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Research Article

Hermite–Hadamard Type Inclusions for m-Polynomial Harmonically Convex Interval-Valued Functions

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ABSTRACT. We introduce the notion of m-polynomial harmonically convex interval-valued function. A relationship between a given interval-valued function and its component real-valued functions is pointed out. Moreover, some new Hermite–Hadamard type results are established for this class of functions. In particular, we show that if a non-negative interval-valued function F, defined on a harmonically convex set $\mathbf{S}$, is m-polynomial harmonically convex with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, then

$$\frac{2^{-1}\, m}{m+2^{-m}-1} F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \supseteq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr \supseteq \frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^{m}\frac{p}{p+1},$$

where F is Lebesgue integrable on $[\alpha,\beta]$. Our results complement and extend existing results in the literature. By taking $m \geq 2$, we derive loads of new and interesting inclusions. We anticipate that the idea outlined herein will trigger further investigations in this direction.

Keywords: Hermite–Hadamard, m-polynomial harmonically convex, interval-valued function.

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1. INTRODUCTION

The Hermite–Hadamard inequality (HHI) stipulates that the average value of a convex function on an interval is bounded below by the value of the function at the midpoint of the interval and above by the average value of the function at the endpoints of the interval. Whenever a new class of function is introduced, researchers want to know if the analogue of the HHI can be established for such class. Loads of articles, in this direction, are bound in the literature. See for example, [3, 4, 9, 10, 12, 13, 14, 15, 22, 23, 26, 24] and the references cited therein. One of such is the harmonically convex function: a set $\mathbf{S} \subset \mathbb{R} \setminus \{0\}$ is called a harmonically convex set if

$$\frac{xy}{\tau x+(1-\tau)y} \in \mathbf{S}$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. In 2014, İşcan [11] proposed and defined a harmonically convex function as follows: a real valued function $f : \mathbf{S} \to \mathbb{R}^+ := (0,\infty)$ is harmonically convex if

$$f\left(\frac{xy}{\tau x+(1-\tau)y}\right) \leq \tau f(y)+(1-\tau)f(x)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. In the same paper, the author established the following Hermite–Hadamard type inequality for this class of functions:

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Theorem 1.1 ([11]). *Let $f : \mathbf{S} \to \mathbb{R}$ be a harmonically convex function. If $\alpha, \beta \in \mathbf{S}$ with $\alpha < \beta$, and f is Lebesgue integrable on $[\alpha, \beta]$, then the following Hermite–Hadamard type inequality holds:*

$$f\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \le \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{f(r)}{r^2}\,dr \le \frac{f(\alpha)+f(\beta)}{2}.$$

Recently, Awan et al. [1] introduced the notion of m-polynomial harmonically convex functions as a generalization of the harmonically convex functions, and then proved, among other things, the result that follows:

Definition 1.1 ([1]). *Let $m \in \mathbb{N}$. Then, a real-valued function $f : \mathbf{S} \to \mathbb{R}^+$ is said to be m-polynomial harmonically convex (concave) if*

$$f\left(\frac{xy}{\tau x+(1-\tau)y}\right) \le (\ge)\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. The sets of all m-polynomial harmonically convex and m-polynomial harmonically concave functions from $\mathbf{S}$ into $\mathbb{R}^+$ is denoted by $\mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$ and $\mathbf{HVP}_m(\mathbf{S}, \mathbb{R}^+)$, respectively.

Theorem 1.2 ([1]). *Let $f : [\alpha, \beta] \to \mathbb{R}^+$ be an m-polynomial harmonically convex function. If f is Riemann integrable on $[\alpha, \beta]$, then*

$$\frac{2^{-1}m}{m+2^{-m}-1}f\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \le \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{f(r)}{r^2}\,dr \le \frac{f(\alpha)+f(\beta)}{m}\sum_{p=1}^{m}\frac{p}{p+1}.$$

In 1966, the late American Mathematician Ramon E. Moore initiated the theory of interval analysis [18]: simply put, the analysis of interval-valued functions. Ever since, this field has received great deal of attention from researchers in various areas of the mathematical sciences (like experts in global optimization and constraint solution algorithms) and has grown steadily in popularity over the past four decades. Interval analysis has been found to be valuable to engineers and scientists interested in scientific computation, especially in reliability, effects of roundoff error, and automatic verification of results, see [7, 8, 6, 5]. With the advent of interval analysis, mathematicians, those who work in the field of mathematical inequalities, want to know if the inequalities in the above mentioned results can be replaced with inclusions. In some cases, the answer to the question is in the affirmative. In lieu of this, E. Sadowska (see also [17]) established the following result for a given interval-valued function:

Theorem 1.3 ([25]). *Let F be a nonnegative continuous convex set-valued function on $[\alpha, \beta]$. Then,*

$$F\left(\frac{\alpha+\beta}{2}\right) \supset \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,dr \supset \frac{F(\alpha)+F(\beta)}{2}. \tag{1.1}$$

Results akin to (1.1), for different classes of set-valued convex functions, have been established. For example, see the papers [28, 27, 21, 16, 7, 8, 6, 5, 2, 29]. Motivated by the above mentioned articles, it is our goal in this article to introduce a new class of interval-valued function called the m-polynomial harmonically convex function and then obtain the interval-valued counterpart of Theorem 1.2. Thereafter, we will establish four more results in this direction. Our results complement and extend known results in the literature. The paper is organized in the following manner: Section 2 contains some brief background information in the theory of interval analysis. In Section 3, we state and prove our main results; followed by an open problem in Section 4.

2. Preliminaries

In this section, we give a brief overview of the theory of interval analysis. For an indepth study of this subject, we invite the interested reader to see the books [18, 19, 20]. We shall call $\mathbb{K}_c$ the class of all bounded closed nonempty intervals in $\mathbb{R}$, i.e.,

$$\mathbb{K}_c := \left\{\left[\alpha^-, \alpha^+\right] \mid \alpha^-, \alpha^+ \in \mathbb{R} \quad \text{and} \quad \alpha^- \leq \alpha^+\right\}.$$

The numbers α^- and α^+ are called the left and right endpoints of $[\alpha^-, \alpha^+]$, respectively. The interval $[\alpha^-, \alpha^+]$ is called degenerated if $\alpha^- = \alpha^+$; positive if $\alpha^- > 0$ and negative if $\alpha^+ < 0$. We denote the sets of all negative intervals and positive intervals in $\mathbb{R}$ by $\mathbb{K}_c^-$ and $\mathbb{K}_c^+$, respectively. That is;

$$\mathbb{K}_c^- := \left\{\left[\alpha^-, \alpha^+\right] \in \mathbb{K}_c \mid \alpha^+ < 0\right\}$$

and

$$\mathbb{K}_c^+ := \left\{\left[\alpha^-, \alpha^+\right] \in \mathbb{K}_c \mid \alpha^- > 0\right\}.$$

Let $A = [\alpha^-, \alpha^+]$, $B = [\beta^-, \beta^+] \in \mathbb{K}_c$ and $\gamma \in \mathbb{R}$. We say $A \subseteq B$ (or $B \supseteq A$) if and only if $\beta^- \leq \alpha^-$ and $\alpha^+ \leq \beta^+$. The following arithmetic operations are defined thus

$$\gamma A = \begin{cases} [\gamma\alpha^-, \gamma\alpha^+] & \text{if} \quad \gamma > 0 \\ \{0\} & \text{if} \quad \gamma = 0 \\ [\gamma\alpha^+, \gamma\alpha^-] & \text{if} \quad \gamma < 0; \end{cases}$$

$$A + B = \left[\alpha^-, \alpha^+\right] + \left[\beta^-, \beta^+\right] := \left[\alpha^- + \beta^-, \alpha^+ + \beta^+\right];$$

$$A - B = \left[\alpha^-, \alpha^+\right] - \left[\beta^-, \beta^+\right] := \left[\alpha^- - \beta^+, \alpha^+ - \beta^-\right];$$

$$A \cdot B := \left[\min\left\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\right\}, \max\left\{\alpha^-\beta^-, \alpha^-\beta^+, \alpha^+\beta^-, \alpha^+\beta^+\right\}\right];$$

$$\frac{A}{B} := \left[\min\left\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\right\}, \max\left\{\frac{\alpha^-}{\beta^-}, \frac{\alpha^-}{\beta^+}, \frac{\alpha^+}{\beta^-}, \frac{\alpha^+}{\beta^+}\right\}\right], \quad 0 \notin B.$$

The Pompieu–Hausdorff distance $d_H : \mathbb{K}_c \times \mathbb{K}_c \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$d_H := \max\left\{\max_{\alpha \in A} d(\alpha, B), \ \max_{\beta \in B} d(\beta, A)\right\} \quad \text{with} \quad d(\beta, A) = \min_{\alpha \in A} |\beta - \alpha|.$$

It is generally known that $(\mathbb{K}_c, d_H)$ is a complete metric space. The concept of a convergent sequence of intervals $(A_n)_{n \in \mathbb{N}}$, $A_n \in \mathbb{K}_c$ is considered in the complete metric space $\mathbb{K}_c$, endowed with the d_H distance. We say that $\lim\limits_{n \to \infty} A_n = A$ if and only if for any real number $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$d_H(A_n, A) < \epsilon \quad \text{for all} \quad n > N_\epsilon.$$

Next, we turn our attention to interval-valued functions.

Definition 2.2. *An interval-valued function is defined to be any $F : [\alpha, \beta] \to \mathbb{K}_c$ with $F(x) = [f^-(x), f^+(x)] \in \mathbb{K}_c$ and $f^-(x) \leq f^+(x)$ for all $x \in [\alpha, \beta]$. We say that F is Lebesgue integrable on $[\alpha, \beta]$ if the real-valued functions f^- and f^+ are Lebesgue integrable on $[\alpha, \beta]$, and then write*

$$\int_\alpha^\beta F(r)\,dr = \left[\int_\alpha^\beta f^-(r)\,dr, \int_\alpha^\beta f^+(r)\,dr\right].$$

3. Main Results

We start by introducing the concept of m-polynomial harmonically convex interval-valued function in the following definition.

Definition 3.3. *Let $\mathbf{S}$ be a harmonically convex set, $F : \mathbf{S} \to \mathbb{K}_c^+$ an interval-valued function and $m \in \mathbb{N}$. We say that F is m-polynomial harmonically convex (concave) if and only if*

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]F(y)\subseteq(\supseteq)F\left(\frac{xy}{\tau x+(1-\tau)y}\right) \tag{3.2}$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. In what follows, we shall denote the sets of all m-polynomial harmonically convex and m-polynomial harmonically concave interval-valued functions from $\mathbf{S}$ into $\mathbb{K}_c^+$ by $\mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ and $\mathbf{HVP}_m(\mathbf{S}, \mathbb{K}_c^+)$, respectively.

Remark 3.1. For a specific value of m, we get a corresponding set inclusion. For instance,

(1) If $m = 1$, then we get the definition of harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right)\supseteq \tau F(x)+(1-\tau)F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

(2) For $m = 2$, we get the following inclusion for a 2-polynomial harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right)\supseteq \frac{3\tau-\tau^2}{2}F(x)+\frac{2-\tau-\tau^2}{2}F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

(3) For $m = 3$, we deduce the succeeding relation for a 3-polynomial harmonically convex interval-valued function:

$$F\left(\frac{xy}{\tau x+(1-\tau)y}\right)\supseteq \frac{6\tau-4\tau^2+\tau^3}{3}F(x)+\frac{3-\tau-\tau^2-\tau^3}{3}F(y)$$

for all $x, y \in \mathbf{S}$ and $\tau \in [0,1]$.

The following theorem gives a relationship between a given interval-valued function F and its component real-valued functions f^- and f^+.

Theorem 3.4. *Let $F : \mathbf{S} \to \mathbb{K}_c^+$ be an interval-valued function such that $F(x) = [f^-(x), f^+(x)] \in \mathbb{K}_c$ and $f^-(x) \le f^+(x)$ for all $x \in [\alpha, \beta]$. Then, $F \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ if and only if $f^- \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$ and $f^+ \in \mathbf{HVP}_m(\mathbf{S}, \mathbb{R}^+)$.*

Proof. Let $x, y \in \mathbf{S}$ and $\tau \in [0,1]$. Then,

$$F \in \mathbf{HXP}_m\left(\mathbf{S}, \mathbb{K}_c^+\right)$$

if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]F(y)\subseteq F\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

if and only if

$$\left[\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^-(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^-(y),\right.$$
$$\left.\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^+(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^+(y)\right]$$
$$\subseteq\left[f^-\left(\frac{xy}{\tau x+(1-\tau)y}\right),f^+\left(\frac{xy}{\tau x+(1-\tau)y}\right)\right]$$

if and only if

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^-(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^-(y)\geq f^-\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

and

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-(1-\tau)^p\right]f^+(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\tau^p\right]f^+(y)\leq f^+\left(\frac{xy}{\tau x+(1-\tau)y}\right)$$

if and only if

$$f^-\in\mathbf{HXP}_m\left(\mathbf{S},\mathbb{R}^+\right)\quad\text{and}\quad f^+\in\mathbf{HVP}_m\left(\mathbf{S},\mathbb{R}^+\right).$$

That completes the proof in both directions. □

Following a similar line of argument, one can easily prove the following result.

Theorem 3.5. *Let $F:\mathbf{S}\to\mathbb{K}_c^+$ be an interval-valued function such that $F(x)=[f^-(x),f^+(x)]\in\mathbb{K}_c$ and $f^-(x)\leq f^+(x)$ for all $x\in[\alpha,\beta]$. Then, $F\in\mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$ if and only if $f^-\in\mathbf{HVP}_m(\mathbf{S},\mathbb{R}^+)$ and $f^+\in\mathbf{HXP}_m(\mathbf{S},\mathbb{R}^+)$.*

For the remaining part of this article, we shall assume that $F:\mathbf{S}\to\mathbb{K}_c^+$ is always of the form $F(x)=[f^-(x),f^+(x)]\in\mathbb{K}_c$ and $f^-(x)\leq f^+(x)$ for all $x\in[\alpha,\beta]$. We are now ready to formulate and prove some Hermite–Hadamard type results for m-polynomial harmonically convex (concave) interval-valued functions.

Theorem 3.6. *Let $F:\mathbf{S}\to\mathbb{K}_c^+$ be an interval-valued function with $\alpha<\beta$ and $\alpha,\beta\in\mathbf{S}$, and Lebesgue integrable on $[\alpha,\beta]$. If $F\in\mathbf{HXP}_m(\mathbf{S},\mathbb{K}_c^+)$, then*

$$\frac{2^{-1}m}{m+2^{-m}-1}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^{m}\frac{p}{p+1}. \tag{3.3}$$

The inclusions are reversed if $F\in\mathbf{HVP}_m(\mathbf{S},\mathbb{K}_c^+)$.

Proof. Assuming $F\in\mathbf{HXP}_m(\mathbf{S},\mathbb{K}_c^+)$, we get from (3.2) the following relation:

$$F\left(\frac{xy}{\frac{1}{2}x+\frac{1}{2}y}\right)\supseteq\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]F(x)+\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]F(y).$$

This implies that for all $x,y\in\mathbf{S}$

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]\Big(F(x)+F(y)\Big)\subseteq F\left(\frac{2xy}{x+y}\right). \tag{3.4}$$

Now, let $x = \frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}$ and $y = \frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}$. Then, (3.4) becomes:

$$\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}\subseteq F\left(\frac{2\alpha\beta}{\alpha+\beta}\right). \tag{3.5}$$

Integrating both sides of (3.5) with respect to τ over $[0,1]$, we get

(3.6)
$$\begin{aligned}
\int_0^1 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau &\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_0^1\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau\\
&=\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_0^1\left\{f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+f^-\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau,\right.\\
&\qquad\left.\int_0^1\left\{f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)+f^+\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\}d\tau\right]\\
&=\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\frac{2\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{f^-(r)}{r^2}dr,\frac{2\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{f^+(r)}{r^2}dr\right]\\
&=\frac{2\alpha\beta}{\beta-\alpha}\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_\alpha^\beta\frac{f^-(r)}{r^2}dr,\int_\alpha^\beta\frac{f^+(r)}{r^2}dr\right]\\
&=\frac{2\alpha\beta}{\beta-\alpha}\frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_\alpha^\beta\frac{F(r)}{r^2}dr.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^1 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau &= \left[\int_0^1 f^-\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau,\int_0^1 f^+\left(\frac{2\alpha\beta}{\alpha+\beta}\right)d\tau\right]\\
&=\left[f^-\left(\frac{2\alpha\beta}{\alpha+\beta}\right),f^+\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\right]\\
&=F\left(\frac{2\alpha\beta}{\alpha+\beta}\right).
\end{aligned}\tag{3.7}$$

Using (3.7) in (3.6), one gets

$$\frac{m}{m+2^{-m}-1}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta\frac{F(r)}{r^2}dr. \tag{3.8}$$

Next, we substitute $x = \alpha$ and $y = \beta$ into (3.2) and integrate the resulting inclusion with respect to τ over $[0, 1]$, to obtain

$$\begin{aligned}\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{F(r)}{r^2}\,dr &= \int_0^1 F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)d\tau \\ &\supseteq \int_0^1 \left\{\frac{1}{m}\sum_{p=1}^m \left[1-(1-\tau)^p\right]F(\alpha)+\frac{1}{m}\sum_{p=1}^m\left[1-\tau^p\right]F(\beta)\right\}d\tau \\ &= \frac{1}{m}\sum_{p=1}^m\int_0^1\left[1-(1-\tau)^p\right]F(\alpha)\,d\tau+\frac{1}{m}\sum_{p=1}^m\int_0^1\left[1-\tau^p\right]F(\beta)\,d\tau \\ &= \frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^m\frac{p}{p+1}.\end{aligned}$$

This gives

$$\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr \supseteq \frac{F(\alpha)+F(\beta)}{m}\sum_{p=1}^m\frac{p}{p+1}. \tag{3.9}$$

Combining (3.8) and (3.9), we get the desired result (3.3). If $F \in \mathbf{HVP}_m(\mathbf{S}, \mathbb{K}_c^+)$, then we establish the reverse inclusions in a similar manner. □

Remark 3.2. Using Theorem 3.6, we obtain the following corollaries:

(1) For $m = 1$, we deduce the result for 1-polynomial harmonically convex interval-valued functions:

$$F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{2}.$$

(2) If $m = 2$, then we obtain the result for 2-polynomial harmonically convex interval-valued functions:

$$\frac{4}{35}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\supseteq\frac{\alpha\beta}{7(\beta-\alpha)}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\frac{F(\alpha)+F(\beta)}{12}.$$

Theorem 3.7. *Let $F : \mathbf{S} \to \mathbb{K}_c^+$ be an interval-valued function with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, and Lebesgue integrable on $[\alpha, \beta]$. If $F \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$, then*

$$\begin{aligned}\frac{1}{4}\left(\frac{m}{m+2^{-m}-1}\right)^2 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) &\supseteq \Omega_1 \supseteq \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta\frac{F(r)}{r^2}\,dr\supseteq\Omega_2 \\ &\supseteq \Big(F(\alpha)+F(\beta)\Big)\frac{m^2+2m+2^{1-m}-2}{2m^2}\sum_{p=1}^m\frac{p}{p+1},\end{aligned} \tag{3.10}$$

where

$$\Omega_1 := \frac{1}{4}\frac{m}{m+2^{-m}-1}\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\};$$

$$\Omega_2 := \frac{1}{2}\left[\frac{F(\alpha)+F(\beta)+2F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^m\frac{p}{p+1}.$$

The inclusions are reversed if $F \in \mathbf{HVP}_m(\mathbf{S}, \mathbb{K}_c^+)$.

Proof. Using the fact that $F \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{K}_c^+)$ and recalling (3.4)

$$\frac{1}{m}\sum_{p=1}^{m}\left[1-\frac{1}{2^p}\right]\Big(F(x)+F(y)\Big) \subseteq F\left(\frac{2xy}{x+y}\right) \tag{3.11}$$

for all $x, y \in \mathbf{S}$. So, in particular for

$$x = \frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda} \quad \text{and} \quad y = \frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}, \quad \text{where} \quad \lambda = \frac{2\alpha\beta}{\alpha+\beta},$$

the inclusion in (3.11) becomes:

$$F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) \supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+F\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}.$$

Integrating both sides of the above relation with respect to τ over $[0, 1]$, one gets

$$\begin{aligned}
F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) &\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\int_0^1\left\{F\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+F\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau \\
&= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left[\int_0^1\left\{f^-\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+f^-\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau,\right. \\
&\qquad \left.\int_0^1\left\{f^+\left(\frac{\alpha\lambda}{\tau\alpha+(1-\tau)\lambda}\right)+f^+\left(\frac{\alpha\lambda}{\tau\lambda+(1-\tau)\alpha}\right)\right\}d\tau\right] \\
&= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\frac{4\alpha\beta}{\beta-\alpha}\left[\int_\alpha^\lambda\frac{f^-(r)}{r^2}\,dr, \int_\alpha^\lambda\frac{f^+(r)}{r^2}\,dr\right] \\
&= \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\alpha^\lambda\frac{F(r)}{r^2}\,dr.
\end{aligned}$$

Thus, we have

$$F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right) \supseteq \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\alpha^\lambda\frac{F(r)}{r^2}\,dr. \tag{3.12}$$

If we also let

$$x = \frac{\beta\lambda}{\tau\lambda+(1-\tau)\beta} \quad \text{and} \quad y = \frac{\beta\lambda}{\tau\beta+(1-\tau)\lambda}$$

and then proceed as outlined above, we obtain

$$F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right) \supseteq \frac{m+2^{-m}-1}{m}\,\frac{4\alpha\beta}{\beta-\alpha}\int_\lambda^\beta\frac{F(r)}{r^2}\,dr. \tag{3.13}$$

Also, by setting $x = \frac{4\alpha\beta}{\alpha+3\beta}$ and $y = \frac{4\alpha\beta}{\beta+3\alpha}$ into (3.11) and then using (3.12) and (3.13), we obtain

$$F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\}$$

$$\supseteq \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\left\{\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\lambda}\frac{F(r)}{r^2}\,dr\right.$$

$$\left. +\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\lambda}^{\beta}\frac{F(r)}{r^2}\,dr\right\} \tag{3.14}$$

$$= \frac{1}{m}\sum_{p=1}^{m}\left(1-\frac{1}{2^p}\right)\frac{m+2^{-m}-1}{m}\frac{4\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr$$

$$= 4\left(\frac{m+2^{-m}-1}{m}\right)^2\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr.$$

From (3.14), we get the following chain of inclusions:

$$\frac{1}{4}\left(\frac{m}{m+2^{-m}-1}\right)^2 F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \supseteq \frac{1}{4}\frac{m}{m+2^{-m}-1}\left\{F\left(\frac{4\alpha\beta}{\alpha+3\beta}\right)+F\left(\frac{4\alpha\beta}{\beta+3\alpha}\right)\right\} \tag{3.15}$$

$$\supseteq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr.$$

Employing the second inclusion of (3.3) from Theorem 3.6 and (3.11), we get

$$\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)}{r^2}\,dr = \frac{1}{2}\left[\frac{2\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\frac{2\alpha\beta}{\alpha+\beta}}\frac{F(r)}{r^2}\,dr + \frac{2\alpha\beta}{\beta-\alpha}\int_{\frac{2\alpha\beta}{\alpha+\beta}}^{\beta}\frac{F(r)}{r^2}\,dr\right]$$

$$\supseteq \frac{1}{2}\left[\frac{F(\alpha)+F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}+\frac{F(\beta)+F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^{m}\frac{p}{p+1}$$

$$= \frac{1}{2}\left[\frac{F(\alpha)+F(\beta)+2F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)}{m}\right]\sum_{p=1}^{m}\frac{p}{p+1} \tag{3.16}$$

$$\supseteq \left[\frac{F(\alpha)+F(\beta)}{2}+\frac{m+2^{-m}-1}{m^2}\Big(F(\alpha)+F(\beta)\Big)\right]\sum_{p=1}^{m}\frac{p}{p+1}$$

$$= \Big(F(\alpha)+F(\beta)\Big)\frac{m^2+2m+2^{1-m}-2}{2m^2}\sum_{p=1}^{m}\frac{p}{p+1}.$$

We get the intended result by putting together (3.15) and (3.16). □

Theorem 3.8. *Let $F, G : \mathbf{S} \to \mathbb{K}_c^+$ be two interval-valued functions with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, and suppose FG is Lebesgue integrable on $[\alpha, \beta]$. If $F \in \mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $G \in \mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$, then*

$$\begin{aligned}
&\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{F(r)\,G(r)}{r^2}\,dr\\
&\supseteq F(\alpha)G(\alpha)\int_0^1\Delta_1(\tau)\,d\tau+F(\alpha)G(\beta)\int_0^1\Delta_2(\tau)\,d\tau\\
&+F(\beta)G(\alpha)\int_0^1\Delta_3(\tau)\,d\tau+F(\beta)G(\beta)\int_0^1\Delta_4(\tau)\,d\tau,
\end{aligned}\tag{3.17}$$

where

$$\Delta_1(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p];$$

$$\Delta_2(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-\tau^p];$$

$$\Delta_3(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p];$$

$$\Delta_4(\tau):=\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-\tau^p].$$

The inclusions are reversed if $F\in\mathbf{HVP}_{m_1}(\mathbf{S},\mathbb{K}_c^+)$ *and* $G\in\mathbf{HVP}_{m_2}(\mathbf{S},\mathbb{K}_c^+)$.

Proof. Given that $F\in\mathbf{HXP}_{m_1}(\mathbf{S},\mathbb{K}_c^+)$ and $G\in\mathbf{HXP}_{m_2}(\mathbf{S},\mathbb{K}_c^+)$, we get

$$\frac{1}{m_1}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\,F(\alpha)+\frac{1}{m_1}\sum_{p=1}^{m_1}[1-\tau^p]\,F(\beta)\subseteq F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\tag{3.18}$$

and

$$\frac{1}{m_2}\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,G(\alpha)+\frac{1}{m_2}\sum_{p=1}^{m_2}[1-\tau^p]\,G(\beta)\subseteq G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right).\tag{3.19}$$

This implies

$$\begin{aligned}
&F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\\
&\supseteq\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,F(\alpha)G(\alpha)\\
&+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-(1-\tau)^p]\sum_{p=1}^{m_2}[1-\tau^p]\,F(\alpha)G(\beta)\\
&+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-(1-\tau)^p]\,F(\beta)G(\alpha)\\
&+\frac{1}{m_1}\frac{1}{m_2}\sum_{p=1}^{m_1}[1-\tau^p]\sum_{p=1}^{m_2}[1-\tau^p]\,F(\beta)G(\beta)\\
&:=\Delta_1(\tau)F(\alpha)G(\alpha)+\Delta_2(\tau)F(\alpha)G(\beta)+\Delta_3(\tau)F(\beta)G(\alpha)+\Delta_4(\tau)F(\beta)G(\beta).
\end{aligned}\tag{3.20}$$

Now, integrating both sides of (3.20) with respect to τ over $[0,1]$, gives

$$\begin{aligned}
&\int_0^1 F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau \\
&= \int_0^1 \left[f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right),\right. \\
&\left. f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right] d\tau \\
&= \left[\int_0^1 f^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^-\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau,\right. \\
&\left.\int_0^1 f^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) g^+\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) d\tau\right] \\
&= \left[\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{f^-(r)g^-(r)}{r^2}\,dr,\ \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{f^+(r)g^+(r)}{r^2}\,dr\right] \\
&= \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{F(r)\,G(r)}{r^2}\,dr \\
&\supseteq F(\alpha)G(\alpha)\int_0^1 \Delta_1(\tau)\,d\tau + F(\alpha)G(\beta)\int_0^1 \Delta_2(\tau)\,d\tau \\
&+F(\beta)G(\alpha)\int_0^1 \Delta_3(\tau)\,d\tau + F(\beta)G(\beta)\int_0^1 \Delta_4(\tau)\,d\tau.
\end{aligned}$$

Hence that completes the proof. □

Theorem 3.9. *Let $F, G : \mathbf{S} \to \mathbb{K}_c^+$ be two interval-valued functions with $\alpha < \beta$ and $\alpha, \beta \in \mathbf{S}$, and suppose FG is Lebesgue integrable on $[\alpha, \beta]$. If $F \in \mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$, $G \in \mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$, $R(\alpha,\beta) = F(\alpha)G(\alpha) + F(\beta)G(\beta)$ and $Q(\alpha,\beta) = F(\alpha)G(\beta) + F(\beta)G(\alpha)$, then*

$$\begin{aligned}
&\frac{m_1 m_2}{(m_1+2^{-m_1}-1)(m_2+2^{-m_2}-1)} F\left(\frac{2\alpha\beta}{\alpha+\beta}\right) G\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \\
&\supseteq \frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta \frac{F(r)\,G(r)}{r^2}\,dr + R(\alpha,\beta)\int_0^1 \left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau) + \tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right] d\tau \\
&+Q(\alpha,\beta)\int_0^1 \left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau) + \tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right] d\tau,
\end{aligned}$$

where $\Lambda_m(\tau) = \frac{1}{m}\sum_{p=1}^m [1-(1-\tau)^p]$ and $\tilde{\Lambda}_m(\tau) = \frac{1}{m}\sum_{p=1}^m [1-\tau^p]$. The inclusions are reversed if $F \in \mathbf{HVP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $G \in \mathbf{HVP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$.

Proof. Let $\tau \in [0,1]$. From the definition of $\tilde{\Lambda}_m$ and Λ_m above, one observes that

$$\tilde{\Lambda}_m\left(\frac{1}{2}\right) = \Lambda_m\left(\frac{1}{2}\right) := P_m := \frac{m+2^{-m}-1}{m}.$$

Hence, from (3.5), one gets

$$P_{m_1}\left\{F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) + F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\} \subseteq F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)$$

and

$$P_{m_2}\left\{G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right) + G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right\} \subseteq G\left(\frac{2\alpha\beta}{\alpha+\beta}\right).$$

Now,

$$\begin{aligned}
&F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)G\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\\
\supseteq & P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+\,F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right.\\
&\left.+\,F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right]\\
\supseteq & P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+\,F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)F(\alpha)+\tilde{\Lambda}_{m_1}(\tau)F(\beta)\right]\left[\Lambda_{m_2}(\tau)G(\beta)+\tilde{\Lambda}_{m_2}(\tau)G(\alpha)\right]\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)F(\beta)+\tilde{\Lambda}_{m_1}(\tau)F(\alpha)\right]\left[\Lambda_{m_2}(\tau)G(\alpha)+\tilde{\Lambda}_{m_2}(\tau)G(\beta)\right]\right\}\\
= & P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+\,F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right][F(\alpha)G(\alpha)+F(\beta)G(\beta)]\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right][F(\alpha)G(\beta)+F(\beta)G(\alpha)]\right\}\\
= & P_{m_1}P_{m_2}\left[F\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)G\left(\frac{\alpha\beta}{\tau\alpha+(1-\tau)\beta}\right)\right.\\
&\left.+\,F\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)G\left(\frac{\alpha\beta}{\tau\beta+(1-\tau)\alpha}\right)\right]\\
&+P_{m_1}P_{m_2}\left\{\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right]R(\alpha,\beta)\right.\\
&\left.+\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right]Q(\alpha,\beta)\right\}.
\end{aligned}\tag{3.21}$$

Integrating with respect to τ over $[0,1]$, we get from (3.21) the following inclusion:

$$\begin{aligned}
&\frac{1}{P_{m_1}P_{m_2}}F\left(\frac{2\alpha\beta}{\alpha+\beta}\right)G\left(\frac{2\alpha\beta}{\alpha+\beta}\right)\\
\supseteq&\frac{2\alpha\beta}{\alpha+\beta}\int_\alpha^\beta\frac{F(r)\,G(r)}{r^2}\,dr+R(\alpha,\beta)\int_0^1\left[\Lambda_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\Lambda_{m_2}(\tau)\right]d\tau\\
&+Q(\alpha,\beta)\int_0^1\left[\Lambda_{m_1}(\tau)\Lambda_{m_2}(\tau)+\tilde{\Lambda}_{m_1}(\tau)\tilde{\Lambda}_{m_2}(\tau)\right]d\tau.
\end{aligned}$$

That completes the proof. □

4. Conclusion

A new class of interval-valued function has been proposed. We show that an interval-valued function $F(x) = [f^-(x), f^+(x)]$ is m-polynomial harmonically convex if and only if its component real-valued functions f^- and f^+ are m-polynomial harmonically convex and m-polynomial harmonically concave, respectively. Furthermore, some new set-inclusions of the Hermite–Hadamard type are hereby established. We therefore pose the following open question:

Open question 1. *Let $m_1, m_2 \in \mathbb{N}$. Is it possible to compare $\mathbf{HXP}_{m_1}(\mathbf{S}, \mathbb{K}_c^+)$ and $\mathbf{HXP}_{m_2}(\mathbf{S}, \mathbb{K}_c^+)$?*

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