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Some results on relative dual Baer property

Research Article

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Abstract: Let R be a ring. In this article, we introduce and study relative dual Baer property. We characterize R-modules M which are R_R-dual Baer, where R is a commutative principal ideal domain. It is shown that over a right noetherian right hereditary ring R, an R-module M is N-dual Baer for all R-modules N if and only if M is an injective R-module. It is also shown that for R-modules $M_1, M_2, \ldots, M_n$ such that M_i is M_j-projective for all $i > j \in \{1, 2, \ldots, n\}$, an R-module N is $\bigoplus_{i=1}^n M_i$-dual Baer if and only if N is M_i-dual Baer for all $i \in \{1, 2, \ldots, n\}$. We prove that an R-module M is dual Baer if and only if $S = End_R(M)$ is a Baer ring and $IM = r_M(l_S(IM))$ for every right ideal I of S.

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1. Introduction

Throughout this paper, R will denote an associative ring with identity, and all modules are unitary right R-modules. Let M be an R-module. We will use the notation $N \ll M$ to indicate that N is small in M (i.e., $L + N \neq M$ for every proper submodule L of M). By $E(M)$ and $End_R(M)$, we denote the injective hull of M and the endomorphism ring of M, respectively. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of rational numbers, integers and natural numbers, respectively. For a prime number p, $\mathbb{Z}(p^\infty)$ denotes the Prüfer p-group.

The concept of Baer rings was first introduced in [6] by Kaplansky. Since then, many authors have studied this kind of rings (see, e.g., [2] and [3]). A ring R is called *Baer* if the right annihilator of any nonempty subset of R is generated by an idempotent. In 2004, Rizvi and Roman extended the notion of Baer rings to a module theoretic version [10]. According to [10], a module M is called a *Baer module* if for every left ideal I of $End_R(M)$, $\cap_{\phi \in I} \text{Ker}\phi$ is a direct summand of M. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module M is said to be *dual Baer* if for every right ideal

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I of $S = End_R(M)$, $\sum_{\phi \in I} \mathrm{Im}\phi$ is a direct summand of M. Equivalently, for every nonempty subset A of S, $\sum_{\phi \in A} \mathrm{Im}\phi$ is a direct summand of M (see [14, Theorem 2.1]).

A module M is said to be *Rickart* if for any $\varphi \in End_R(M)$, $\mathrm{Ker}\varphi$ is a direct summand of M (see [7]). The notion of dual Rickart modules was studied recently in [8] by Lee-Rizvi-Roman. A module M is said to be *dual Rickart* if for every $\varphi \in End_R(M)$, $\mathrm{Im}\varphi$ is a direct summand of M. In [8], it was introduced the notion of relative dual Rickart property which was used in the study of direct sums of dual Rickart modules. Let N be an R-module. An R-module M is called N-*dual Rickart* if for every homomorphism $\varphi : M \to N$, $\mathrm{Im}\varphi$ is a direct summand of N (see [8]). Similarly, we introduce in this paper the concept of relative dual Baer property. A module M is called N-*dual Baer* if for every subset A of $\mathrm{Hom}_R(M, N)$, $\sum_{f \in A} \mathrm{Im}f$ is a direct summand of N. It is clear that if M is N-dual Baer, then M is N-dual Rickart.

We determine the structure of modules M which are R_R-dual Baer for a commutative principal ideal domain R (Proposition 2.7). Then we show that for an R-module M, R_R is M-dual Baer if and only if M is a semisimple module (Proposition 2.9). It is shown that over a right noetherian right hereditary ring R, an R-module M is N-dual Baer for all R-modules N if and only if M is an injective R-module (Corollary 2.17). We prove that if $\{M_i\}_I$ is a family of R-modules, then for each $j \in I$, $\bigoplus_{i \in I} M_i$ is M_j-dual Baer if and only if M_i is M_j-dual Baer for all $i \in I$ (Corollary 2.24). It is also shown that for R-modules M_1, $M_2, \ldots, M_n$ such that M_i is M_j-projective for all $i > j \in \{1, 2, \ldots, n\}$, an R-module N is $\bigoplus_{i=1}^n M_i$-dual Baer if and only if N is M_i-dual Baer for all $i \in \{1, 2, \ldots, n\}$ (Theorem 2.25). We conclude this paper by showing that an R-module M is dual Baer if and only if $S = End_R(M)$ is a Baer ring and $IM = r_M(l_S(I))$ for every right ideal I of S, where $l_S(I) = \{\varphi \in S \mid \varphi I = 0\}$, $r_M(l_S(I)) = \{m \in M \mid l_S(I)m = 0\}$ and $IM = \sum_{f \in I} \mathrm{Im}f$ (Theorem 2.31).

2. Main results

Definition 2.1. Let N be an R-module. An R-module M is called N-*dual Baer* if, for every subset A of $\mathrm{Hom}_R(M, N)$, $\sum_{f \in A} \mathrm{Im}f$ is a direct summand of N.

Obviously, an R-module M is dual Baer if and only if M is M-dual Baer.

Example 2.2. (1) Let N be a semisimple R-module. Then for every R-module M, M is N-dual Baer.

(2) If M and N are R-modules such that $Hom_R(M, N) = 0$, then M is N-dual Baer. It follows that for any couple of different maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of a commutative noetherian ring R, $E(R/\mathfrak{m}_1)$ is $E(R/\mathfrak{m}_2)$-dual Baer (see [12, Proposition 4.21]).

(3) Let p be a prime number. Note that $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ are dual Baer $\mathbb{Z}$-modules. On the other hand, it is clear that $\mathbb{Z}(p^\infty)$ is $\mathbb{Z}/p\mathbb{Z}$-dual Baer but $\mathbb{Z}/p\mathbb{Z}$ is not $\mathbb{Z}(p^\infty)$-dual Baer.

Recall that a module M is said to have the *strong summand sum property*, denoted briefly by *SSSP*, if the sum of any family of direct summands of M is a direct summand of M.

Following [8, Definition 2.14], a module M is called N-*d-Rickart* if, for every homomorphism $\varphi : M \to N$, $\mathrm{Im}\varphi$ is a direct summand of N.

Proposition 2.3. *Let M and N be two R-modules. If M is N-dual Baer, then M is N-d-Rickart. The converse holds when N has the SSSP.*

Proof. This follows from the definitions of "M is N-d-Rickart" and "M is N-dual Baer". □

The next example shows that the assumption "N has the *SSSP*" is not superfluous in Proposition 2.3.

Example 2.4. Let R be a von Neumann regular ring which is not semisimple (e.g., $R = \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$). By [8, Proposition 2.26], the R-module R_R does not have the *SSSP*. On the other hand, R_R is R_R-d-Rickart, but it is not R_R-dual Baer (see [14, Corollary 2.9] and [8, Remark 2.2]).

Proposition 2.5. *Let N be an indecomposable R-module. Then the following conditions are equivalent for an R-module M.*

(i) *M is N-dual Baer;*

(ii) *M is N-d-Rickart;*

(iii) *Every nonzero $\varphi \in Hom_R(M, N)$ is an epimorphism.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear.

(ii) $\Rightarrow$ (iii) Let $0 \neq \varphi \in Hom_R(M, N)$. By assumption, $\mathrm{Im}\varphi$ is a direct summand of N. But N is indecomposable. Then $\mathrm{Im}\varphi = N$. This completes the proof. $\square$

Proposition 2.6. *Let M and N be modules such that $Hom_R(M, N) \neq 0$ (e.g., N is M-generated). Then the following conditions are equivalent:*

(i) *M is N-dual Baer and N is indecomposable;*

(ii) *Every nonzero homomorphism $\varphi \in \mathrm{Hom}_R(M, N)$ is an epimorphism.*

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2.5.

(ii) $\Rightarrow$ (i) It is clear that M is N-dual Baer. Now let K be a nonzero direct summand of N. Let K' be a submodule of N such that $N = K \oplus K'$. Since $Hom_R(M, N) \neq 0$, there exists a nonzero homomorphism $\varphi \in Hom_R(M, N)$. Let $\pi' : N \to K'$ be the projection map and let $i' : K' \to N$ be the inclusion map. Then $i'\pi'\varphi \in Hom_R(M, N)$. Assume that $i'\pi'\varphi \neq 0$. By hypothesis, $\mathrm{Im}i'\pi'\varphi = N$. So $K' = N$. Thus $K = 0$, a contradiction. Therefore $i'\pi'\varphi = 0$. Hence $K' = 0$ and $K = N$. It follows that N is indecomposable. $\square$

The following result describes the structure of R-modules which are R_R-dual Baer, where R is a commutative principal ideal domain which is not a field.

Proposition 2.7. *Let R be a commutative principal ideal domain which is not a field. Then the following conditions are equivalent for an R-module M:*

(i) *M is R_R-dual Baer;*

(ii) *M is R_R-d-Rickart;*

(iii) *M has no nonzero cyclic torsion-free direct summands;*

(iv) $Hom_R(M, R_R) = 0$.

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Assume that M has an element x such that xR is a direct summand of M and $R_R \cong xR$. Let $\pi : M \to xR$ be the projection map and let $f : xR \to R_R$ be an isomorphism. Then $f\pi : M \to R_R$ is an epimorphism. Let α be a nonzero element of R which is not invertible. Consider the homomorphism $g : R_R \to R_R$ defined by $g(r) = \alpha r$ for all $r \in R$. Then $gf\pi \in Hom_R(M, R_R)$ and $\mathrm{Im}gf\pi = \alpha R$. It is clear that $\alpha R \neq 0$ and $\alpha R \neq R$. Thus αR is not a direct summand of R. So M is not R_R-d-Rickart, a contradiction.

(iii) $\Rightarrow$ (iv) Assume that $Hom_R(M, R_R) \neq 0$. So there exists a nonzero homomorphism $f : M \to R_R$. Thus $\mathrm{Im}f = aR$ for some nonzero $a \in R$ since R is a principal ideal domain. Then $M/\mathrm{Ker}f \cong aR \cong R_R$ is a projective R-module. It follows that $\mathrm{Ker}f$ is a direct summand of M. Let Y be a submodule of M such that $M = \mathrm{Ker}f \oplus Y$. Therefore $Y \cong R_R$. This contradicts our assumption. Hence $Hom_R(M, R_R) = 0$.

(iv) $\Rightarrow$ (i) This is immediate. $\square$

Example 2.8. Consider a $\mathbb{Z}$-module $M = \mathbb{Q}^{(I)} \oplus T$, where T is a torsion $\mathbb{Z}$-module and I is an index set. Suppose that M is not $\mathbb{Z}$-dual Baer. By Proposition 2.7, there exists a cyclic submodule L of M such that $L \cong \mathbb{Z}$ and L is a direct summand of M. Let N be a submodule of M such that $M = L \oplus N$.

Since T is the torsion submodule of M, we have $T \subseteq N$. Hence T is a direct summand of N. Let K be a submodule of N such that $N = K \oplus T$. Thus $M = L \oplus K \oplus T$. Therefore $L \oplus K \cong \mathbb{Q}^{(I)}$. So L is injective, a contradiction. It follows that M is $\mathbb{Z}$-dual Baer. On the other hand, note that if $T \cong \mathbb{Z}(2^\infty) \oplus \mathbb{Z}/8\mathbb{Z}$, then M is not a dual Baer module (see [14, Corollary 3.5].

In Proposition 2.7, we studied when an R-module M is R_R-dual Baer. Next, we investigate when R_R is M-dual Baer for an R-module M.

Proposition 2.9. *The following conditions are equivalent for an R-module M:*

(i) *The R-module R_R is M-dual Baer;*

(ii) *M is a semisimple module.*

Proof. (i) $\Rightarrow$ (ii) Let $x \in M$. Consider the R-homomorphism $\varphi : R \to M$ defined by $\varphi(r) = xr$ for all $r \in R$. Then $\text{Im}\varphi = xR$. Since R_R is M-dual Baer, it follows that for any submodule L of M, $L = \sum_{x \in L} xR$ is a direct summand of M. Therefore M is semisimple.

(ii) $\Rightarrow$ (i) is obvious. □

Corollary 2.10. *The following conditions are equivalent for a ring R:*

(i) *The R-module R_R is dual Baer;*

(ii) *The R-module R_R is $E(R)$-dual Baer;*

(iii) *R is a semisimple ring.*

Proof. (i) $\Leftrightarrow$ (iii) By [14, Corollary 2.9].

(ii) $\Leftrightarrow$ (iii) This follows from Proposition 2.9. □

Remark 2.11. If K is a submodule of an R-module M such that K is M-dual Baer, then K is a direct summand of M. In particular, if the R-module M is $E(M)$-dual Baer, then M is an injective module.

The next example shows that even if a module M is injective, the module M need not be M-dual Baer.

Example 2.12. Let R be a self injective ring which is not semisimple (e.g., $R = \prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$). Then $E(R_R) = R_R$. By [14, Corollary 2.9], the R-module R_R is not R_R-dual Baer.

Next, we will be concerned with the modules M which are N-dual Baer for all modules N. We begin with the following proposition which provides a class of rings R whose semisimple modules are N-dual Baer for any R-module N.

Proposition 2.13. *Let R be a right noetherian right V-ring and let M be a semisimple R-module. Then M is N-dual Baer for every R-module N.*

Proof. Let N be an R-module. It is clear that for any $\varphi \in Hom_R(M, N)$, $\text{Im}\varphi$ is semisimple. Let A be a subset of $Hom_R(M, N)$. Then $\sum_{f \in A} \text{Im} f$ is a semisimple submodule of N. Since R is a right noetherian right V-ring, $\sum_{f \in A} \text{Im} f$ is injective by [4, Proposition 1]. Therefore $\sum_{f \in A} \text{Im} f$ is a direct summand of N. So M is N-dual Baer. □

The next example shows that the condition "R is a right noetherian ring" in the hypothesis of Proposition 2.13 is not superfluous.

Example 2.14. Let F be a field and let $R = \prod_{n \in \mathbb{N}} F_n$ such that $F_n = F$ for all $n \in \mathbb{N}$. Then R is a commutative V-ring which is not noetherian. Note that $Soc(R) = \oplus_{n \in \mathbb{N}} F_n$ is an essential proper ideal of R. In particular, $Soc(R)$ is not a direct summand of R. So $Soc(R)$ is not R_R-dual Baer.

Following [13], a module M is called *noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, Imf is not a small submodule of N.

Proposition 2.15. *Let M be a module. Assume that M is N-dual Baer for every R-module N. Then every factor module of M is injective. In particular, M is a noncosingular module.*

Proof. Let L be a submodule of M. Let $\pi : M \to M/L$ be the natural epimorphism and let $\mu : M/L \to E(M/L)$ be the inclusion map. Then $\mu\pi \in Hom_R(M, E(M/L))$ and Im$\mu\pi = M/L$. Since M is $E(M/L)$-dual Baer, M/L is a direct summand of $E(M/L)$. So M/L is injective. This completes the proof. □

Proposition 2.16. *Let R be a right noetherian ring. Then the following conditions are equivalent for an R-module M:*

(i) *M is N-dual Baer for all R-modules N;*

(ii) *Every factor module of M is an injective R-module.*

Proof. (i) ⇒ (ii) By Proposition 2.15.

(ii) ⇒ (i) Let N be an R-module. It is clear that Imφ is injective for every $\varphi \in Hom_R(M, N)$. Since the ring R is right noetherian, $\sum_{f \in A}$ Imf is injective for every subset A of Hom $_R(M, N)$ by [1, Proposition 18.13]. Therefore $\sum_{f \in A}$ Imf is a direct summand of N. This proves the proposition. □

Recall that a ring R is called *right hereditary* if each of its right ideals is projective. It is well known that a ring R is right hereditary if and only if every factor module of an injective right R-module is injective (see, for example [16, 39.16]). The next result is a direct consequence of Proposition 2.16. It determines the structure of R-modules M which are N-dual Baer for all R-modules N, where R is a right noetherian right hereditary ring.

Corollary 2.17. *Let R be a right noetherian right hereditary ring* (*e.g., R is a Dedekind domain*)*. Then the following conditions are equivalent for an R-module M:*

(i) *M is N-dual Baer for any R-module N;*

(ii) *M is an injective R-module.*

Example 2.18. Let M be a $\mathbb{Z}$-module. It is easily seen from Corollary 2.17 that M is N-dual Baer for any $\mathbb{Z}$-module N if and only if M is a direct sum of $\mathbb{Z}$-modules each isomorphic to the additive group of rational numbers $\mathbb{Q}$ or to $\mathbb{Z}(p^\infty)$ (for various primes p).

Combining Corollary 2.17 and [8, Corollary 2.30], we obtain the following result.

Corollary 2.19. *The following conditions are equivalent for a ring R:*

(i) *Every injective R-module is dual Baer;*

(ii) *Every injective module is N-dual Baer for every R-module N;*

(iii) *R is a right noetherian right hereditary ring.*

The next characterization extends [14, Corollary 2.5].

Theorem 2.20. *Let M and N be two R-modules. Then M is N-dual Baer if and only if for any direct summand M' of M and any submodule N' of N, M' is N'-dual Baer.*

Proof. Let $M' = eM$ for some $e^2 = e \in End_R(M)$ and let N' be a submodule of N. Let $\{\varphi_i\}_I$ be a family of homomorphisms in Hom $_R(M', N')$. Since $\varphi_i e(M) = \varphi_i(M') \subseteq N' \subseteq N$ for every $i \in I$ and M is N-dual Baer, $\sum_{i \in I} \varphi_i e(M)$ is a direct summand of N. Therefore $\sum_{i \in I} \varphi_i(M')$ is a direct summand of N'. It follows that M' is N'-dual Baer. The converse is obvious. □

Corollary 2.21. *The following conditions are equivalent for a module M:*

(i) *M is a dual Baer module;*

(ii) *For any direct summand K of M and any submodule N of M, K is N-dual Baer.*

From [14, Example 3.1 and Theorem 3.4], it follows that a direct sum of dual Baer modules is not dual Baer, in general. Next, we focus on when a direct sum of N-dual Baer modules is also N-dual Baer for some module N.

Proposition 2.22. *Let N be a module having the SSSP and let $\{M_i\}_I$ be a family of modules. Then $\bigoplus_{i\in I} M_i$ is N-dual Baer if and only if M_i is N-dual Baer for all $i \in I$.*

Proof. Suppose that $\bigoplus_{i\in I} M_i$ is N-dual Baer. By Theorem 2.20, M_i is N-dual Baer for all $i \in I$. Conversely, assume that M_i is N-dual Baer for all $i \in I$. Let $\{\varphi_\lambda\}_\Lambda$ be a family of homomorphisms in $\text{Hom}_R(\bigoplus_{i\in I} M_i, N)$. For each $i \in I$, let $\mu_i : M_i \to \bigoplus_{i\in I} M_i$ denote the inclusion map. Then for every $i \in I$ and every $\lambda \in \Lambda$, $\varphi_\lambda \mu_i \in Hom_R(M_i, N)$. Since M_i is N-dual Baer for every $i \in I$, it follows that $\text{Im}(\varphi_\lambda \mu_i)$ is a direct summand of N for every $(i, \lambda) \in I \times \Lambda$. Note that for each $\lambda \in \Lambda$, $\text{Im}\varphi_\lambda = \sum_{i\in I} \text{Im}(\varphi_\lambda \mu_i)$. As N has the $SSSP$, $\sum_{\lambda\in\Lambda} \text{Im}\varphi_\lambda = \sum_{\lambda\in\Lambda} \sum_{i\in I} \text{Im}(\varphi_\lambda \mu_i)$ is a direct summand of N. Therefore $\bigoplus_{i\in I} M_i$ is N-dual Baer. □

The following result is taken from [14, Theorem 2.1].

Theorem 2.23. *The following conditions are equivalent for a module M and $S = End_R(M)$:*

(i) *M is a dual Baer module;*

(ii) *For every nonempty subset A of S, $\sum_{f\in A} Imf = e(M)$ for some idempotent $e \in S$;*

(iii) *M has the $SSSP$ and for every $\varphi : M \to M$, $Im\varphi$ is a direct summand of M.*

Corollary 2.24. *Let $\{M_i\}_I$ be a family of modules and let $j \in I$. Then $\bigoplus_{i\in I} M_i$ is M_j-dual Baer if and only if M_i is M_j-dual Baer for all $i \in I$.*

Proof. The necessity follows from Theorem 2.20. Conversely, by assumption, we have M_j is M_j-dual Baer. Then M_j is a dual Baer module. By Theorem 2.23, M_j has the $SSSP$. Applying Proposition 2.22, $\bigoplus_{i\in I} M_i$ is M_j-dual Baer. □

In the following result, we present conditions under which a module N is $\bigoplus_{i=1}^n M_i$-dual Baer for some modules M_i $(1 \le i \le n)$.

Theorem 2.25. *Let $M_1, \ldots, M_n$ be R-modules, where $n \in \mathbb{N}$. Assume that M_i is M_j-projective for all $i > j \in \{1, 2, \ldots, n\}$. Then for any R-module N, N is $\bigoplus_{i=1}^n M_i$-dual Baer if and only if N is M_i-dual Baer for all $i \in \{1, 2, \ldots, n\}$.*

Proof. The necessity follows from Theorem 2.20. Conversely, suppose that N is M_i-dual Baer for all $i \in \{1, 2, \ldots, n\}$. We will show that N is $\bigoplus_{i=1}^n M_i$-dual Baer. By induction on n and taking into account [9, Proposition 4.33], it is sufficient to prove this for the case $n = 2$. Assume that N is M_i-dual Baer for $i = 1, 2$ and M_2 is M_1-projective. Let $\{\phi_\lambda\}_\Lambda$ be a family of homomorphisms in $\text{Hom}_R(N, M_1 \oplus M_2)$. Let $\pi_2 : M_1 \oplus M_2 \to M_2$ be the projection of $M_1 \oplus M_2$ on M_2 along M_1. We want to prove that $\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda$ is a direct summand of $M_1 \oplus M_2$. Since N is M_2-dual Baer, $\sum_{\lambda\in\Lambda} \pi_2\phi_\lambda(N)$ is a direct summand of M_2. So $\sum_{\lambda\in\Lambda} \pi_2\phi_\lambda(N)$ is M_1-projective by [9, Proposition 4.32]. As $M_1 + \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right) = M_1 \oplus \left(\sum_{\lambda\in\Lambda} \pi_2\phi_\lambda(N)\right)$ is a direct summand of $M_1 \oplus M_2$, there exists a submodule $L \le \sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda$ such that $M_1 + \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right) = M_1 \oplus L$ by [9, Lemma 4.47]. Thus $\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda = \left(M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right)\right) \oplus L$ by modularity. It is easily seen that $\sum_{\lambda\in\Lambda} \pi_2\phi_\lambda(N)$ is a direct summand of M_2. Let K_2 be a submodule of M_2 such that $M_2 = K_2 \oplus \left(\sum_{\lambda\in\Lambda} \pi_2\phi_\lambda(N)\right)$. Therefore $M_1 \oplus M_2 = M_1 \oplus L \oplus K_2$. Let $\pi_1 : M_1 \oplus (L \oplus K) \to M_1$

be the projection of $M_1 \oplus M_2$ on M_1 along $L \oplus K$. Then $\pi_1\phi_\lambda \in \text{Hom}_R(N, M_1)$ for every $\lambda \in \Lambda$. Moreover, we have

$$\sum_{\lambda\in\Lambda} \pi_1\phi_\lambda(N) = \pi_1\left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right) = \left(\left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right) + (L \oplus K)\right) \cap M_1.$$

But $\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda = \left(M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right)\right) \oplus L$. Then,

$$\sum_{\lambda\in\Lambda} \pi_1\phi_\lambda(N) = \left(\left(M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right)\right) \oplus L \oplus K\right) \cap M_1 = M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right).$$

Since N is M_1-dual Baer, $\sum_{\lambda\in\Lambda} \pi_1\phi_\lambda(N) = M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right)$ is a direct summand of M_1. It follows that $\left(M_1 \cap \left(\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda\right)\right) \oplus L$ is a direct summand of $M_1 \oplus L \oplus K_2$. So $\sum_{\lambda\in\Lambda} \text{Im}\phi_\lambda$ is a direct summand of $M_1 \oplus M_2$. Consequently, N is $M_1 \oplus M_2$-dual Baer. This completes the proof. □

Corollary 2.26. *Let $M_1, \ldots, M_n$ be R-modules, where $n \in \mathbb{N}$. Assume that M_i is M_j-projective for all $i > j \in \{1, 2, \ldots, n\}$. Then $M = \bigoplus_{i=1}^n M_i$ is a dual Baer module if and only if M_i is M_j-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$.*

Proof. The necessity follows from Theorem 2.20. Conversely, suppose that M_i is M_j-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$. By Corollary 2.24, M is M_j-dual Baer for all $j \in \{1, 2, \ldots, n\}$. Since M_i is M_j-projective for all $i > j \in \{1, 2, \ldots, n\}$, M is $\bigoplus_{i=1}^n M_i$-dual Baer by Theorem 2.25. Thus M is a dual Baer module. □

Note that the sufficiency in Corollary 2.26 can be proved by using [14, Theorem 3.10].

Following [8, Definition 5.7], a module M is called N-D_2 (or relatively D_2 to N) if for any submodule M' of M, M/M' is isomorphic to a direct summand of N implies that M' is a direct summand of M.

Proposition 2.27. *Let $M_1, \ldots, M_n$ be R-modules, where $n \in \mathbb{N}$. Assume that M_i is M_j-D_2 for all $i, j \in \{1, 2, \ldots, n\}$. Then $\bigoplus_{i=1}^n M_i$ is a dual Baer module if and only if M_i is M_j-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$ and M has the SSSP.*

Proof. ($\Rightarrow$) By [8, Theorem 5.11], M_i is M_j-d-Rickart for all $i, j \in \{1, 2, \ldots, n\}$. Note that M_i has the $SSSP$ for every $i \in \{1, 2, \ldots, n\}$ (see Theorem 2.23). Applying Proposition 2.3, it follows that M_i is M_j-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$.

($\Leftarrow$) This follows easily from [8, Theorem 5.11], Proposition 2.3 and Theorem 2.23. □

Theorem 2.28. *Let $M = \bigoplus_{i\in I} M_i$ be the direct sum of fully invariant submodules M_i. Then M is a dual Baer module if and only if M_i is a dual Baer module for all $i \in I$.*

Proof. The necessity follows from [14, Corollary 2.5]. Conversely, let $S = End_R(M)$ and let $\{\varphi_\lambda\}_\Lambda$ be a family of homomorphisms in S. For each $i \in I$, let $\pi_i : M \to M_i$ be the projection map and let $\mu_i : M_i \to M$ be the inclusion map. Note that for each $\lambda \in \Lambda$, $\varphi_\lambda(M) = \sum_{i\in I} \varphi_\lambda\mu_i(M_i)$. Since each M_i ($i \in I$) is fully invariant in M, it follows that $\varphi_\lambda(M) = \sum_{i\in I} \pi_i\varphi_\lambda\mu_i(M_i)$ for all $\lambda \in \Lambda$. For every $i \in I$ and every $\lambda \in \Lambda$, let $N_{i,\lambda} = \pi_i\varphi_\lambda\mu_i(M_i)$. Therefore,

$$\sum_{\lambda\in\Lambda} \varphi_\lambda(M) = \sum_{\lambda\in\Lambda}\sum_{i\in I} \pi_i\varphi_\lambda\mu_i(M_i) = \sum_{\lambda\in\Lambda}\left(\sum_{i\in I} N_{i,\lambda}\right) = \bigoplus_{i\in I}\left(\sum_{\lambda\in\Lambda} N_{i,\lambda}\right).$$

Since each M_i ($i \in I$) is dual Baer, each M_i ($i \in I$) has the $SSSP$ by Theorem 2.23. Thus $\sum_{\lambda\in\Lambda} N_{i,\lambda}$ is a direct summand of M_i for every $i \in I$. So $\sum_{\lambda\in\Lambda} \varphi_\lambda(M)$ is a direct summand of M. Consequently, M is a dual Baer module. □

We conclude this paper by showing a new characterization of dual Baer modules.

Let M be an R-module with $S = End_R(M)$. Then for every nonempty subset A of S, we denote $l_S(A) = \{\varphi \in S \mid \varphi A = 0\}$ and $r_M(A) = \{m \in M \mid Am = 0\}$. We also denote $l_S(N) = \{\varphi \in S \mid \varphi(N) = 0\}$ for any submodule N of M.

Recall that a ring R is called a *Baer ring* if for every nonempty subset $I \subseteq R$, there exists an idempotent $e \in R$ such that $l_S(I) = Re$.

Proposition 2.29. ([5, Proposition 2.3]) *For an R-module M, $S = End_R(M)$ is a Baer ring if and only if $r_M(l_S(\sum_{\varphi \in A} Im\varphi))$ is a direct summand of M for all nonempty subsets A of S.*

The next example shows that if M is a module such that $S = End_R(M)$ is a Baer ring, then M is not a dual Baer module, in general.

Example 2.30. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Then $S = End_{\mathbb{Z}}(M) \cong \mathbb{Z}$. Clearly, $\mathbb{Z}$ is a Baer ring. On the other hand, it is easily seen that M is not a dual Baer module.

Note that if M is an R-module with $S = End_R(M)$, then for any nonempty subset A of S, $l_S(A) = l_S(AM)$, where $AM = \sum_{f \in A} \mathrm{Im} f$. The next result can be considered as an analogue of [8, Theorem 3.5].

Theorem 2.31. *The following are equivalent for an R-module M and $S = End_R(M)$:*

(i) *M is a dual Baer module;*

(ii) *S is a Baer ring and $AM = r_M(l_S(AM))$ for every nonempty subset A of S;*

(iii) *S is a Baer ring and $IM = r_M(l_S(IM))$ for every right ideal I of S.*

Proof. (i) ⇒ (ii) From [15, Theorem 3.6], it follows that S is a Baer ring. Moreover, we have $r_M(l_S(AM)) = r_M(l_S(A)) = r_M(S(1-e)) = e(M) = AM$ for all nonempty subsets A of S.

(ii) ⇒ (iii) This is obvious.

(iii) ⇒ (i) Let I be a right ideal of S. Since S is a Baer ring, $r_M(l_S(IM))$ is a direct summand of M by Proposition 2.29. But $IM = r_M(l_S(IM))$. Then IM is a direct summand of M. By Theorem 2.23, it follows that M is a dual Baer module. □

Combining Theorem 2.31 and [10, Theorem 4.1], we get the following result.

Corollary 2.32. *Let M be an R-module such that $IM = r_M(l_S(IM))$ for every right ideal I of $S = End_R(M)$. If M is a Baer module, then M is a dual Baer module.*

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