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Research Article

A STUDY ON GENERALIZED 5-PRIMES NUMBERS

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Abstract

In this paper, we introduce the generalized 5-primes numbers sequences and we deal with, in detail, three special cases which we call them 5-primes, Lucas 5-primes and modified 5-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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1. Introduction

In this paper, we define the generalized 5-primes sequences and we investigate, in detail, three special cases which we call them 5-primes, Lucas-5-primes and modified 5-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences produce several nice and interesting sequences.

A generalized Pentanacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \tag{1.1}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [4], [5], [6]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0 \tag{1.2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda$. Note that we have the following identities:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta + \lambda &= r, \\
\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \alpha\delta + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta &= -s, \\
\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\gamma + \alpha\gamma\delta + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta &= t, \\
\alpha\beta\lambda\gamma + \alpha\beta\lambda\delta + \alpha\beta\gamma\delta + \alpha\lambda\gamma\delta + \beta\lambda\gamma\delta &= -u \\
\alpha\beta\gamma\delta\lambda &= v.
\end{aligned}$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet's formula

$$\begin{aligned}
W_n &= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\
&\quad + \frac{b_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{b_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}
b_1 &= W_4 - (\beta+\gamma+\delta+\lambda)W_3 + (\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2 - (\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1 + (\beta\lambda\gamma\delta)W_0, \\
b_2 &= W_4 - (\alpha+\gamma+\delta+\lambda)W_3 + (\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2 - (\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1 + (\alpha\lambda\gamma\delta)W_0, \\
b_3 &= W_4 - (\alpha+\beta+\delta+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2 - (\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1 + (\alpha\beta\lambda\delta)W_0, \\
b_4 &= W_4 - (\alpha+\beta+\gamma+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2 - (\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1 + (\alpha\beta\lambda\gamma)W_0, \\
b_5 &= W_4 - (\alpha+\beta+\gamma+\delta)W_3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1 + (\alpha\beta\gamma\delta)W_0.
\end{aligned}$$

Usually, it is customary to choose r, s, t, u, v so that the Equ. (1.2) has at least one real (say α) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [1]. This result of Howard and Saidak [1] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = 2$, $s = 3, t = 5, u = 7, v = 11$ and in this case we write $V_n = W_n$. A generalized 5-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} + 7V_{n-4} + 11V_{n-5} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{7}{11}V_{-(n-1)} - \frac{5}{11}V_{-(n-2)} - \frac{3}{11}V_{-(n-3)} - \frac{2}{11}V_{-(n-4)} + \frac{1}{11}V_{-(n-5)}$$

for $n = 1, 2, 3, ...$Therefore, recurrence (1.4) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized 5-primes numbers. Binet formula of generalized 5-primes numbers can be given as

$$\begin{aligned} V_n &= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad +\frac{b_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{b_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

where

$$\begin{aligned} b_1 &= V_4-(\beta+\gamma+\delta+\lambda)V_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1+(\beta\lambda\gamma\delta)V_0 \quad (1.5) \\ b_2 &= V_4-(\alpha+\gamma+\delta+\lambda)V_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1+(\alpha\lambda\gamma\delta)V_0, \\ b_3 &= V_4-(\alpha+\beta+\delta+\lambda)V_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1+(\alpha\beta\lambda\delta)V_0, \\ b_4 &= V_4-(\alpha+\beta+\gamma+\lambda)V_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1+(\alpha\beta\lambda\gamma)V_0, \\ b_5 &= V_4-(\alpha+\beta+\gamma+\delta)V_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1+(\alpha\beta\gamma\delta)V_0. \end{aligned}$$

Here, $\alpha,\beta,\gamma,\delta$ and λ are the roots of the equation

$$x^5-2x^4-3x^3-5x^2-7x-11=0. \quad (1.6)$$

Moreover, the approximate value of the roots $\alpha,\beta,\gamma,\delta$ and λ of Equation (1.6) are given by

$$\begin{aligned} \alpha &= 3.501101503801069 \\ \beta &= 0.3060834095195042+1.329047329711188i \\ \gamma &= 0.3060834095195042-1.329047329711188i \\ \delta &= -1.056634161420038+0.7567376493934506i \\ \lambda &= -1.056634161420038-0.7567376493934506i \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\lambda &= 2, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -3, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 5, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -7 \\ \alpha\beta\gamma\delta\lambda &= 11. \end{aligned}$$

The first few generalized 5-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized 5-primes numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{11}V_4-\frac{5}{11}V_1-\frac{3}{11}V_2-\frac{2}{11}V_3-\frac{7}{11}V_0$
2	V_2	$\frac{2}{121}V_1-\frac{6}{121}V_0-\frac{1}{121}V_2+\frac{25}{121}V_3-\frac{7}{121}V_4$
3	V_3	$\frac{64}{1331}V_0+\frac{19}{1331}V_1+\frac{293}{1331}V_2-\frac{65}{1331}V_3-\frac{6}{1331}V_4$
4	V_4	$\frac{2903}{14\,641}V_1-\frac{239}{14\,641}V_0-\frac{907}{14\,641}V_2-\frac{194}{14\,641}V_3+\frac{64}{14\,641}V_4$
5	$11V_0+7V_1+5V_2+3V_3+2V_4$	$\frac{33\,606}{161\,051}V_0-\frac{8782}{161\,051}V_1-\frac{1417}{161\,051}V_2+\frac{1182}{161\,051}V_3-\frac{239}{161\,051}V_4$
6	$22V_0+25V_1+17V_2+11V_3+7V_4$	$\frac{33\,606}{1771\,561}V_4-\frac{183\,617}{1771\,561}V_1-\frac{87\,816}{1771\,561}V_2-\frac{69\,841}{1771\,561}V_3-\frac{331\,844}{1771\,561}V_0$

Now we define three special cases of the sequence $\{V_n\}$. 5-primes sequence$\{G_n\}_{n\geq 0}$, Lucas 5-primes sequence $\{H_n\}_{n\geq 0}$ and modified 5-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+5}=2G_{n+4}+3G_{n+3}+5G_{n+2}+7G_{n+1}+11G_n, \qquad G_0=0,G_1=0,G_2=0,G_3=1,G_4=2, \tag{1.7}$$

$$H_{n+5}=2H_{n+4}+3H_{n+3}+5H_{n+2}+7H_{n+1}+11H_n, \qquad H_0=5,H_1=2,H_2=10,H_3=41,H_4=150, \tag{1.8}$$

and

$$E_{n+5}=2E_{n+4}+3E_{n+3}+5E_{n+2}+7E_{n+1}+11E_n, \qquad E_0=0,E_1=0,E_2=0,E_3=1,E_4=1, \tag{1.9}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n}=-\frac{7}{11}G_{-(n-1)}-\frac{5}{11}G_{-(n-2)}-\frac{3}{11}G_{-(n-3)}-\frac{2}{11}G_{-(n-4)}+\frac{1}{11}G_{-(n-5)}, \tag{1.10}$$

$$H_{-n}=-\frac{7}{11}H_{-(n-1)}-\frac{5}{11}H_{-(n-2)}-\frac{3}{11}H_{-(n-3)}-\frac{2}{11}H_{-(n-4)}+\frac{1}{11}H_{-(n-5)} \tag{1.11}$$

and

$$E_{-n}=-\frac{7}{11}E_{-(n-1)}-\frac{5}{11}E_{-(n-2)}-\frac{3}{11}E_{-(n-3)}-\frac{2}{11}E_{-(n-4)}+\frac{1}{11}E_{-(n-5)} \tag{1.12}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.10), (1.11) and (1.12) hold for all integer n.

Note that the sequences G_n, H_n and E_n are not indexed in [7] yet. Next, we present the first few values of the 5-primes, Lucas 5-primes and modified 5-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special fifth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9
G_n	0	0	0	1	2	7	25	88	311	1082
G_{-n}		0	$\frac{1}{11}$	$-\frac{7}{121}$	$-\frac{6}{1331}$	$\frac{64}{14\,641}$	$-\frac{239}{161\,051}$	$\frac{33606}{1771561}$	$-\frac{331844}{19\,487\,171}$	$\frac{303121}{214358881}$
H_n	5	2	10	41	150	542	1831	6435	22574	79052
H_{-n}		$-\frac{7}{11}$	$-\frac{61}{121}$	$-\frac{277}{1331}$	$-\frac{2813}{14\,641}$	$\frac{148\,908}{161\,051}$	$-\frac{727\,195}{1771\,561}$	$-\frac{2234\,183}{19\,487\,171}$	$\frac{5014\,051}{214\,358\,881}$	$-\frac{85824736}{2357947691}$
E_n	0	0	0	1	1	5	18	63	223	771
E_{-n}		$-\frac{1}{11}$	$\frac{18}{121}$	$-\frac{71}{1331}$	$-\frac{130}{14\,641}$	$\frac{943}{161\,051}$	$-\frac{36\,235}{1771561}$	$\frac{701510}{19487171}$	$-\frac{3953405}{214358881}$	$-\frac{2169506}{2357947691}$

For all integers n, 5-primes, Lucas 5-primes and modified 5-primes numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n, \\ E_n &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{(\delta-1)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{(\lambda-1)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

respectively.

2. Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1. *Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized 5-primes sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3 + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0)x^4}{1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5}. \tag{2.1}$$

Proof. Using the definition of generalized 5-primes numbers, and substracting $2x\sum_{n=0}^{\infty} V_n x^n$, $3x^2\sum_{n=0}^{\infty} V_n x^n$, $5x^3\sum_{n=0}^{\infty} V_n x^n, 7x^4\sum_{n=0}^{\infty} V_n x^n$ and $11x^5\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned} (1-2x-3x^2-5x^3-7x^4-11x^5)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - 2x\sum_{n=0}^{\infty} V_n x^n - 3x^2\sum_{n=0}^{\infty} V_n x^n \\ &\quad - 5x^3\sum_{n=0}^{\infty} V_n x^n - 7x^4\sum_{n=0}^{\infty} V_n x^n - 11x^5\sum_{n=0}^{\infty} V_n x^n \\ &= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=0}^{\infty} V_n x^{n+1} - 3\sum_{n=0}^{\infty} V_n x^{n+2} \\ &\quad - 5\sum_{n=0}^{\infty} V_n x^{n+3} - 7\sum_{n=0}^{\infty} V_n x^{n+4} - 11\sum_{n=0}^{\infty} V_n x^{n+5} \\ &= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=1}^{\infty} V_{n-1} x^n - 3\sum_{n=2}^{\infty} V_{n-2} x^n \\ &\quad - 5\sum_{n=3}^{\infty} V_{n-3} x^n - 7\sum_{n=4}^{\infty} V_{n-4} x^n - 11\sum_{n=5}^{\infty} V_{n-5} x^n \end{aligned}$$

and so

$$\begin{aligned}(1-2x-3x^2-5x^3-7x^4-11x^5)\sum_{n=0}^{\infty}V_nx^n &= (V_0+V_1x+V_2x^2+V_3x^3+V_4x^4)-2(V_0x+V_1x^2+V_2x^3+V_3x^4)\\ &\quad -3(V_0x^2+V_1x^3+V_2x^4)-5(V_0x^3+V_1x^4)-7V_0x^4\\ &\quad +\sum_{n=5}^{\infty}(V_n-2V_{n-1}-3V_{n-2}-5V_{n-3}-7V_{n-4}-11V_{n-5})x^n\\ &= V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3\\ &\quad +(V_4-2V_3-3V_2-5V_1-7V_0)x^4.\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}V_nx^n=\frac{V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3+(V_4-2V_3-3V_2-5V_1-7V_0)x^4}{1-2x-3x^2-5x^3-7x^4-11x^5}.$$

The previous lemma gives the following results as particular examples.

COROLLARY 2. *Generated functions of 5-primes, Lucas 5-primes and modified 5-primes numbers are*

$$\sum_{n=0}^{\infty}G_nx^n=\frac{x^3}{1-2x-3x^2-5x^3-7x^4-11x^5},$$

and

$$\sum_{n=0}^{\infty}H_nx^n=\frac{5-8x-9x^2-10x^3-7x^4}{1-2x-3x^2-5x^3-7x^4-11x^5},$$

and

$$\sum_{n=0}^{\infty}E_nx^n=\frac{x^3-x^4}{1-2x-3x^2-5x^3-7x^4-11x^5},$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized 5-primes numbers $\{V_n\}$ by the use of generating function for V_n.

THEOREM 3. *(Binet formula of generalized 5-primes numbers)*

$$\begin{aligned}V_n &= \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \quad (3.1)\\ &\quad +\frac{d_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{d_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\end{aligned}$$

where

$$\begin{aligned}d_1 &= V_0\alpha^4+(V_1-2V_0)\alpha^3+(V_2-2V_1-3V_0)\alpha^2+(V_3-2V_2-3V_1-5V_0)\alpha+(V_4-2V_3-3V_2-5V_1-7V_0),\\ d_2 &= V_0\beta^4+(V_1-2V_0)\beta^3+(V_2-2V_1-3V_0)\beta^2+(V_3-2V_2-3V_1-5V_0)\beta+(V_4-2V_3-3V_2-5V_1-7V_0),\\ d_3 &= V_0\gamma^4+(V_1-2V_0)\gamma^3+(V_2-2V_1-3V_0)\gamma^2+(V_3-2V_2-3V_1-5V_0)\gamma+(V_4-2V_3-3V_2-5V_1-7V_0),\\ d_4 &= V_0\delta^4+(V_1-2V_0)\delta^3+(V_2-2V_1-3V_0)\delta^2+(V_3-2V_2-3V_1-5V_0)\delta+(V_4-2V_3-3V_2-5V_1-7V_0),\\ d_5 &= V_0\lambda^4+(V_1-2V_0)\lambda^3+(V_2-2V_1-3V_0)\lambda^2+(V_3-2V_2-3V_1-5V_0)\lambda+(V_4-2V_3-3V_2-5V_1-7V_0).\end{aligned}$$

Proof. Let

$$h(x)=1-2x-3x^2-5x^3-7x^4-11x^5.$$

Then for some $\alpha,\beta,\gamma,\delta$ and λ we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)$$

i.e.,

$$1-2x-3x^2-5x^3-7x^4-11x^5=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) \tag{3.2}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta}$ and $\frac{1}{\lambda}$ are the roots of $h(x)$. This gives $\alpha,\beta,\gamma,\delta$ and λ as the roots of

$$h(\frac{1}{x})=1-\frac{2}{x}-\frac{3}{x^2}-\frac{5}{x^3}-\frac{7}{x^4}-\frac{11}{x^5}=0.$$

This implies $x^5-2x^4-3x^3-5x^2-7x-11=0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty}V_nx^n=\frac{V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3+(V_4-2V_3-3V_2-5V_1-7V_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)}.$$

Then we write

$$\begin{aligned}&\frac{V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3+(V_4-2V_3-3V_2-5V_1-7V_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)} \\ =\;&\frac{A_1}{(1-\alpha x)}+\frac{A_2}{(1-\beta x)}+\frac{A_3}{(1-\gamma x)}+\frac{A_4}{(1-\delta x)}+\frac{A_5}{(1-\lambda x)}.\end{aligned} \tag{3.3}$$

So

$$\begin{aligned}&V_0+(V_1-2V_0)x+(V_2-2V_1-3V_0)x^2+(V_3-2V_2-3V_1-5V_0)x^3+(V_4-2V_3-3V_2-5V_1-7V_0)x^4\\ =\;&A_1(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)+A_2(1-\alpha x)(1-\gamma x)(1-\delta x)(1-\lambda x)\\ &+A_3(1-\alpha x)(1-\beta x)(1-\delta x)(1-\lambda x)+A_4(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x)\\ &+A_5(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get

$V_0+(V_1-2V_0)\frac{1}{\alpha}+(V_2-2V_1-3V_0)\frac{1}{\alpha^2}+(V_3-2V_2-3V_1-5V_0)\frac{1}{\alpha^3}+(V_4-2V_3-3V_2-5V_1-7V_0)\frac{1}{\alpha^4}=A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})(1-\frac{\lambda}{\alpha})$.

This gives

$$\begin{aligned}A_1&=\frac{\alpha^4(V_0+(V_1-2V_0)\frac{1}{\alpha}+(V_2-2V_1-3V_0)\frac{1}{\alpha^2}+(V_3-2V_2-3V_1-5V_0)\frac{1}{\alpha^3}+(V_4-2V_3-3V_2-5V_1-7V_0)\frac{1}{\alpha^4})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}\\ &=\frac{V_0\alpha^4+(V_1-2V_0)\alpha^3+(V_2-2V_1-3V_0)\alpha^2+(V_3-2V_2-3V_1-5V_0)\alpha+(V_4-2V_3-3V_2-5V_1-7V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}A_2&=\frac{V_0\beta^4+(V_1-2V_0)\beta^3+(V_2-2V_1-3V_0)\beta^2+(V_3-2V_2-3V_1-5V_0)\beta+(V_4-2V_3-3V_2-5V_1-7V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)},\\ A_3&=\frac{V_0\gamma^4+(V_1-2V_0)\gamma^3+(V_2-2V_1-3V_0)\gamma^2+(V_3-2V_2-3V_1-5V_0)\gamma+(V_4-2V_3-3V_2-5V_1-7V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)},\\ A_4&=\frac{V_0\delta^4+(V_1-2V_0)\delta^3+(V_2-2V_1-3V_0)\delta^2+(V_3-2V_2-3V_1-5V_0)\delta+(V_4-2V_3-3V_2-5V_1-7V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)},\\ A_5&=\frac{V_0\lambda^4+(V_1-2V_0)\lambda^3+(V_2-2V_1-3V_0)\lambda^2+(V_3-2V_2-3V_1-5V_0)\lambda+(V_4-2V_3-3V_2-5V_1-7V_0)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}.\end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1} + A_4(1-\delta x)^{-1} + A_5(1-\lambda x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} V_n x^n &= A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n + A_5 \sum_{n=0}^{\infty} \lambda^n x^n \\ &= \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n) x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n$$

and then we get (3.1).

Note that from (1.5) and (3.1) we have

$$\begin{aligned}
& V_4 - (\beta+\gamma+\delta+\lambda)V_3 + (\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2 - (\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1 + (\beta\lambda\gamma\delta)V_0 \\
= \; & V_0\alpha^4 + (V_1 - 2V_0)\alpha^3 + (V_2 - 2V_1 - 3V_0)\alpha^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)\alpha + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0), \\
& V_4 - (\alpha+\gamma+\delta+\lambda)V_3 + (\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2 - (\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1 + (\alpha\lambda\gamma\delta)V_0 \\
= \; & V_0\beta^4 + (V_1 - 2V_0)\beta^3 + (V_2 - 2V_1 - 3V_0)\beta^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)\beta + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0), \\
& V_4 - (\alpha+\beta+\delta+\lambda)V_3 + (\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2 - (\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1 + (\alpha\beta\lambda\delta)V_0 \\
= \; & V_0\gamma^4 + (V_1 - 2V_0)\gamma^3 + (V_2 - 2V_1 - 3V_0)\gamma^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)\gamma + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0), \\
& V_4 - (\alpha+\beta+\gamma+\lambda)V_3 + (\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2 - (\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1 + (\alpha\beta\lambda\gamma)V_0 \\
= \; & V_0\delta^4 + (V_1 - 2V_0)\delta^3 + (V_2 - 2V_1 - 3V_0)\delta^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)\delta + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0), \\
& V_4 - (\alpha+\beta+\gamma+\delta)V_3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1 + (\alpha\beta\gamma\delta)V_0 \\
= \; & V_0\lambda^4 + (V_1 - 2V_0)\lambda^3 + (V_2 - 2V_1 - 3V_0)\lambda^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)\lambda + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0).
\end{aligned}$$

Next, using Theorem 3, we present the Binet formulas of 5-primes, Lucas 5-primes and modified 5-primes sequences.

COROLLARY 4. *Binet formulas of 5-primes, Lucas 5-primes and modified 5-primes sequences are*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n,$$

and

$$\begin{aligned} E_n &= \frac{(\alpha-1)\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta-1)\beta^{n}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{(\gamma-1)\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{(\delta-1)\delta^{n}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{(\lambda-1)\lambda^{n}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [3]. Take $k = i = 5$ in Corollary 3.1 in [3]. Let

$$\Lambda = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^4 & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^4 & \delta^3 & \delta^2 & \delta & 1 \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^3 & \delta^2 & \delta & 1 \\ \lambda^{n-1} & \lambda^3 & \lambda^2 & \lambda & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^4 & \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^4 & \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^4 & \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^4 & \delta^{n-1} & \delta^2 & \delta & 1 \\ \lambda^4 & \lambda^{n-1} & \lambda^2 & \lambda & 1 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^4 & \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^4 & \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^4 & \delta^3 & \delta^{n-1} & \delta & 1 \\ \lambda^4 & \lambda^3 & \lambda^{n-1} & \lambda & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^4 & \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^4 & \delta^3 & \delta^2 & \delta^{n-1} & 1 \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda^{n-1} & 1 \end{pmatrix}, \Lambda_5 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^4 & \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^4 & \delta^3 & \delta^2 & \delta & \delta^{n-1} \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda & \lambda^{n-1} \end{pmatrix}.$$

Then the Binet formula for 5-primes numbers is

$$\begin{aligned} G_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{5} G_{6-j} \det(\Lambda_j) = \frac{1}{\Lambda}(G_5 \det(\Lambda_1) + G_4 \det(\Lambda_2) + G_3 \det(\Lambda_3) + G_2 \det(\Lambda_4) + G_1 \det(\Lambda_5)) \\ &= \frac{1}{\det(\Lambda)}(7 \det(\Lambda_1) + 2 \det(\Lambda_2) + \det(\Lambda_3)) \end{aligned}$$

Similarly, we obtain the Binet formula for Lucas 5-primes and modified 5-primes numbers as

$$\begin{aligned} H_n &= \frac{1}{\det(\Lambda)}(H_5 \det(\Lambda_1) + H_4 \det(\Lambda_2) + H_3 \det(\Lambda_3) + H_2 \det(\Lambda_4) + H_1 \det(\Lambda_5)) \\ &= \frac{1}{\det(\Lambda)}(542 \det(\Lambda_1) + 150 \det(\Lambda_2) + 41 \det(\Lambda_3) + 10 \det(\Lambda_4) + 2 \det(\Lambda_5)) \end{aligned}$$

and

$$\begin{aligned} E_n &= \frac{1}{\det(\Lambda)}(E_5 \det(\Lambda_1) + E_4 \det(\Lambda_2) + E_3 \det(\Lambda_3) + E_2 \det(\Lambda_4) + E_1 \det(\Lambda_5)) \\ &= \frac{1}{\det(\Lambda)}(5 \det(\Lambda_1) + \det(\Lambda_2) + \det(\Lambda_3)) \end{aligned}$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 5-primes sequence $\{V_n\}_{n\geq 0}$.

Theorem 5 (Simson Formula of Generalized 5-primes Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = 11^n \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \tag{4.1}$$

Proof. (4.1) is given in Soykan [8].

The previous theorem gives the following results as particular examples.

Corollary 6. *For all integers n, Simson formula of 5-primes, Lucas 5-primes and modified 5-primes numbers are given as*

$$\begin{vmatrix} G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \end{vmatrix} = 11^{n-3} \tag{4.2}$$

and

$$\begin{vmatrix} H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \end{vmatrix} = 409 \times 431 \times 1103 \times 11^{n-4} \tag{4.3}$$

and

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = 27 \times 11^{n-4} \tag{4.4}$$

respectively.

5. Some Identities

In this section, we obtain some identities of 5-primes, Lucas 5-primes and modified 5-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 7. *The following equalities are true:*

$$\begin{aligned} 1331H_n &= -277G_{n+6} - 117G_{n+5} + 1326G_{n+4} + 11\,747G_{n+3} - 2813G_{n+2}, \qquad (5.1) \\ 121H_n &= -61G_{n+5} + 45G_{n+4} + 942G_{n+3} - 432G_{n+2} - 277G_{n+1}, \\ 11H_n &= -7G_{n+4} + 69G_{n+3} - 67G_{n+2} - 64G_{n+1} - 61G_n, \\ H_n &= 5G_{n+3} - 8G_{n+2} - 9G_{n+1} - 10G_n - 7G_{n-1}, \\ H_n &= 2G_{n+2} + 6G_{n+1} + 15G_n + 28G_{n-1} + 55G_{n-2}, \end{aligned}$$

and

$$
\begin{aligned}
2138793107G_n &= 18571H_{n+6} + 1176092H_{n+5} - 1191254H_{n+4} - 21714675H_{n+3} + 39754441H_{n+2} \\
194435737G_n &= 110294H_{n+5} - 103231H_{n+4} - 1965620H_{n+3} + 3625858H_{n+2} + 18571H_{n+1} \\
194435737G_n &= 117357H_{n+4} - 1634738H_{n+3} + 4177328H_{n+2} + 790629H_{n+1} + 1213234H_n \\
194435737G_n &= -1400024H_{n+3} + 4529399H_{n+2} + 1377414H_{n+1} + 2034733H_n + 1290927H_{n-1} \\
194435737G_n &= 1729351H_{n+2} - 2822658H_{n+1} - 4965387H_n - 8509241H_{n-1} - 15400264H_{n-2}
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$H_n = a \times G_{n+6} + b \times G_{n+5} + c \times G_{n+4} + d \times G_{n+3} + e \times G_{n+2}$$

and solving the system of equations

$$
\begin{aligned}
H_0 &= a \times G_6 + b \times G_5 + c \times G_4 + d \times G_3 + e \times G_2 \\
H_1 &= a \times G_7 + b \times G_6 + c \times G_5 + d \times G_4 + e \times G_3 \\
H_2 &= a \times G_8 + b \times G_7 + c \times G_6 + d \times G_5 + e \times G_4 \\
H_3 &= a \times G_9 + b \times G_8 + c \times G_7 + d \times G_6 + e \times G_5 \\
H_4 &= a \times G_{10} + b \times G_9 + c \times G_8 + d \times G_7 + e \times G_6
\end{aligned}
$$

we find that $a = -\frac{277}{1331}, b = -\frac{117}{1331}, c = \frac{1326}{1331}, d = \frac{11\,747}{1331}, e = -\frac{2813}{1331}$. The other equalities can be proved similarly.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 8. *The following equalities are true:*

$$
\begin{aligned}
1331E_n &= -71G_{n+6} + 340G_{n+5} - 304G_{n+4} + 3G_{n+3} - 130G_{n+2}, \\
121E_n &= 18G_{n+5} - 47G_{n+4} - 32G_{n+3} - 57G_{n+2} - 71G_{n+1}, \\
11E_n &= -G_{n+4} + 2G_{n+3} + 3G_{n+2} + 5G_{n+1} + 18G_n, \\
E_n &= G_n - G_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
297G_n &= -16E_{n+6} + 43E_{n+5} + 37E_{n+4} + 36E_{n+3} + 13E_{n+2}, \\
27G_n &= E_{n+5} - E_{n+4} - 4E_{n+3} - 9E_{n+2} - 16E_{n+1}, \\
27G_n &= E_{n+4} - E_{n+3} - 4E_{n+2} - 9E_{n+1} + 11E_n, \\
27G_n &= E_{n+3} - E_{n+2} - 4E_{n+1} + 18E_n + 11E_{n-1}, \\
27G_n &= E_{n+2} - E_{n+1} + 23E_n + 18E_{n-1} + 11E_{n-2}.
\end{aligned}
$$

Note that all the identities in the above Lemma can be proved by induction as well.

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

LEMMA 9. *The following equalities are true:*

$$\begin{aligned}
3267H_n &= 217E_{n+6} - 1864E_{n+5} - 1300E_{n+4} + 23049E_{n+3} + 9866E_{n+2}, \\
297H_n &= -130E_{n+5} - 59E_{n+4} + 2194E_{n+3} + 1035E_{n+2} + 217E_{n+1}, \\
27H_n &= -29E_{n+4} + 164E_{n+3} + 35E_{n+2} - 63E_{n+1} - 130E_n, \\
27H_n &= 106E_{n+3} - 52E_{n+2} - 208E_{n+1} - 333E_n - 319E_{n-1}, \\
27H_n &= 160E_{n+2} + 110E_{n+1} + 197E_n + 423E_{n-1} + 1166E_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
23526724177E_n &= -39550160H_{n+6} + 92241613H_{n+5} + 93222517H_{n+4} - 26985426H_{n+3} + 954441363H_{n+2}, \\
2138793107E_n &= 1194\,663H_{n+5} - 2311633H_{n+4} - 20430566H_{n+3} + 61599113H_{n+2} - 39550160H_{n+1}, \\
194435737E_n &= 7063H_{n+4} - 1531507H_{n+3} + 6142948H_{n+2} - 2835229H_{n+1} + 1194663H_n, \\
194435737E_n &= -1517381H_{n+3} + 6164137H_{n+2} - 2799914H_{n+1} + 1244104H_n + 77693H_{n-1}, \\
194435737E_n &= 3129375H_{n+2} - 7352057H_{n+1} - 6342801H_n - 10543974H_{n-1} - 16691191H_{n-2}.
\end{aligned}$$

We now present a few special identities for the modified 5-primes sequence $\{E_n\}$.

THEOREM 10. *(Catalan's identity) For all integers n and m, the following identity holds*

$$\begin{aligned}
E_{n+m}E_{n-m} - E_n^2 &= (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2 \\
&= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2}) + G_{n-2}(-G_m + G_{m-1})) \\
&\quad (G_n(G_{-m} - G_{1-m}) + G_{n-1}(-G_{-m} + G_{-m-2}) + G_{n-2}(-G_{-m} + G_{-m-1})) \\
&\quad -(G_n - G_{n-1})^2
\end{aligned}$$

Proof. We use the identity

$$E_n = G_n - G_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified 5-primes sequnce

COROLLARY 11. *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = G_n - G_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified 5-primes sequence $\{E_n\}$.

THEOREM 12. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).$$

(b): *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n-1} - G_{n-2})(G_{n-2} - G_{n-3}) - (G_n - G_{n-1})^4.$$

(c): *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+1} - G_n)(G_{n+2} - G_{n+1})(G_{n+6} - G_{n+5}) - (G_{n+3} - G_{n+2})^3.$$

Proof. Use the identity $E_n = G_n - G_{n-1}$.

6. Linear Sums

The following proposition presents some formulas of generalized 5-primes numbers with positive subscripts.

PROPOSITION 13. *If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} V_k = \frac{1}{27}(V_{n+5} - V_{n+4} - 4V_{n+3} - 9V_{n+2} - 16V_{n+1} - V_4 + V_3 + 4V_2 + 9V_1 + 16V_0)$.
- **(b):** $\sum_{k=0}^{n} V_{2k} = \frac{1}{27}(-V_{2n+2} + 4V_{2n+1} + 25V_{2n} + 3V_{2n-1} + 22V_{2n-2} + V_4 - 4V_3 + 2V_2 - 3V_1 + 5V_0)$.
- **(c):** $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{27}(2V_{2n+2} + 22V_{2n+1} - 2V_{2n} + 15V_{2n-1} - 11V_{2n-2} - 2V_4 + 5V_3 + 2V_2 + 12V_1 + 11V_0)$.

Proof. Take $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 2.1 in [9].

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 5-primes numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

COROLLARY 14. *For $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} G_k = \frac{1}{27}(G_{n+5} - G_{n+4} - 4G_{n+3} - 9G_{n+2} - 16G_{n+1} - 1)$.
- **(b):** $\sum_{k=0}^{n} G_{2k} = \frac{1}{27}(-G_{2n+2} + 4G_{2n+1} + 25G_{2n} + 3G_{2n-1} + 22G_{2n-2} - 2)$.
- **(c):** $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{27}(2G_{2n+2} + 22G_{2n+1} - 2G_{2n} + 15G_{2n-1} - 11G_{2n-2} + 1)$.

Second one presents some summing formulas of Lucas 5-primes numbers (take $G_n = H_n$ with $H_0 = 5$, $H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$).

COROLLARY 15. *For $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} H_k = \frac{1}{27}(H_{n+5} - H_{n+4} - 4H_{n+3} - 9H_{n+2} - 16H_{n+1} + 29)$.
- **(b):** $\sum_{k=0}^{n} H_{2k} = \frac{1}{27}(-H_{2n+2} + 4H_{2n+1} + 25H_{2n} + 3H_{2n-1} + 22H_{2n-2} + 25)$.
- **(c):** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{27}(2H_{2n+2} + 22H_{2n+1} - 2H_{2n} + 15H_{2n-1} - 11H_{2n-2} + 4)$.

Third one presents some summing formulas of modified 5-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

COROLLARY 16. *For $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} E_k = \frac{1}{27}(E_{n+5} - E_{n+4} - 4E_{n+3} - 9E_{n+2} - 16E_{n+1})$.
- **(b):** $\sum_{k=0}^{n} E_{2k} = \frac{1}{27}(-E_{2n+2} + 4E_{2n+1} + 25E_{2n} + 3E_{2n-1} + 22E_{2n-2} - 3)$.
- **(c):** $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{27}(2E_{2n+2} + 22E_{2n+1} - 2E_{2n} + 15E_{2n-1} - 11E_{2n-2} + 3)$.

The following proposition presents some formulas of generalized 5-primes numbers with negative subscripts.

PROPOSITION 17. *If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 1$ we have the following formulas:*

- **(a):** $\sum_{k=1}^{n} V_{-k} = \frac{1}{27}(-V_{-n+4} + V_{-n+3} + 4V_{-n+2} + 9V_{-n+1} + 16V_{-n} + V_4 - V_3 - 9V_1 - 4V_2 - 16V_0)$.
- **(b):** $\sum_{k=1}^{n} V_{-2k} = \frac{1}{27}(-2V_{-2n+3} + 5V_{-2n+2} + 2V_{-2n+1} + 12V_{-2n} + 11V_{-2n-1} - V_4 + 4V_3 - 2V_2 + 3V_1 - 5V_0)$.
- **(c):** $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{27}(V_{-2n+3} - 4V_{-2n+2} + 2V_{-2n+1} - 3V_{-2n} - 22V_{-2n-1} + 2V_4 - 5V_3 - 2V_2 - 12V_1 - 11V_0)$.

Proof. Take $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 3.1 in [9].

From the above proposition, we have the following corollary which gives sum formulas of 5-primes numbers (take $G_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

Corollary 18. *For $n \geq 1$, 5-primes numbers have the following properties.*

(a): $\sum_{k=1}^{n} G_{-k} = \frac{1}{27}(-G_{-n+4} + G_{-n+3} + 4G_{-n+2} + 9G_{-n+1} + 16G_{-n} + 1)$.

(b): $\sum_{k=1}^{n} G_{-2k} = \frac{1}{27}(-2G_{-2n+3} + 5G_{-2n+2} + 2G_{-2n+1} + 12G_{-2n} + 11G_{-2n-1} + 2)$.

(c): $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{27}(G_{-2n+3} - 4G_{-2n+2} + 2G_{-2n+1} - 3G_{-2n} - 22G_{-2n-1} - 1)$.

Taking $G_n = H_n$ with $H_0 = 5$, $H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$ in the last proposition, we have the following corollary which presents sum formulas of 5-primes -Lucas numbers.

Corollary 19. *For $n \geq 1$, 5-primes -Lucas numbers have the following properties.*

(a): $\sum_{k=1}^{n} H_{-k} = \frac{1}{27}(-H_{-n+4} + H_{-n+3} + 4H_{-n+2} + 9H_{-n+1} + 16H_{-n} - 29)$.

(b): $\sum_{k=1}^{n} H_{-2k} = \frac{1}{27}(-2H_{-2n+3} + 5H_{-2n+2} + 2H_{-2n+1} + 12H_{-2n} + 11H_{-2n-1} - 25)$.

(c): $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{27}(H_{-2n+3} - 4H_{-2n+2} + 2H_{-2n+1} - 3H_{-2n} - 22H_{-2n-1} - 4)$.

From the above proposition, we have the following corollary which gives sum formulas of modified 5-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

Corollary 20. *For $n \geq 1$, modified 5-primes numbers have the following properties.*

(a): $\sum_{k=1}^{n} E_{-k} = \frac{1}{27}(-E_{-n+4} + E_{-n+3} + 4E_{-n+2} + 9E_{-n+1} + 16E_{-n})$.

(b): $\sum_{k=1}^{n} E_{-2k} = \frac{1}{27}(-2E_{-2n+3} + 5E_{-2n+2} + 2E_{-2n+1} + 12E_{-2n} + 11E_{-2n-1} + 3)$.

(c): $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{27}(E_{-2n+3} - 4E_{-2n+2} + 2E_{-2n+1} - 3E_{-2n} - 22E_{-2n-1} - 3)$.

7. Matrices Related with Generalized 5-primes Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \tag{7.1}$$

For matrix formulation (7.1), see [2]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ r & s & t & u & v \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

We define the square matrix A of order 5 as:

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 11$. From (1.4) we have

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = G_n$ in (7.2) we have

$$\begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+3} & 3G_{n+2}+5G_{n+1}+7G_n+11G_{n-1} & 5G_{n+2}+7G_{n+1}+11G_n & 7G_{n+2}+11G_{n+1} & 11G_{n+2} \\ G_{n+2} & 3G_{n+1}+5G_n+7G_{n-1}+11G_{n-2} & 5G_{n+1}+7G_n+11G_{n-1} & 7G_{n+1}+11G_n & 11G_{n+1} \\ G_{n+1} & 3G_n+5G_{n-1}+7G_{n-2}+11G_{n-3} & 5G_n+7G_{n-1}+11G_{n-2} & 7G_n+11G_{n-1} & 11G_n \\ G_n & 3G_{n-1}+5G_{n-2}+7G_{n-3}+11G_{n-4} & 5G_{n-1}+7G_{n-2}+11G_{n-3} & 7G_{n-1}+11G_{n-2} & 11G_{n-1} \\ G_{n-1} & 3G_{n-2}+5G_{n-3}+7G_{n-4}+11G_{n-5} & 5G_{n-2}+7G_{n-3}+11G_{n-4} & 7G_{n-2}+11G_{n-3} & 11G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+3} & 3V_{n+2}+5V_{n+1}+7V_n+11V_{n-1} & 5V_{n+2}+7V_{n+1}+11V_n & 7V_{n+2}+11V_{n+1} & 11V_{n+2} \\ V_{n+2} & 3V_{n+1}+5V_n+7V_{n-1}+11V_{n-2} & 5V_{n+1}+7V_n+11V_{n-1} & 7V_{n+1}+11V_n & 11V_{n+1} \\ V_{n+1} & 3V_n+5V_{n-1}+7V_{n-2}+11V_{n-3} & 5V_n+7V_{n-1}+11V_{n-2} & 7V_n+11V_{n-1} & 11V_n \\ V_n & 3V_{n-1}+5V_{n-2}+7V_{n-3}+11V_{n-4} & 5V_{n-1}+7V_{n-2}+11V_{n-3} & 7V_{n-1}+11V_{n-2} & 11V_{n-1} \\ V_{n-1} & 3V_{n-2}+5V_{n-3}+7V_{n-4}+11V_{n-5} & 5V_{n-2}+7V_{n-3}+11V_{n-4} & 7V_{n-2}+11V_{n-3} & 11V_{n-2} \end{pmatrix}.$$

THEOREM 21. *For all integer $m, n \geq 0$, we have*

(a): $B_n = A^n$

(b): $C_1 A^n = A^n C_1$

(c): $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a): By expanding the vectors on the both sides of (7.3) to 5-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3} + 7A^{n-4} + 11A^{n-5}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 11^n$$

for all integer m and n.

THEOREM 22. *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_n G_{m+3} + V_{n-1}(3G_{m+2} + 5G_{m+1} + 7G_m + 11G_{m-1}) + V_{n-2}(5G_{m+2} \\ &\quad +7G_{m+1} + 11G_m) + V_{n-3}(7G_{m+2} + 11G_{m+1}) + 11V_{n-4}G_{m+2} \end{aligned} \tag{7.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

REMARK 23. *By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n, (7.4) is true.*

COROLLARY 24. *For all integers m, n, we have*

$$
\begin{aligned}
G_{n+m} &= G_nG_{m+3} + G_{n-1}(3G_{m+2} + 5G_{m+1} + 7G_m + 11G_{m-1}) + G_{n-2}(5G_{m+2} + 7G_{m+1} + 11G_m) \\
&\quad + G_{n-3}(7G_{m+2} + 11G_{m+1}) + 11G_{n-4}G_{m+2}, \\
H_{n+m} &= H_nG_{m+3} + H_{n-1}(3G_{m+2} + 5G_{m+1} + 7G_m + 11G_{m-1}) + H_{n-2}(5G_{m+2} + 7G_{m+1} + 11G_m) \\
&\quad + H_{n-3}(7G_{m+2} + 11G_{m+1}) + 11H_{n-4}G_{m+2}, \\
E_{n+m} &= E_nG_{m+3} + E_{n-1}(3G_{m+2} + 5G_{m+1} + 7G_m + 11G_{m-1}) + E_{n-2}(5G_{m+2} + 7G_{m+1} + 11G_m) \\
&\quad + E_{n-3}(7G_{m+2} + 11G_{m+1}) + 11E_{n-4}G_{m+2}.
\end{aligned}
$$

REFERENCES

[1] Howard, F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200, 225-237, 2010.

[2] Kalman, D., Generalized Fibonacci Numbers By Matrix Methods, Fibonacci Quarterly, 20(1), 73-76, 1982.

[3] Kiliç, E., Stanica, P., A Matrix Approach for General Higher Order Linear Recurrences, Bulletin of the Malaysian Mathematical Sciences Society, (2) 34(1), 51.67, 2011.

[4] Melham, R. S., Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$,Fibonacci Quarterly, 305-311, 1999.

[5] Natividad, L. R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3 (2), 2013.

[6] Rathore, G.P.S., Sikhwal, O., Choudhary, R., Formula for finding nth Term of Fibonacci-Like Sequence of Higher Order, International Journal of Mathematics And its Applications, 4 (2-D), 75-80, 2016.

[7] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[8] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[9] Soykan, Y., Sum Formulas For Generalized Fifth-Order Linear Recurrence Sequences, Journal of Advances in Mathematics and Computer Science, 34(5), 1-14, 2019, Article no.JAMCS.53303, ISSN: 2456-9968, DOI: 10.9734/JAMCS/2019/v34i530224.