Konuralp Journal of Mathematics, 9 (1) (2021) 137-142

Konuralp Journal of Mathematics

Research Paper

https://dergipark.org.tr/en/pub/konuralpjournalmath

e-ISSN: 2147-625X

Certain New Subclasses of t-fold Symmetric Bi-univalent Function Using Q-derivative Operator

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Abstract

In this current study, we introduced and investigated two new subclasses of the bi-univalent functions associated with q-derivative operator; both $f(z)$ and $f^{-1}(z)$ are t-fold symmetric holomorphic functions in the open unit disk. Among other results, upper bounds for the coefficients $|\rho_{t+1}|$ and $|\rho_{2t+1}|$ are found in this study. Also certain special cases are indicated.

Keywords: *t-fold symmetric bi-univalent functions, analytic functions, univalent function*
2010 Mathematics Subject Classification: *30C45, 30C50*

1. Introduction

Let $\mathscr{H}$ be the family of holomorphic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ which is of the form:

$$f(z) = z + \rho_2 z^2 + \rho_3 z^3 + \cdots \tag{1.1}$$

in the open unit disk $\Omega = \{z : |z| < 1\}$. We indicate by $\mathscr{G}$ the subclass of functions in $\mathscr{H}$ which are univalent in Ω (for more details see [10]). The Keobe-One Quarter Theorem [10] state that the image of Ω under all univalent function $f \in \mathscr{H}$ contains a disk of radius $\frac{1}{4}$. Hence all univalent function $f \in \mathscr{H}$ has an inverse f^{-1} satisfy $f^{-1}(f(z))$ and $f(f^{-1}(\upsilon)) = \upsilon$ $(|\upsilon| < r_0(f),\ r_0(f) \geq \frac{1}{4})$, where

$$g(\upsilon) = f^{-1}(\upsilon) = \upsilon - \rho_2\upsilon^2 + (2\rho_2^2 - \rho_3)\upsilon^3 - (5\rho_2^3 - 5\rho_2\rho_3 + \rho_4)\upsilon^4 + \cdots. \tag{1.2}$$

A function $f \in \mathscr{H}$ denoted by $\mathscr{E}$ is said to be bi-univalent in Ω if both $f^{-1}(z)$ ans $f(z)$ are univalent in Ω (see for details [15, 7, 26, 11, 17, 30, 31, 32, 33, 28, 23, 24, 21]).

Jackson [13, 14] introduced the q-derivative operator $\mathscr{D}_q$ of a function as follows:

$$\mathscr{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \tag{1.3}$$

and $\mathscr{D}_q f(z) = f'(0)$. In case $f(z) = z^\phi$ for ϕ is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathscr{D}_q z^\phi = \frac{z^\phi - (zq)^\phi}{(q-1)z} = [\phi]_q z^{\phi-1}.$$

As $q \longrightarrow 1^-$ and $\phi \in \mathscr{N}$, we get

$$[\phi]_q = \frac{1 - q^\phi}{1 - q} = 1 + q + \cdots + q^\phi \longrightarrow \phi \tag{1.4}$$

where $(z \neq 0,\ q \neq 0)$, for more details on the concepts of q-derivative (see [16, 6, 3, 4, 19, 20, 9, 17, 22]).

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A domain Ψ is said to be t-fold symmetric if a rotation of Ψ about the origin through an angle $2\pi/t$ carries Ψ on itself. Therefore, a function $f(z)$ holomorphic in Ω is said to be t-fold symmetric if

$$f\left(e^{\frac{2\pi i}{t}}z\right)=e^{\frac{2\pi i}{t}}f(z).$$

A function is said to be t-fold symmetric if it has the following normalized form

$$f(z)=z+\sum_{\phi=1}^{\infty}\rho_{t\phi+1}z^{t\phi+1}\qquad (z\in\Omega,\ t\in\mathcal{N}). \tag{1.5}$$

Let $\mathcal{S}_t$ the class of t-fold symmetric univalent functions in Ω, that are normalized by (1.5). In which, the functions in the class $\mathcal{S}$ are *one*-fold symmetric. Similar to the concept of t-fold symmetric univalent functions, we introduced the concept of t-fold symmetric bi-univalent functions which is denoted by $\mathcal{E}_t$. Each of the function $f\in\mathcal{E}$ produces t-fold symmetric bi-univalent function for each integer $t\in\mathcal{N}$. The normalized form of $f(z)$ is given as in (1.5) and the series expansion for $f^{-1}(z)$, which has been investigated by Srivastava et al. [27], is given below:

$$g(\upsilon)=f^{-1}(\upsilon)=\upsilon-\rho_{t+1}\upsilon^{t+1}+\left[(t+1)\rho_{t-1}^{2}-\rho_{2t+1}\right]\upsilon^{2t+1}-\left[\frac{1}{2}(t+1)(3t+2)\rho_{t+1}^{3}-(3t+2)\rho_{t+1}\rho_{2t+1}+\rho_{3t+1}\right]. \tag{1.6}$$

Here are some examples of t-fold symmetric bi-univalent functions:

$$\left\{\frac{1}{2}\log\left(\frac{1+z^t}{1-z^t}\right)^{\frac{1}{t}}\right\},\quad \left[-\log(1-z^t)\right]^{\frac{1}{t}},\quad \left\{\frac{z^t}{1-z^t}\right\}^{\frac{1}{t}},$$

for more details on t-fold symmetric analytic bi-univalent functions (see [2, 5, 8, 12, 27, 29, 17, 25]).

In this current research, we introduced two new subclasses denoted by $\mathcal{T}_{\mathcal{E},t}^{q,\gamma}(\delta;\varphi)$ and $\mathcal{T}_{\mathcal{E},t}^{q,\xi}(\delta;\varphi)$ of the function class $\mathcal{E}_t$ and obtain estimates coefficient $|\rho_{t+1}|$ and $|\rho_{2t+1}|$ for functions in these two new subclasses.

Lemma 1.1. *[18] Let the function $\omega\in\mathcal{P}$ be given by the following series:*

$$\omega(z)=1+\omega_1 z+\omega_2 z^2+\cdots\quad (z\in\mathcal{U}).$$

The sharp estimate given by

$$|\omega_n|\le 2\quad (n\in\mathcal{N}),$$

holds true.

2. Main Results

Definition 2.1. *A function $f(z)$ given by (1.5) is said to be in the class $\mathcal{T}_{\mathcal{E},t}^{q,\gamma}(\delta;\varphi)$ $(0<q<1,\ 0<\gamma\le 1,\ 0\le\delta\le 1,\ \varphi\in\mathcal{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathcal{N})$ if the following condition are fulfilled*

$$f\in\mathcal{E}_t\quad \textit{and}\quad \left|\arg\left[(1-\delta)\left(\mathcal{D}_q f(z)\right)^{\varphi}+\delta\left(z(\mathcal{D}_q f(z))'+\mathcal{D}_q f(z)\right)(\mathcal{D}_q f(z))^{\varphi-1}\right]\right|<\frac{\gamma\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left[(1-\delta)\left(\mathcal{D}_q g(\upsilon)\right)^{\varphi}+\delta\left(\upsilon(\mathcal{D}_q g(\upsilon))'+\mathcal{D}_q g(\upsilon)\right)(\mathcal{D}_q g(\upsilon))^{\varphi-1}\right]\right|<\frac{\gamma\pi}{2} \tag{2.2}$$

where $g(\upsilon)$ is given by (1.6).

Remark 2.2. *We have $\lim_{q\longrightarrow 1^{-1}}\mathcal{T}_{\mathcal{E},t}^{q,\gamma}(\delta;\varphi)=\mathcal{T}_{\mathcal{E},t}^{\gamma}(\delta;\varphi)$ and for one-fold case $\mathcal{T}_{\mathcal{E},1}^{\gamma}(0;1)=\mathcal{T}_{\mathcal{E}}^{\gamma}$ which was introduced by Srivastava et al. [26].*

Theorem 2.3. *Let $f(z)\in\mathcal{T}_{\mathcal{E},t}^{q,\gamma}(\delta;\varphi)$, $(0<q<1,\ 0<\gamma\le 1,\ 0\le\delta\le 1,\ \varphi\in\mathcal{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathcal{N})$ be given (1.5). Then*

$$|\rho_{t+1}|\le\frac{2\gamma}{\sqrt{(t+1)(\varphi+2t\delta)[2t+1]_q\gamma+(\varphi+2t\delta)(\varphi-1)[t+1]_q^2\gamma-(\gamma-1)(\varphi+t\delta)^2[t+1]_q^2}} \tag{2.3}$$

and

$$|\rho_{2t+1}|\le\frac{2\gamma}{(\varphi+2\delta t)[2t+1]_q}+\frac{2(t+1)\gamma^2}{(\varphi+t\delta)^2[t+1]_q^2}. \tag{2.4}$$

Proof. Using inequalities (2.1) and (2.2), we get

$$(1-\delta)\left(\mathscr{D}_q f(z)\right)^{\varphi}+\delta\left(z(\mathscr{D}_q f(z))'+\mathscr{D}_q f(z)\right)(\mathscr{D}_q f(z))^{\varphi-1}=[x(z)]^{\gamma}\qquad (z\in\Omega) \tag{2.5}$$

and

$$(1-\delta)\left(\mathscr{D}_q g(\upsilon)\right)^{\varphi}+\delta\left(\upsilon(\mathscr{D}_q g(\upsilon))'+\mathscr{D}_q g(\upsilon)\right)(\mathscr{D}_q g(\upsilon))^{\varphi-1}=[y(\upsilon)]^{\gamma}\qquad (\upsilon\in\Omega) \tag{2.6}$$

where $x(z)$ and $y(\upsilon)$ in $\mathscr{P}$ are given by the following series

$$x(z)=1+x_t z^t+x_{2t}z^{2t}+x_{3t}z^{3t}+\cdots \tag{2.7}$$

and

$$y(\upsilon)=1+y_t\upsilon^t+y_{2t}\upsilon^{2t}+y_{3t}\upsilon^{3t}+\cdots. \tag{2.8}$$

Now, comparing the coefficients in (2.5) and (2.6), we get

$$(\varphi+t\delta)[t+1]_q\rho_{t+1}=\gamma x_t \tag{2.9}$$

$$(\varphi+2t\delta)[2t+1]_q\rho_{2t+1}+\frac{(\varphi+2t\delta)(\varphi-1)[t+1]_q^2}{2}\rho_{t+1}^2=\gamma x_{2t}+\frac{\gamma(\gamma-1)}{2}x_{2t}^2 \tag{2.10}$$

$$-(\varphi+t\delta)[t+1]_q\rho_{t+1}=\gamma y_t \tag{2.11}$$

$$(\varphi+2t\delta)[2t+1]_q\left([t+1]\rho_{t+1}^2-\rho_{2t+1}\right)+\frac{(\varphi+2t\delta)(\varphi-1)[t+1]_q^2}{2}\rho_{t+1}^2=\gamma y_{2t}+\frac{\gamma(\gamma-1)}{2}y_{2t}^2. \tag{2.12}$$

From (2.9) and (2.11), we obtain

$$x_t=-y_t \tag{2.13}$$

and

$$2(\varphi+t\delta)^2[t+1]_q^2\rho_{t+1}^2=\gamma^2(x_t^2+y_t^2). \tag{2.14}$$

Further from (2.10), (2.12) and (2.14), we obtain that

$$(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{t+1}^2+(\varphi+2t\delta)(\varphi-1)[t+1]_q^2\rho_{t+1}^2=\gamma(x_{2t}+y_{2t})+\frac{\gamma(\gamma-1)}{2}\left[\frac{2(\varphi+t\delta)^2[t+1]_q^2\rho_{t+1}^2}{\gamma^2}\right].$$

Therefore, we have

$$\rho_{t+1}^2=\frac{(x_{2t}+y_{2t})\gamma^2}{[t+1](\varphi+2t\delta)[2t+1]_q\gamma+(\varphi+2t\delta)(\varphi-1)[t+1]_q^2\gamma-(\gamma-1)(\varphi+t\delta)^2[t+1]_q^2}.$$

By applying Lemma 1.1 for the coefficients x_{2t} and y_{2t}, then we have

$$|\rho_{t+1}|\le\frac{2\gamma}{\sqrt{[t+1](\varphi+2t\delta)[2t+1]_q\gamma+(\varphi+2t\delta)(\varphi-1)[t+1]_q^2\gamma-(\gamma-1)(\varphi+t\delta)^2[t+1]_q^2}}.$$

Also, to find the bound on $|\rho_{2t+1}|$, using the relation (2.12) and (2.10), we obtain

$$2(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{2t+1}-(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{t+1}^2=\gamma(x_{2t}-y_{2t})+\frac{\gamma(\gamma-1)}{2}\left(x_t^2-y_t^2\right). \tag{2.15}$$

It follows from (2.13), (2.14) and (2.15), we get

$$\rho_{2t+1}=\frac{\gamma(x_{2t}-y_{2t})}{2(\varphi+2t\delta)[2t+1]_q}+\frac{[t+1]\gamma^2(x_t^2+y_t^2)}{4(\varphi+t\delta)^2[2t+1]_q^2}. \tag{2.16}$$

Applying Lemma 1.1 for the coefficients x_t, x_{2t}, y_t, y_{2t}, then we have

$$|\rho_{2t+1}|\le\frac{2\gamma}{(\varphi+2\delta t)[2t+1]_q}+\frac{2(t+1)\gamma^2}{(\varphi+t\delta)^2[t+1]_q^2}.$$

□

Remark 2.4. *We note that $\mathscr{T}^{q,\gamma}_{\mathscr{E},1}(\delta;\varphi)=\mathscr{T}^{q,\gamma}_{\mathscr{E}}(\delta;\varphi)$, which is the one-fold case introduced by Akgül [1].*

Choosing $q\longrightarrow 1^{-1}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.5. *Let $f(z)\in\mathscr{T}^{\gamma}_{\mathscr{E},t}(\delta;\varphi)$, $(0<\gamma\leq 1,\ 0\leq\delta\leq 1,\ \varphi\in\mathscr{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathscr{N})$ be given (1.5). Then*

$$|\rho_{t+1}|\leq\frac{2\gamma}{\sqrt{(t+1)(\varphi+2t\delta)[2t+1]\gamma+(\varphi+2t\delta)(\varphi-1)[t+1]^2\gamma-(\gamma-1)(\varphi+t\delta)^2[t+1]^2}}\tag{2.17}$$

and

$$|\rho_{2t+1}|\leq\frac{2\gamma}{(\varphi+2\delta t)[2t+1]}+\frac{2\gamma^2}{(\varphi+t\delta)^2[t+1]}.\tag{2.18}$$

Remark 2.6. *We note that $\lim_{q\longrightarrow 1^{-1}}\mathscr{T}^{q,\gamma}_{\mathscr{E},1}(0;1)=\mathscr{T}^{\gamma}_{\mathscr{E}}$, which is the one-fold case and obtain the corollary 2.7 as follows:*

Corollary 2.7. *[26] Let $f(z)\in\mathscr{T}^{\gamma}_{\mathscr{E}}$, $(0<\gamma\leq 1,\ z,\upsilon\in\Omega)$ be given (1.1). Then*

$$|\rho_2|\leq\gamma\sqrt{\frac{2}{\gamma+2}}\tag{2.19}$$

and

$$|\rho_3|\leq\frac{\gamma(3\gamma+2)}{3}\tag{2.20}$$

Definition 2.8. *A function $f(z)$ given by (1.5) is said to be in the class $\mathscr{T}^{q,\xi}_{\mathscr{E},t}(\delta;\varphi)$ $(0<q<1,\ 0\leq\xi<1,\ 0\leq\delta\leq 1,\ \varphi\in\mathscr{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathscr{N})$ if the following condition are fulfilled*

$$f\in\mathscr{E}_t\quad\text{and}\quad\Re\left\{(1-\delta)\left(\mathscr{D}_qf(z)\right)^{\varphi}+\delta\left(z(\mathscr{D}_qf(z))'+\mathscr{D}_qf(z)\right)(\mathscr{D}_qf(z))^{\varphi-1}\right\}>\xi\tag{2.21}$$

and

$$\Re\left\{(1-\delta)\left(\mathscr{D}_qg(\upsilon)\right)^{\varphi}+\delta\left(\upsilon(\mathscr{D}_qg(\upsilon))'+\mathscr{D}_qg(\upsilon)\right)(\mathscr{D}_qg(\upsilon))^{\varphi-1}\right\}>\xi\tag{2.22}$$

where $g(\upsilon)$ is given by (1.6).

Remark 2.9. *We have the class $\lim_{q\longrightarrow 1^{-1}}\mathscr{T}^{q,\xi}_{\mathscr{E},t}(\delta;\varphi)=\mathscr{T}^{\xi}_{\mathscr{E},t}(\delta;\varphi)$ and for one-fold case $\mathscr{T}^{\xi}_{\mathscr{E},1}(0;1)=\mathscr{T}^{\xi}_{\mathscr{E}}$ which was introduced by Srivastava et al. [26].*

Theorem 2.10. *Let $f(z)\in\mathscr{T}^{q,\xi}_{\mathscr{E},t}(\delta;\varphi)$, $(0<q<1,\ 0\leq\xi<1,\ 0\leq\delta\leq 1,\ \varphi\in\mathscr{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathscr{N})$ be given (1.5). Then*

$$|\rho_{t+1}|\leq\min\left\{\frac{2(1-\xi)}{(\varphi+t\delta)^2[t+1]_q},\ 2\sqrt{\frac{1-\xi}{[2t+1]_q(t+1)(\varphi+2t\delta)+(\varphi-1)(\varphi+2t\delta)[t+1]_q^2}}\right\}\tag{2.23}$$

and

$$|\rho_{2t+1}|\leq\frac{2(1-\xi)}{(\varphi+2\delta t)[2t+1]_q}.\tag{2.24}$$

Proof. Using inequalities (2.21) and (2.22), we get

$$(1-\delta)\left(\mathscr{D}_qf(z)\right)^{\varphi}+\delta\left(z(\mathscr{D}_qf(z))'+\mathscr{D}_qf(z)\right)(\mathscr{D}_qf(z))^{\varphi-1}=\xi+(1-\xi)x(z)\qquad(z\in\Omega)\tag{2.25}$$

and

$$(1-\delta)\left(\mathscr{D}_qg(\upsilon)\right)^{\varphi}+\delta\left(\upsilon(\mathscr{D}_qg(\upsilon))'+\mathscr{D}_qg(\upsilon)\right)(\mathscr{D}_qg(\upsilon))^{\varphi-1}=\xi+(1-\xi)y(z)\qquad(\upsilon\in\Omega)\tag{2.26}$$

where the functions $x(z)$ and $y(\upsilon)$ are given by (2.7) and (2.8). Now comparing the coefficients in (2.25) and (2.26), we get

$$(\varphi+t\delta)[t+1]_q\rho_{t+1}=(1-\xi)x_t\tag{2.27}$$

$$(\varphi+2t\delta)[2t+1]_q\rho_{2t+1}+\frac{(\varphi+2t\delta)(\varphi-1)[t+1]_q^2}{2}\rho_{t+1}^2=(1-\xi)x_{2t}\tag{2.28}$$

$$-(\varphi+t\delta)[t+1]_q\rho_{t+1}=(1-\xi)y_t\tag{2.29}$$

$$(\varphi+2t\delta)[2t+1]_q\left([t+1]\rho_{t+1}^2-\rho_{2t+1}\right)+\frac{(\varphi+2t\delta)(\varphi-1)[t+1]_q^2}{2}\rho_{t+1}^2=(1-\xi)y_{2t}. \tag{2.30}$$

From (2.27) and (2.29), we obtain

$$x_t=-y_t \tag{2.31}$$

and

$$2(\varphi+t\delta)^2[t+1]_q^2\rho_{t+1}^2=(1-\xi)^2(x_t^2+y_t^2). \tag{2.32}$$

Also, from (2.28) and (2.30), we get

$$(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{t+1}^2+(\varphi+2t\delta)(\varphi-1)[t+1]_q^2\rho_{t+1}^2=(1-\xi)(x_{2t}+y_{2t}). \tag{2.33}$$

Applying the Lemma 1.1 for the coefficients x_t, x_{2t}, y_t, y_{2t}, we find that

$$|\rho_{t+1}|\leq 2\sqrt{\frac{1-\xi}{[2t+1]_q(t+1)(\varphi+2t\delta)+(\varphi-1)(\varphi+2t\delta)[t+1]_q^2}}.$$

Also, to find the bound on $|\rho_{2t+1}|$, using the relation (2.30) and (2.28), we obtain

$$-(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{t+1}^2+2(\varphi+2t\delta)[2t+1]_q[t+1]\rho_{2t+1}=(1-\xi)(x_{2t}-y_{2t}) \tag{2.34}$$

or equivalently

$$\rho_{2t+1}=\frac{(1-\xi)(x_{2t}-y_{2t})}{2[1+2t]_q(\varphi+2t\delta)}+\frac{(t+1)}{2}\rho_{t+1}^2 \tag{2.35}$$

By substituting the value of ρ_{t+1}^2 from (2.32), we have

$$\rho_{2t+1}=\frac{(1-\xi)(x_{2t}-y_{2t})}{2[1+2t]_q(\varphi+2t\delta)}+\frac{(1-\xi)^2(t+1)(x_t^2+y_t^2)}{4(\varphi+t\delta)^2[1+t]_q^2}. \tag{2.36}$$

Applying the Lemma 1.1 for the coefficients x_t, x_{2t}, y_t, y_{2t}, we get

$$|\rho_{2t+1}|\leq\frac{2(1-\xi)}{[1+2t]_q(\varphi+2t\delta)}+\frac{2(1-\xi)^2(t+1)}{(\varphi+t\delta)^2[1+t]_q^2}.$$

Also, by using (2.33) and (2.35), and applying Lemma 1.1 we obtain

$$|\rho_{2t+1}|\leq\frac{2(1-\xi)}{[1+2t]_q(\varphi+2t\delta)}$$

which is the desired bounds on coefficients $|\rho_{2t+1}|$ as asserted in Theorem 2.10.

□

Taking $\lim_{q\longrightarrow 1^{-1}}$ in Theorem 2.10, we have Corollary 2.11 as follows:

Corollary 2.11. *Let $f(z)\in\mathscr{T}_{\mathscr{E},t}^{\xi}(\delta;\varphi)$, $(0\leq\xi<1,\ 0\leq\delta\leq 1,\ \varphi\in\mathscr{N}_0,\ z,\upsilon\in\Omega,\ t\in\mathscr{N})$ be given (1.5). Then*

$$|\rho_{t+1}|\leq\begin{cases}2\sqrt{\frac{1-\xi}{[2t+1](t+1)(\varphi+2t\delta)+(\varphi-1)(\varphi+2t\delta)[t+1]^2}} & 0\leq\xi\leq\frac{t}{1+2t}\\ \frac{2(1-\xi)}{(\varphi+t\delta)^2[t+1]} & \frac{t}{1+2t}\leq\xi<1\end{cases} \tag{2.37}$$

and

$$|\rho_{2t+1}|\leq\frac{2(1-\xi)}{(\varphi+2\delta t)[2t+1]}. \tag{2.38}$$

Remark 2.12. *For one-fold case, Corollary 2.11 gives the following Corollary for the bounds on Coefficients $|\rho_2|$ and $|\rho_3|$.*

Corollary 2.13. *Let $f(z)\in\mathscr{T}_{\mathscr{E}}^{\xi}(\delta;\varphi)$, $(0\leq\xi<1,\ 0\leq\delta\leq 1,\ \varphi\in\mathscr{N}_0,\ z,\upsilon\in\Omega)$ be given (1.1). Then*

$$|\rho_2|\leq\begin{cases}\sqrt{\frac{2(1-\xi)}{3(\varphi+2t\delta)+2(\varphi-1)(\varphi+2t\delta)}} & 0\leq\xi\leq\frac{1}{3}\\ \frac{(1-\xi)}{(\varphi+t\delta)^2} & \frac{1}{3}\leq\xi<1\end{cases} \tag{2.39}$$

and

$$|\rho_3|\leq\frac{2(1-\xi)}{3(\varphi+2\delta t)}. \tag{2.40}$$

Remark 2.14. *Putting $\delta=0$ and $\varphi=1$ in Corollary 2.13, we obtain the following corollary:*

Corollary 2.15. *[9] Let $f(z) \in \mathscr{T}_{\mathscr{E}}^{\xi}$, $(0 \leq \xi < 1,\ z, \upsilon \in \Omega)$ be given (1.1). Then*

$$|\rho_2| \leq \begin{cases} \sqrt{\frac{2(1-\xi)}{3}} & 0 \leq \xi \leq \frac{1}{3} \\ 1-\xi & \frac{1}{3} \leq \xi < 1 \end{cases} \tag{2.41}$$

and

$$|\rho_3| \leq \frac{2(1-\xi)}{3}. \tag{2.42}$$

Remark 2.16. *Corollary 2.15 is an improvement of the estimates for coefficients on $|\rho_2|$ and $|\rho_3|$ obtained by Srivastava et al. [26].*

Corollary 2.17. *[26] Let $f(z) \in \mathscr{T}_{\mathscr{E}}^{\xi}$, $(0 \leq \xi < 1,\ z, \upsilon \in \Omega)$ be given (1.1). Then*

$$|\rho_2| \leq \sqrt{\frac{2(1-\xi)}{3}} \tag{2.43}$$

and

$$|\rho_3| \leq \frac{(1-\xi)(5-3\xi)}{3}. \tag{2.44}$$

Acknowledgement

The author thank the referees(s) for their relevant contributions which improved this research.

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