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A Highly Approximate Pseudo-Spectral Method for the Solution of Convection-Diffusion Equations

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Abstract

The main purpose of this paper is to compute a highly accurate numerical solution of two dimensional convection–diffusion equations with variable coefficients by using Legendre pseudo-spectral method based on Legendre-Gauss-Lobatto nodes. The Kronecker product is used here to formulate a linear system of differentiation matrices; this system was reduced to be more accurate with less memory usage. Error analysis with test problems are introduced to show that the suggested scheme of the spectral method has high accuracy.

1. Introduction

Given a simply connected, domain $\Omega \equiv [a,\ b] \ \times [c,\ d]$ in R^2 , with Lipschitz boundary $\partial\Omega$, a time interval $I \equiv [0,T]$ and the differential operator

$$\mathscr{K} u = -\nabla \cdot (A(\mathbf{x},t)\nabla u) + p(\mathbf{x})u + q \cdot \nabla u, \tag{1.1}$$

where $\mathbf{x} = (x,y)$,∇ and $\nabla\cdot$ denote the gradient and divergence operators, respectively and $A(x)$ represents a matrix-value function. In this paper we are concerned with the numerical approximation of the linear convection- or advection-diffusion problem: Find $u : Q \equiv \Omega \times I \to R$ such that:

$$\partial_t u + \mathscr{K} u = f(\mathbf{x},t) \quad in\ Q, \tag{1.2}$$

$$u = u_B(\mathbf{x}) \quad On\ \partial\Omega \times I, \tag{1.3}$$

$$u(\mathbf{x},0) = u_0(x) \quad in\ \Omega, \tag{1.4}$$

where $q \equiv (\alpha, \beta)$ is the convection vector, u_B is the boundary function and f represents the source or reaction term. Here and elsewhere we shall assume that all coefficients occurring in these equations are uniformly bounded.
For solving this problem, we construct a solution using pseudo-spectral method which may be considered as an extension to the work in [1, 2] There are two steps to form this solution [3, 4, 5]. First, we use polynomial interpolation of the solution based on some suitable nodes as discrete representation of the solution; here we chose Legendre nodes. Secondly, we form a system of algebraic equations from the previous discretization after collocating it; this step replaces differential operators by matrix approximations (see[3, 4, 5, 6]).
A brief outline of the paper is as follows: Section 2 describes some necessary notations about the continuous problem. Section 3 summarizes the steps of solving the problem (1.1)-(1.4) by using Legendre pseudo–spectral method and modifying the resulted system using matrices properties. As a result a system of algebraic linear equations is formed and a solution of the considered problem is discussed. In Section 4 we shall prove error estimates of the scheme given in Section 3. In Section 5, some numerical examples are presented to show the effectiveness of the proposed scheme.

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2. The continuous problem

Throughout this paper we use standard notations that will be used in the sequel [7]. We use the functional spaces $L_\infty(\Omega), L_2(\Omega), V = H_0^1(\Omega), C(I; L_2(\Omega)), L_2(I; L_2(\Omega))$ (see e.g. [6, 8]). By $(.,.)$ we shall denote either the inner product in $L_2(\Omega)$. We denote by $|\ .\ |, \|.\|_\infty, \|.\|$, the norms in $L_2(\Omega), L_\infty(\Omega)$ and V, respectively. All the constants which occur in the course of this paper will be denoted by C.(ε is small and $C_\varepsilon = C(\varepsilon^{-1})$)
We shall consider the collocation points $\{x_i\}_{i=0}^N$ which is the set of $(N+1)$ Legendre-Gauss-Lobatto nodes [4]. Also we define the family $V_N = span\left\{\ell_p(s) : 1 \le p \le\ N-1\right\}$ of finite dimensional subspace of V where $\{\ell_p\}$ are the Lagrange basis polynomials associated with $\{x_i\}_{i=0}^N$.
Now we list the following assumptions on the coefficients and data in (1.1)-(1.4):
(A1) $A(\mathbf{x},t)$ is symmetric, Lipschitz continuous in t and uniformly positive definite matrix in Q, i.e.,

$$(A\xi, \xi) \ge c\,|\xi|^2,$$

$$\left|\left|A(\mathbf{x},t) - A\left(\mathbf{x},t'\right)\right|\right|_M \le c\left|t-t'\right| \quad \forall t,\, t' \in I,$$

where $||\cdot||_M$ is a matrix norm.
(A2) $p,\ \nabla\cdot q \in L_\infty(\Omega),\ with\ p>0, \quad p\cdot n \ge 0\ \ on\ \ \partial\Omega$ and

$$(p\xi, \xi) \ge c\,|\xi|^2. \tag{2.1}$$

(A3) $p - \frac{1}{2}\nabla\cdot q \ge 0\ \ in\ \Omega.$
(A4) $f : Q \to R$ and $u_B(\mathbf{x}) :\ \partial\Omega\times I \to R$ are Lipschitz continuous in the sense of

$$\left|f(\mathbf{x},t) - f\left(\mathbf{x},t'\right)\right| \le c\left|t-t'\right| \quad \forall t,\, t' \in I,$$

and analogously for $u_B(\mathbf{x})$.
(A5) $u_0(\mathbf{x}) \in V$.
Under these assumptions, the weak solution of (1.1)-(1.4) is defined as:
Find $u : Q \to R$ such that

$$(\partial_t u, \chi) + ((u,\chi)) = (f,\chi), \qquad \forall\chi\in V, \quad a.e.t \in I, \tag{2.2}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x})\ \ in\ \Omega.$$

Here we use the notation$((u,\chi))$ to represent the bilinear form corresponding to the differential operator $\mathscr{K}$; i.e.

$$((u,\chi)) = (A(x)\nabla u, \nabla\chi) + (q\cdot\nabla u + p(\mathbf{x})u, \chi). \tag{2.3}$$

Without loss of generality, we may assume a homogeneous Dirichlet boundary condition on $\partial\Omega$, i.e., $u_B(\mathbf{x}) = 0$. Otherwise, we write the bilinear form (2.1) in terms of u and $u_B(\mathbf{x})$ in place of u.
The existence of a weak solution $u \in L_2(I; L_2(\Omega)) \cap C(I; L_2(\Omega))$ is guaranteed by Lax-Milgram theorem. In fact the use of (A2) and (A3) imply that the bilinear form $((.,.))$ is bounded and V-elliptic, i.e.,

$$|((u,\chi))| \le C\|u\|\ \|\chi\| \qquad \forall u,\chi \in L_2(I; L_2(\Omega)) \cap C(I; L_2(\Omega)),$$

$$((u,u)) \ge C\|u\|^2 \qquad \forall u \in L_2(I; L_2(\Omega)) \cap C(I; L_2(\Omega)). \tag{2.4}$$

However, to prove the coercivity condition (2.4), it requires to use (A3) with the application of Gauss's theorem:

$$2(q\cdot\nabla\omega, \omega) = (\omega(q\cdot n), \omega) - (\omega(\nabla\cdot q), \omega).$$

3. The suggested approximation scheme

For a given integer number $N_z \ge 0$, we denote by $z_0,\ z_1,\ \cdots, z_N$, the nodes of the shifted N_z+1-point integration formula of Gauss, Gauss-Radau or Gauss-Lobatto type, and by $w_0,\ w_1,\ \cdots, w_N$ the corresponding weights [4]. In order to solve problem (1.1)-(1.4) by pseudo-spectral method we use the notation

$$\hat{\varphi}(z) = \varphi\left(\frac{2(z-\alpha)}{\beta-\alpha} - 1\right) \qquad \forall z \in [\alpha,\ \beta],$$

to approximate the function $u(\mathbf{x},t)$ in Q by using cubic grid consisting of $(N_x+1)\times\left(N_y+1\right)\times(N_t+1)$ nodes as

$$u(\mathbf{x},t) \approx \sum_i^{N_x}\sum_j^{N_y}\sum_k^{N_t} U_{i,j,k}\hat{\varphi}_i(x)\,\hat{\varphi}_j(y)\,\hat{\varphi}_k(t), \tag{3.1}$$

where

$$U_{i,j,k} = u\left(x_i^{(N_x)}, y_j^{(N_y)}, t_k^{(N_t)}\right).$$

Here we have used Lagrange's interpolant functions which are defined by

$$\varphi_i(z) = \prod_{j=0, j\neq i}^{N_z} \frac{z - z_j}{z_i - z_j} \quad , \ \forall 0 \leq i \leq N_z.$$

By using the Kronecker product [9], equation (3.1) can be expressed in the following matrix form:

$$u(x,y,t) \approx \left(\hat{\Phi}^{(N_x)}_{[a,b]}(x) \otimes \hat{\Phi}^{(N_y)}_{[c,d]}(y) \otimes \hat{\Phi}^{(N_t)}_{[0,T]}(t)\right) \bar{U},$$

or simply

$$u(x,y,t) \approx \left(\hat{\Phi}(x) \otimes \hat{\Phi}(y) \otimes \hat{\Phi}(t)\right) \bar{U},$$

where $\bar{U}$ is a column vector given by:

$$\bar{U} = \left[u_{0,0,0}, u_{0,0,1}, u_{0,0,2}, \ldots, u_{0,0,N_t+1}, u_{0,1,0}, \ldots, u_{0,1,N_t+1}, \ldots\ldots, u_{N_x+1,N_y+1,N_t+1}\right]^T.$$

By using a general form of the first differentiation matrix

$$D_{N_z+1} = \left(D_{ij} = \frac{d}{dz}\phi_j(z)\Big|_{z=z_i}\right)_{0\leq i,j\leq N_z}$$

from Lagrange interpolation with a modification [6] to improve accuracy

$$D_{im} = \begin{cases} \frac{\prod_{k\neq i,m}^{N_z}(z_i - z_k)}{\prod_{k\neq m}^{N_z}(z_m - z_k)} & i < m, i \neq 0, N_z, \\ -D_{N_z-i, N_z-m} & i > m, i \neq 0, N_z, \\ -\sum_{k\neq i}^{N_z} D_{ik} & i = m \neq 0, N_z, \\ -\frac{N_z(N_z+1)}{4} & i = m = 0, \\ \frac{N_z(N_z+1)}{4} & i = m = N_z, \end{cases}$$

with the classical form of the second order differentiation matrix $D^2_{N_z+1}$ [1]:

$$D^2_{im} = \begin{cases} 2D_{im}\left(D_{ii} - \frac{1}{z_i - z_m}\right) & i \neq m, \\ -\sum_{k=0,k\neq i}^{N_z} D^2_{ik} & i = m, \end{cases}$$

we can rewrite equation (1.2) in discrete form

$$\begin{aligned} &\Big[\tfrac{2}{T}\left(\hat{\Phi}(x)\otimes\hat{\Phi}(y)\otimes\hat{\Phi}(t)D_{N_t+1}\right) + \tfrac{2\lambda_1(x,y,t)}{b-a}\left(\hat{\Phi}(x)D_{N_x+1}\otimes\hat{\Phi}(y)\otimes\hat{\Phi}(t)\right) \\ &+\tfrac{2\lambda_2(x,y,t)}{d-c}\left(\hat{\Phi}(x)\otimes\hat{\Phi}(y)D_{N_y+1}\otimes\hat{\Phi}(t)\right) - \tfrac{4\lambda_3(x,y,t)}{(b-a)^2}\left(\hat{\Phi}(x)D^2_{N_x+1}\otimes\hat{\Phi}(y)\otimes\hat{\Phi}(t)\right) \\ &-\tfrac{4\lambda_4(x,y,t)}{(d-c)^2}\left(\hat{\Phi}(x)\otimes\hat{\Phi}(y)\, D^2_{N_y+1}\otimes\hat{\Phi}(t)\right) \\ &-\tfrac{4\lambda_5(x,y,t)}{(b-a)(d-c)}\left(\hat{\Phi}(x)D_{N_x+1}\otimes\hat{\Phi}(y)\, D_{N_y+1}\otimes\hat{\Phi}(t)\right) \\ &+p(x,y)\left(\hat{\Phi}(x)\otimes\hat{\Phi}(y)\otimes\hat{\Phi}(t)\right)\Big]\bar{U} = F(x,y,t), \end{aligned} \tag{3.2}$$

where $\lambda_i, i = 1,2,\ldots,5$ are the coefficients of $\partial_x u$, $\partial_y u$, $\partial_x^2 u$, $\partial_y^2 u$ and $\partial_{xy} u$ respectively.
Then, by collocating equation (3.2) at any interior collocation point $\left\{(x_i, y_j, t_k)\right\}_{i,j,k}$, $1 \leq i \leq N_x - 1, 1 \leq j \leq N_y - 1$, and $1 \leq k \leq N_t$, equation (3.2) becomes:

$$\begin{aligned} &\Big[\tfrac{2}{T}\left(e_i^{N_x+1}\otimes e_j^{N_y+1}\otimes D_{N_t+1}e_k^{N_t+1}\right) + \tfrac{2\lambda^1_{ijk}}{b-a}\left(D_{N_x+1}e_i^{N_x+1}\otimes e_j^{N_y+1}\otimes e_k^{N_t+1}\right) \\ &+\tfrac{2\lambda^2_{ijk}}{d-c}\left(e_i^{N_x+1}\otimes D_{N_y+1}e_j^{N_y+1}\otimes e_k^{N_t+1}\right) - \tfrac{4\lambda^3_{ijk}}{(b-a)^2}\left(D^2_{N_x+1}e_i^{N_x+1}\otimes e_j^{N_y+1}\otimes e_k^{N_t+1}\right) \\ &-\tfrac{4\lambda^4_{ijk}}{(d-c)^2}\left(e_i^{N_x+1}\otimes D^2_{N_y+1}e_j^{N_y+1}\otimes e_k^{N_t+1}\right) - \tfrac{4\lambda^5_{ijk}}{(b-a)(d-c)}\left(D_{N_x+1}e_i^{N_x+1}\otimes D_{N_y+1}e_j^{N_y+1}\otimes e_k^{N_t+1}\right) \\ &+ p_{ij}\left(e_i^{N_x+1}\otimes e_j^{N_y+1}\otimes e_k^{N_t+1}\right)\Big]\bar{U} = F_{ijk}, \end{aligned} \tag{3.3}$$

where e_r^M is the r-th row of the identity matrix I_M. Rewriting equation (3.3) in matrix form yields

$$\begin{aligned} &\Big[\tfrac{2}{T}\left([I_{N_x+1}]_{1:N_x-1,:}\otimes\left[I_{N_y+1}\right]_{1:N_y-1,:}\otimes[D_{N_t+1}]_{1:N_t,:}\right) \\ &+\tfrac{2}{b-a}\tilde{\lambda}_1\left([D_{N_x+1}]_{1:N_x-1,:}\otimes\left[I_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right) \\ &+\tfrac{2}{d-c}\tilde{\lambda}_2\left([I_{N_x+1}]_{1:N_x-1,:}\otimes\left[D_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right) \\ &-\tfrac{4}{(b-a)^2}\tilde{\lambda}_3\left(\left[D^2_{N_x+1}\right]_{1:N_x-1,:}\otimes\left[I_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right) \\ &-\tfrac{4}{(d-c)^2}\tilde{\lambda}_4\left([I_{N_x+1}]_{1:N_x-1,:}\otimes\left[D^2_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right) \\ &-\tfrac{4}{(b-a)(d-c)}\tilde{\lambda}_5\left([D_{N_x+1}]_{1:N_x-1,:}\otimes\left[D_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right) \\ &\tilde{p}\left([I_{N_x+1}]_{1:N_x-1,:}\otimes\left[I_{N_y+1}\right]_{1:N_y-1,:}\otimes[I_{N_t+1}]_{2:N_t,:}\right)\Big]\bar{U} = \overset{\vee}{F}, \end{aligned}$$

or

$$\mathrm{A}\bar{U}=\check{F}, \tag{3.4}$$

where $\check{F}=\left[f_{1,1,1},f_{1,1,2},\ldots,f_{1,1,N_t1},f_{1,2,1},\ldots,f_{1,2,N_t},\ldots\ldots,f_{N_x-1,N_y-1,N_t}\right]^T$, the notation $[A_M]_{i:j,:}$ represents the intersection of rows from i to j and all columns from any matrix $A_{M\times M}$,and the notation $\widetilde{v}$ is a diagonal matrix with $\{v\}_{i,j,k}$ as diagonal elements for interior points $1\le i\le N_x-1, 1\le j\le N_y-1$, and $1\le k\le N_t$.
The system (3.4) represents $(N_x-1)\times\left(N_y-1\right)\times N_t$ equations in $(N_x+1)\times\left(N_y+1\right)\times(N_t+1)$ variables, which contain variables at both interior and initial-boundary nodes. As we only interested in the unknown interior points $\left\{\left(x_i,y_j,t_k\right)\right\}_{i,j,k}$, $1\le i\le N_x-1, 1\le j\le N_y-1$, and $1\le k\le N_t$, the incomplete system (3.4) can be rewritten as

$$\mathrm{A}\left(\bar{V}+\bar{W}\right)=\check{F}, \tag{3.5}$$

with

$$v_{i,j,k}=\begin{cases} u_{i,j,k} & \forall\ 1\le i\le N_x-1,\ 1\le j\le N_y-1,\ 1\le k\le N_t,\\ 0 & \textit{otherwise},\end{cases}$$

and

$$\bar{W}=\bar{U}-\bar{V}.$$

After separating the unknown variables from the known ones, the system (3.5) could be rewritten as

$$A_1\bar{\bar{\mathrm{V}}}=\check{K}-A_2\bar{\bar{W}},$$

where $\bar{\bar{\mathrm{V}}},\bar{\bar{W}}$ represent the columns of the nonzero elements of $\bar{V},\bar{W}$ and the matrices A_1,A_2 represent the remaining matrices of A after eliminating columns corresponding to zeros elements of $\bar{V}$ and $\bar{W}$respectively.

4. Error estimates

This section is devoted to proving the convergence of the proposed scheme and estimating its accuracy. In order to that we will need the following two lemmas:

Lemma 4.1. *[10] Assume that $(N+1\ )-$ point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to Legendre weight is used to integrate the product uv where $u\in H^m\left(\Omega\right)$ and $v\in P_N$. Then there exists a constant C independent of N such that*

$$\left|\left(u\ ,v\right)-\left(u\ ,v\right)_N\right|\ \le CN^{-m}\left\|u\right\|_{H^m(\Omega)}\left|v\right|_{L_2(\Omega)},$$

where

$$\left(u\ ,v\right)_N=\sum_{i=0}^{N}u\left(x_i\right)v\left(x_i\right)w_i$$

Lemma 4.2. *(Gronwall inequality) [10] If a non-negative integrable function $u(t)$ satisfies*

$$u\left(t\right)\le\alpha\left(t\right)+C\int_{-1}^{t}u\left(s\right)ds, -1\le t\le 1,$$

where $\alpha\left(t\right)$ is an integrable function, then

$$\left\|u\left(t\right)\right\|_{L^p}\le C\left\|\alpha\left(t\right)\right\|_{L^p}, p\ge 1 \textit{ or } p=\infty.$$

Now, we have the following main result.

Theorem 4.3. *Under the assumptions (A1)-(A5), the solution of problem (1.1)-(1.4) is $u\in L_2(I,V)\cap L_\infty(I,V)$ with $\partial_t u\in L_2(I,H^m(\Omega))$, and there exists a positive constant C independent of u,f,N,m and r such that the estimate*

$$\begin{aligned}&\left|e_u\right|^2_{L_\infty(I,L_2(\Omega))}+\left\|e_u\right\|^2_{L_2(I,V)}+\left\|e_u\right\|^2_{L_\infty(I,V)}\le CN^{-r}\left\|f\right\|^2_{L_2(I,H^r(\Omega))}\\ &+CN^{-m}\left(\left|\partial_t u\right|^2_{L_2(I,H^m(\Omega))}+\left\|u\right\|^2_{L_2(I,H^m(\Omega))}\right)\end{aligned}$$

holds uniformly for any integer numbers $r,m\ge 1$.

Proof. On applying (2.2) at x_i, $0\le i\le N$, we get

$$\left(\partial_t U^N,\chi\right)_N+\left(\left(U^N,\chi\right)\right)_N=\left(f,\chi\right)_N,\quad \forall\chi\in V,\ \ a.e.\, t\in I \tag{4.1}$$

where $U^N\in L_2\left(I;V\right)\cap C\left(I;L_2\left(\Omega\right)\right)$ Subtracting equation(4.1) from (2.2) yields

$$\left(\partial_t u,\chi\right)-\left(\partial_t U^N,\chi\right)_N+\left(\left(u,\chi\right)\right)-\left(\left(U^N,\chi\right)\right)_N=\left(f,\chi\right)-\left(f,\chi\right)_N$$

Denoting $e_u=u-U^N$, we get

$$\left(\partial_t e_u,\chi\right)+\left(\left(e_u,\chi\right)\right)=\left(f,\chi\right)-\left(f,\chi\right)_N-\left(\left(\partial_t U^N,\chi\right)-\left(\partial_t U^N,\chi\right)_N\right)-\left(\left(\left(U^N,\chi\right)\right)-\left(\left(U^N,\chi\right)\right)_N\right). \tag{4.2}$$

Putting $\chi = e_u$ and then $\chi = \partial_t e_u$ in equation (4.2) respectively and summing up we get

$$\begin{aligned}
&\tfrac{1}{2}\partial_t |e_u|^2 + \|e_u\|_V^2 + |\partial_t e_u|^2 + \tfrac{1}{2}\partial_t \|e_u\|_V^2 \\
&= (f,e_u) - (f,e_u)_N - \left((\partial_t U^N, e_u) - (\partial_t U^N, e_u)_N\right) \\
&- \left(((U^N,e_u)) - ((U^N,e_u))_N\right) + (f,\partial_t e_u) - (f,\partial_t e_u)_N \\
&- \left((\partial_t U^N, \partial_t e_u) - (\partial_t U^N, \partial_t e_u)_N\right) \\
&- \left(((U^N,\partial_t e_u)) - ((U^N,\partial_t e_u))_N\right).
\end{aligned}$$

From equation (2.3) and Lemma 4.1

$$\begin{aligned}
&\tfrac{1}{2}\partial_t |e_u|^2 + \|e_u\|_V^2 + |\partial_t e_u|^2 + \tfrac{1}{2}\partial_t \|e_u\|_V^2 \le \\
&CN^{-r}\|f\|_{H^r(\Omega)}|e_u|_{L_2(\Omega)} + CN^{-r}\|f\|_{H^r(\Omega)}|\partial_t e_u|_{L_2(\Omega)} \\
&+CN^{-m}\left\|\partial_t U^N\right\|_{H^m(\Omega)}|e_u|_{L_2(\Omega)} + CN^{-m}\left\|\partial_t U^N\right\|_{H^m(\Omega)}|\partial_t e_u|_{L_2(\Omega)} \\
&+N^{-m}\left\|U^N\right\|_{H^m(\Omega)}|e_u|_{L_2(\Omega)} + CN^{-m}\left\|U^N\right\|_{H^m(\Omega)}|\partial_t e_u|_{L_2(\Omega)}.
\end{aligned} \tag{4.3}$$

The first two terms on are estimated by the use of Peter-Paul inequality [11] as

$$CN^{-r}\|f\|_{H^r(\Omega)}|e_u|_{L_2(\Omega)} \le CN^{-r}\left(C_\varepsilon\|f\|^2_{H^r(\Omega)} + \varepsilon|e_u|^2_{L_2(\Omega)}\right),$$

$$CN^{-r}\|f\|_{H^r(\Omega)}|\partial_t e_u|_{L_2(\Omega)} \le CN^{-r}\left(C_\varepsilon\|f\|^2_{H^r(\Omega)} + \varepsilon|\partial_t e_u|^2_{L_2(\Omega)}\right),$$

where ε a very small constant. The application of Young's inequality estimates the third term of (4.3)

$$CN^{-m}\left\|\partial_t U^N\right\|_{H^m(\Omega)}|e_u|_{L_2(\Omega)} \le CN^{-m}\left(C_\varepsilon|\partial_t u|^2_{H^m(\Omega)} + \frac{1}{2}|\partial_t e_u|^2_{H^m(\Omega)} + \frac{1}{2}|e_u|^2_{L_2(\Omega)}\right).$$

Similarly, we estimate the last three terms of (4.3). Finally, choosing ε sufficiently small we get

$$\begin{aligned}
&\tfrac{1}{2}\partial_t |e_u|^2 + \|e_u\|_V^2 + \tfrac{1}{2}\partial_t \|e_u\|_V^2 \\
&\le CN^{-r}C_\varepsilon\|f\|^2_{H^r(\Omega)} \\
&+CN^{-m}\left(C_\varepsilon|\partial_t u|^2_{H^m(\Omega)} + C_\varepsilon\|u\|^2_{H^m(\Omega)} + |e_u|^2_{L_2(\Omega)}\right).
\end{aligned} \tag{4.4}$$

The integration of equation (4.4) over the interval I with the use of Grönwall's inequality yields the asserted estimate. □

5. Numerical examples

To test our proposed method, we apply the proposed scheme for various examples whose exact solutions are provided in each case. For all examples, we take $N_x = N_y = N_t = N$, to study the convergence behavior of the presented method, we also calculated the following norms [8] for errors with different values of N at $t = T$:

1. The L_2-norm (Frobenius norm) defined for any matrix A by:

$$A_2 = \left(\sum_{i,j}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

2. The L_∞-norm defined for A by

$$A_\infty = \max_{i,j}|a_{ij}|.$$

All the computations are carried out in double precision arithmetic using Matlab7.12.0.635 (R2011a). The code was executed on an Intel Core i7-3610QM, 2.3 GHz Laptop.

Example 5.1. *Consider equation (1.2) with constant coefficients.*

$$\partial_t u - \Delta u = e^t\left(x^2 + y^2 - 4\right), \qquad x,y,t \in [0,1],$$

with initial and boundary conditions as follows:

$$u(x,y,0) = x^2 + y^2,$$

$$u(x,0,t) = x^2 e^t,\ u(x,1,t) = \left(1+x^2\right)e^t,$$

$$u(0,y,t) = y^2 e^t,\ u(1,y,t) = \left(1+y^2\right)e^t.$$

and the exact solution

$$u_{exact}(x,y,t) = e^t\left(x^2+y^2\right).$$

Table 1 *shows the Max error and Frobenius error at different values of N,* Figure 5.1 *shows the similarity in results of the present Method with the exact solution and surface and contour plots for the error $u_{ij} - u^{ex}_{ij}$ at $t = T$ respectively.*

Example 5.2. *Consider another example for convection diffusion equation (1.2) with constant coefficients*

$$\partial_t u + \nabla \mathrm{u} - \Delta \mathrm{u} = e^t \left(3\cos(x+y) - 2\sin(x+y)\right), \qquad x, y \in [0,\pi], \ t \in [0,1],$$

with initial and boundary conditions as follows

$$u(x,y,0) = \cos(x+y),$$

$$u(x,0,t) = e^t \cos x, \ u(x,\pi,t) = -e^t \cos x,$$

$$u(0,y,t) = e^t \cos y, \ u(\pi,y,t) = -e^t \cos y.$$

and the exact solution

$$u_{exact}(x,y,t) = e^t \cos(x+y)$$

Table 2 *shows the Max error and Frobenius error at different values of N.* Figure 5.2 *shows the similarity in results of the present Method with the exact solution and the surface and contour plots for the error* $u_{ij} - u^{ex}_{ij}$ *at* $t = T$ *respectively.*

Example 5.3. *Now, consider the following differential equation:*

$$\partial_t u + x^2 \partial_x u + y^3 \partial_y u - xye^{-(x+y)} \partial_{xx} u - e^{-(x+y)} \partial_{yy} u = f(x,y,t), \qquad x, y \in [0,\pi], \ t \in [0,1],$$

where

$$f(x,y,t) = e^t \left(\left(1 + (1+xy)e^{-(x+y)}\right) \cos(x+y) - \left(x^2 + y^3\right) \sin(x+y) \right)$$

with initial and boundary conditions as follows:

$$u(x,y,0) = \cos(x+y)$$

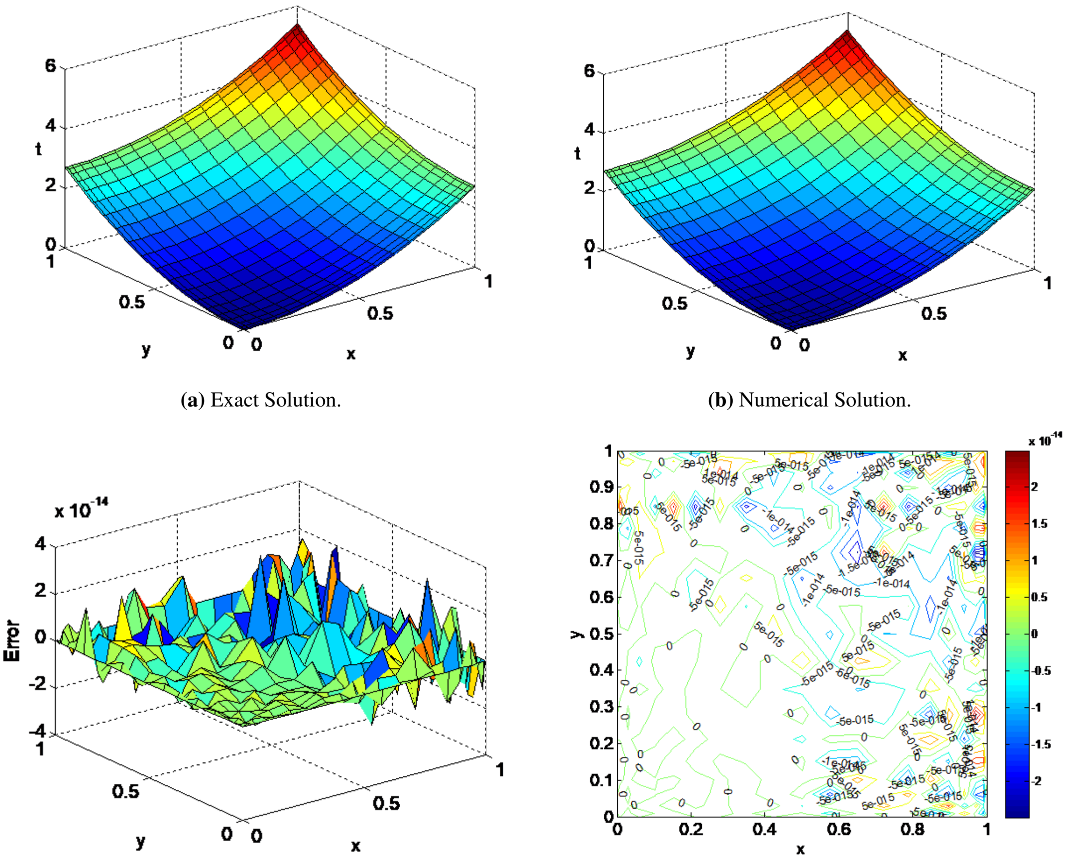
$$u(x,0,t) = e^t \cos x, \ u(x,\pi,t) = -e^t \cos x,$$

$$u(0,y,t) = e^t \cos y, \ u(\pi,y,t) = -e^t \cos y.$$

and the exact solution

$$u_{exact}(x,y,t) = e^t \cos(x+y).$$

then the results are given in Table 3, *again the similarity in results of the present Method with the exact solution, and the surface and contour plots for the error* $u_{ij} - u^{ex}_{ij}$ *at* $t = T$ *are given in* Figure 5.3 *respectively.*



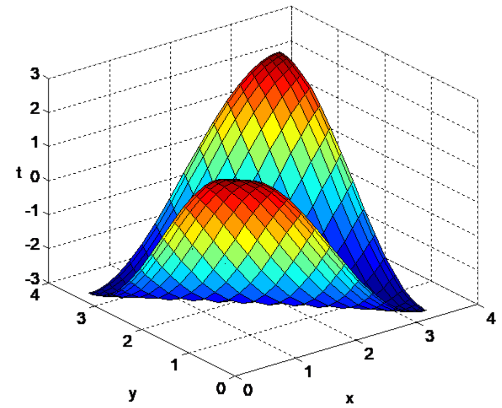
(a) Exact Solution.

(b) Numerical Solution.

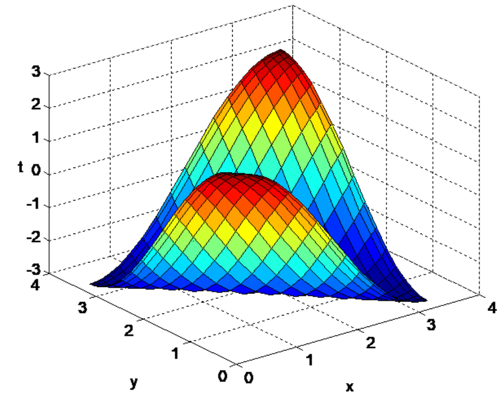
(c) Surface plot for error $u_i j - u_i j^e x$ at $t = T$.

(d) Contour plot for error $u_i j - u_i j^e x$ at $t = T$.

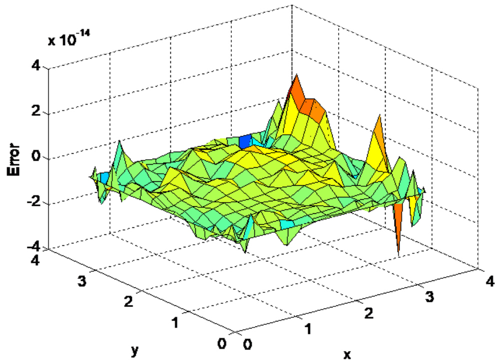
Figure 5.1: Comparison between exact and numerical solutions of Example 5.1 with surface and counter plots of the error.



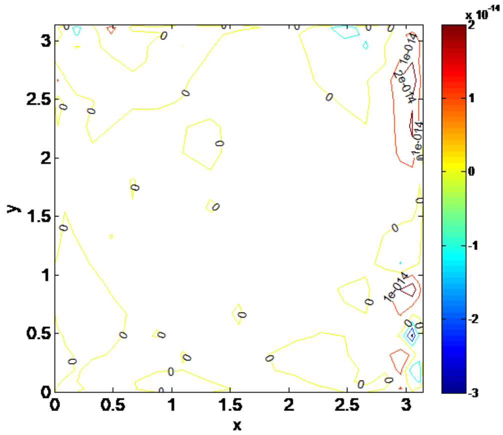
(a) Exact Solution.



(b) Numerical Solution.



(c) Surface plot for error $u_i j - u_i j^e x$ at $t = T$.



(d) Contour plot for error $u_i j - u_i j^e x$ at $t = T$.

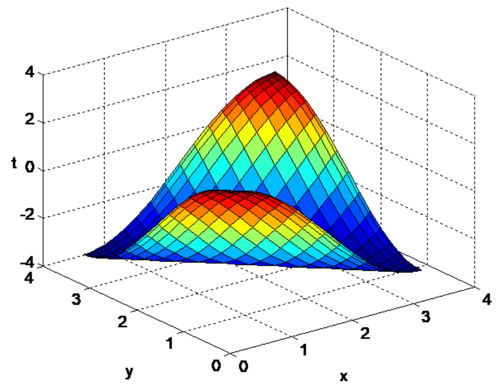
Figure 5.2: Comparison between exact and numerical solutions of Example 5.2 with surface and counter plots of the error.

Table 1: Max error and Frobenius error at different valuse of N for Example 5.1.

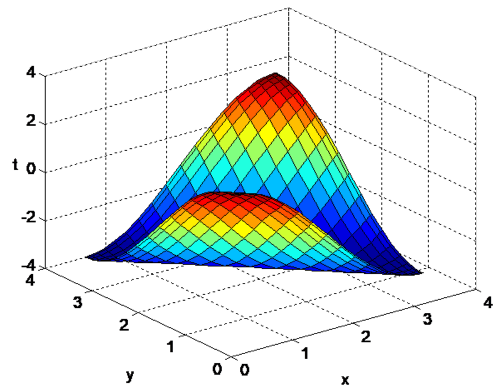
N	L_∞	L_2
6	5.1338e-007	1.5710e-006
8	5.7054e-010	2.0571e-009
10	4.2011e-013	1.6188e-012
12	8.8041e-014	3.7847e-013
14	2.1450e-013	7.0891e-013
16	2.5818e-013	1.0988e-012
18	4.8848e-013	1.8580e-012
20	3.7392e-013	2.0660e-012

Table 2: Max error and Frobenius error at different valuse of N for Example 5.2.

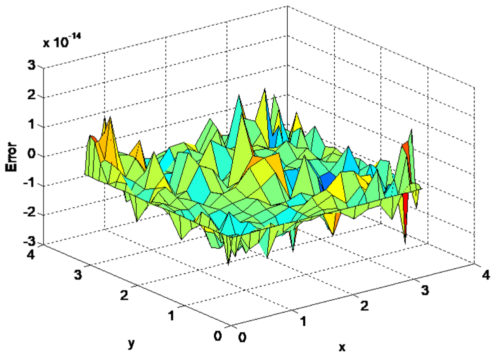
N	L_∞	L_2
6	1.2460e-004	2.5773e-004
8	7.5038e-007	2.2982e-006
10	3.4863e-009	1.3228e-008
12	1.1002e-011	5.3229e-011
14	3.0198e-014	1.6745e-013
16	2.9754e-014	9.1839e-014
18	1.9096e-014	7.2120e-014
20	3.1974e-014	1.1562e-013



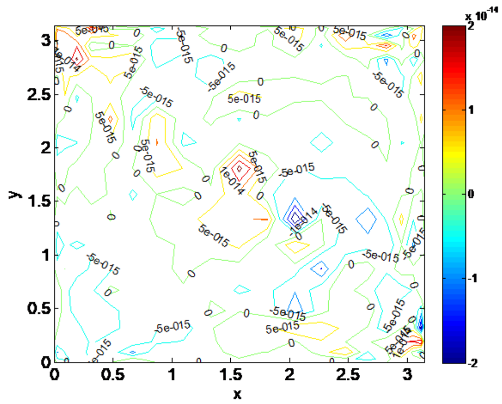
(a) Exact Solution.



(b) Numerical Solution.



(c) Surface plot for error $u_i j - u_i j^e x$ at $t = T$.



(d) Contour plot for error $u_i j - u_i j^e x$ at $t = T$.

Figure 5.3: Comparison between exact and numerical solutions of Example 5.3 with surface and counter plots of the error.

6. Conclusion

In this paper, Legendre collocation method was applied to calculate approximated solution of 2D Advection-Diffusion Equation with variable Coefficients. By using Kronecker product and modified differentiation matrices with reducing the resulting system, Error analysis and numerical results for this equation show that the suggested method is a high accuracy method.

Table 3: Max error and Frobenius error at different values of N for Example 5.3.

N	L_∞	L_2
6	0.0017	0.0033
8	1.2645e-005	2.9322e-005
10	9.7233e-008	2.9858e-007
12	2.6289e-010	9.5973e-010
14	9.8699e-013	3.5146e-012
16	1.9718e-013	5.7126e-013
18	1.4566e-013	6.6142e-013
20	2.2049e-013	8.2462e-013

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