.

Theorem 4.5. Central Delannoy numbers $D(k)$ for $k \geq 0$ can be determinantly expressed by

$$D(k) = \frac{(-1)^k}{k!} \left| \mathcal{P}_{(k+1) \times 1} \quad \mathcal{B}_{(k+1) \times k} \right|_{(k+1) \times (k+1)}, \tag{4.4}$$

where

$$\mathcal{P}_{(k+1) \times 1} = (\langle k \rangle_0 \quad \langle k \rangle_1 \quad \langle k \rangle_2 \quad \dots \quad \langle k \rangle_k)^T$$

and

$$\mathcal{B}_{(k+1) \times k} = \left(\binom{i-1}{j-1} (-1)^{i-j} \langle k+1 \rangle_{i-j} \right)_{(k+1) \times k}.$$

5. Remarks

Finally we list several remarks as follows.

1. The determinantal forms (4.2) and (4.4) in Theorem 4.3 and Theorem 4.5 recover [19, Theorem 2.1].
2. The formula (4.3) coincides with the first result in [19, Theorem 3.1]. Letting $m = n$ in (4.3) recovers the second result in [19, Theorem 3.1].
3. The determinantal form (4.4) in Theorem 4.5 is different from

$$D(k) = (-1)^k \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k-2} & a_{k-3} & a_{k-4} & \cdots & a_1 & 1 & 0 \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_2 & a_1 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad (5.1)$$

where $k \geq 1$ and

$$a_k = \frac{(-1)^{k+1}}{6^k} \sum_{\ell=1}^k (-1)^\ell 6^{2\ell} \frac{(2\ell-3)!!}{(2\ell)!!} \binom{\ell}{k-\ell}.$$

The determinantal form (5.1) was established in [26, Theorem 1.1]. See also related texts in the papers [19].

4. This paper is a modified version of the electronic preprint [28].

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Conflict of interest

The authors declare that they have no conflict of interest.

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