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The Convergence of Ishikawa Iteration for Generalized Φ-contractive Mappings

Linxin Li[a], Dingping Wu[a],

[a] Department of Mathematics,Chengdu University of Information Technology,Chengdu,China.

Abstract

Charles[1] proved the convergence of Picard-type iterative for generalized $\Phi-$accretive non-self mappings in a real uniformly smooth Banach space. Based on the theorems of the zeros of strongly $\Phi-$quasi-accretive and fixed points of strongly $\Phi-$hemi-contractions, we extend the results to Ishikawa iterative and Ishikawa iteration process with errors for generalized $\Phi-$hemi-contractive mappings.

Keywords: strongly $\Phi-$quasi-accretive strongly $\Phi-$hemi-contractions Ishikawa iteration process with errors unique solution.
2010 MSC: 54H25, 47H10.

1. Introduction

In[2], we can see that the convergence theorems of Ishikawa iterative process with errors for $\Phi-$hemi-contractive mappings in uniformly smooth Banach spaces; In[3], we can see that strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces; In[4], we can see that Mann and Ishikawa-type iterative schemes for approximating fixed points of multi-valued non-self mappings; In[5], we can see that convergence analysis of the Picard–Ishikawa hybrid iterative process with applications.

In 2009, Charles[1] proved the convergence of Picard-type iterative for generalized $\Phi-$accretive non-self maps in a real uniformly smooth Banach space. In this paper, we consider that the Ishikawa iteration process and Ishikawa iteration process with errors will be extended from the results of Charles [1].

In 1974, Ishikawa[6] introduced the Ishikawa iteration process as follows: For a convex subset C of a Banach space E and a mapping T from C into itself, for any given $x_0 \in C$, the sequence $\{x_n\}$ in C is

Email addresses: `1224020519@qq.com` (Linxin Li), `wdp68@163.com` (Dingping Wu)

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defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\,x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)\,x_n + \beta_n T x_n, \quad n \geq 0 \end{aligned} \tag{0.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1] satisfying the conditions $0 \leq \alpha_n, \beta_n \leq 1$ for all n , $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n = \infty$.

In 1995, Liu[7] introduced what he called the Ishikawa iteration process with errors.

In 1998, Xu[8] introduced the following alternative definitions:

Let K be a nonempty convex subset of E and $T : K \to K$ be a nonlinear mapping. For any given $x_0, u_0, v_0 \in K$,the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\beta_n-\gamma_n)\,x_n + \beta_n T y_n + \gamma_n u_n, n \geq 0, \\ y_n &= (1-a_n-b_n)\,x_n + a_n T x_n + b_n v_n \\ &= x_n - a_n\,(I-T)\,x_n - b_n\,(x_n - v_n)\ , n \geq 0, \end{aligned} \tag{0.2}$$

where $\{u_n\}$, $\{v_n\}$ are any bounded sequences in K; $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$ are four real sequences in [0,1] and satisfy $\beta_n + \gamma_n \leq 1$, $a_n + b_n \leq 1$, for all $n \geq 0$.

2. Preliminaries

Definition 1. *[1] Given a gauge function φ, the mapping $J_\varphi : E \to 2^{E^*}$ defined by*

$$J_\varphi x := \{u^* \in E^* : \langle x, u^*\rangle = \|x\|\,\|u^*\|\,;\ \|u^*\| = \varphi(\|x\|)\}$$

is called the duality map with gauge function φ where X is any normed space.

In the particular case $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the normalized duality map.

Proposition 2. *[9] If a Banach space E has a uniformly Gateaux differentiable norm, then $J : E \to E^*$ is uniformly continuous on bounded subsets of E from the strong topology of E to the weak*topology of E^*.*

Definition 3. *[10] Let E be an arbitrary real normed linear space. A mapping $T : D(T) \subseteq E \to E$ is called strongly hemi-contractive if $F(T) \neq \emptyset$, and there exists $t > 1$ such that for all $r > 0$,*

$$\|x - x^*\| \leq \|(1+r)\,(x-x^*) - rt\,(Tx - x^*)\| \tag{0.1}$$

holds for all $x \in D(T)$, $x^* \in F(T)$. If $t = 1$, then T is called hemi-contractive. Finally, T is called generalized $\Phi-$hemi-contractive, if for all $x \in D(T)$, $x^* \in F(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that

$$\langle (I-T)\,x - (I-T)\,x^*, j\,(x-x^*)\rangle \geq \Phi(\|x-x^*\|)\,. \tag{0.2}$$

It follows from inequality (2.2) that T is generalized $\Phi-$hemi-contractive if and only if

$$\langle Tx - x^*, j\,(x-x^*)\rangle \leq \|x-x^*\|^2 - \Phi(\|x-x^*\|)\,,\ \ \forall n \geq 0. \tag{0.3}$$

Definition 4. *[1] Let $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$. The mapping $T : D(T) \subseteq E \to E$ is called generalized $\Phi-$quasi-accretive if, for all $x \in E$, $x^* \in N(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that*

$$\langle Tx - Tx^*, j\,(x-x^*)\rangle \geq \Phi(\|x-x^*\|)\,. \tag{0.4}$$

Proposition 5. *[1] If $F(T) = \{x \in E : Tx = x\} \neq \emptyset$, the mapping $T : E \to E$ is strongly hemi-contractive if and only if $(I-T)$is strongly quasi-accretive; it is strongly $\varphi-$hemi-contractive if and only if $(I-T)$ is strongly $\varphi-$quasi-accretive; and T is generalized $\Phi-$hemi-contractive if and only if $(I-T)$ is generalized $\Phi-$quasi-accretive.*

Proposition 6. *[1] Let E be a uniformly smooth real Banach space, and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$$

for all $x, y \in E$.

Proposition 7. *[1] Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers and $\{\alpha_n\}$ be a sequence of positive numbers satisfying the conditions $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \le \lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, n = 1, 2, ...$$

be given where $\psi : [0,\infty) \to [0,\infty)$ is strictly increasing continuous function such that it is positive on $(0,\infty)$ and $\psi(0) = 0$. Then $\lambda_n \to 0$, as $n \to \infty$.

3. Main Results

In this section, we will consider to extend the result of Charles[1] to Ishikawa iterative and Ishikawa iteration process with errors under the following assumptions. First, we extend the result of Charles[1] to Ishikawa iterative.

Theorem 8. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T : D \to D$ is a bounded generalized $\Phi-$hemi-contractive map with strictly increasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that $\Phi(0) = 0$ and $x^* \in F(T) \ne \emptyset$. For arbitrary $x_0 \in D$, $\{x_n\}$ be an Ishikawa iterative sequence defined by (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ $\subseteq [0,1]$, $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n, \beta_n \le d_0$, $\{x_n\}$ converges strongly to the unique fixed point x^* of T.*

Proof. Let r be sufficiently large such that $x_1 \in B_r(x^*)$. Define $G := \overline{B_r(x^*)} \cap D$. Then, since T is bounded we have that $(I-T)(G)$ is bounded.

Let $M = \max\{\sup \|(I-T)x_n\|\ ,\ \sup \|Ty_n - x^*\| : x_n\ ,\ y_n \in G\}$. As j is uniformly continuous on bounded subsets of E, for $\varepsilon_0 := \frac{\Phi(\frac{r}{4})}{16M}$, there exists a $\delta > 0$ such that $x, y \in D(T)$,$\|x-y\| < \delta$ implies $\|j(x) - j(y)\| < \varepsilon_0$.

Set $d_0 = \min\left\{1, \frac{r}{4M}, \frac{r}{2(M+r)}\ ,\ \frac{\delta}{2M}, \frac{\delta}{2(M+r)}\ ,\ \frac{\Phi(\frac{r}{4})}{2r^2}\right\}$.

Claim1: $\{x_n\}$ is bounded.

Suffices to show that x_n is in G for all $n \ge 1$. The proof is by induction. By our assumption, $x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in D$ $\forall n \ge 1$, we have that $\|x_{n+1} - x^*\| > r$.

We have the following estimates:

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T x_n - x^*\| \\ &\le \|x_n - x^*\| + \beta_n \|(I-T)x_n\| \\ &\le r + d_0 M \\ &\le \frac{3}{2} r\ ,\end{aligned}$$

$$\begin{aligned}\|(x_n - x^*) - (y_n - x^*)\| &= \beta_n \|(I-T)x_n\| \\ &\le d_0 M \\ &\le \frac{\delta}{2} < \delta\ ,\end{aligned}$$

now

$$\begin{aligned}\|x_n - x^*\| &\geq \|x_{n+1} - x^*\| - \alpha_n \|Ty_n - x_n\| \\ &\geq \|x_{n+1} - x^*\| - \alpha_n [\|Ty_n - x^*\| + \|x_n - x^*\|] \\ &\geq r - d_0 (M + r) \\ &\geq r - \frac{r}{2} = \frac{r}{2} ,\end{aligned}$$

$$\begin{aligned}\|y_n - x^*\| &\geq \|x_n - x^*\| - \beta_n \|(I - T) x_n\| \\ &\geq \frac{r}{2} - d_0 M \\ &\geq \frac{r}{4} ,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|Ty_n - x^*\| \\ &\leq (1 - \alpha_n) r + \alpha_n M \\ &\leq \frac{3}{2} r ,\end{aligned}$$

$$\begin{aligned}\|(x_{n+1} - x^*) - (x_n - x^*)\| &\leq \alpha_n \|Ty_n - x_n\| \\ &\leq \alpha_n [\|Ty_n - x^*\| + \|x_n - x^*\|] \\ &\leq \alpha_n (M + r) \\ &\leq \frac{\delta}{2} < \delta ,\end{aligned}$$

therefore,

$$\|j (x_n - x^*) - j (y_n - x^*)\| < \varepsilon_0 ,$$

$$\|j (x_{n+1} - x^*) - j (x_n - x^*)\| < \varepsilon_0 .$$

Using Proposition 2.6 and the above formulas, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n) x_n + \alpha_n T y_n - x^*\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Ty_n - x^*, j (y_n - x^*)\rangle \\ &\quad + 2\alpha_n \langle Ty_n - x^*, j (x_{n+1} - x^*) - j (x_n - x^*)\rangle \\ &\quad + 2\alpha_n \langle Ty_n - x^*, j (x_n - x^*) - j (y_n - x^*)\rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \left[\|y_n - x^*\|^2 - \Phi (\|y_n - x^*\|)\right] \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j (x_{n+1} - x^*) - j (x_n - x^*)\| \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j (x_n - x^*) - j (y_n - x^*)\| \\ &\leq (1 - \alpha_n)^2 r^2 + 4\alpha_n \cdot M \cdot \varepsilon_0 \\ &\quad + 2\alpha_n \left[\|y_n - x^*\|^2 - \Phi (\|y_n - x^*\|)\right] ,\end{aligned} \tag{0.1}$$

$$\begin{aligned}\|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n (I-T) x_n\|^2 \\ &\le \|x_n - x^*\|^2 - 2\beta_n \langle (I-T) x_n, j(x_n - x^*)\rangle \\ &\quad - 2\beta_n \langle (I-T) x_n, j(y_n - x^*) - j(x_n - x^*)\rangle \\ &\le \|x_n - x^*\|^2 - 2\beta_n \Phi(\|x_n - x^*\|) \\ &\quad + 2\beta_n \|(I-T) x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\ &\le r^2 + 2\beta_n \cdot M \cdot \varepsilon_0. \end{aligned} \tag{0.2}$$

Substitute (3.2) into (3.1), since $0 < \alpha_n , \beta_n \le d_0$ and $d_0 = \min\left\{1, \frac{r}{4M}, \frac{r}{2(M+r)}, \frac{r}{2M}, \frac{\delta}{2(M+r)}, \frac{\Phi(\frac{r}{4})}{2r^2}\right\}$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2 r^2 + 4\alpha_n \cdot M \cdot \varepsilon_0 + 2\alpha_n \left[r^2 + 2\beta_n \cdot M \cdot \varepsilon_0 - \Phi(\|y_n - x^*\|)\right] \\ &\le r^2 + 2\alpha_n \left[\frac{\alpha_n}{2} r^2 + 2M\varepsilon_0 + 2\beta_n M \varepsilon_0\right] - 2\alpha_n \Phi\left(\frac{r}{4}\right) \\ &\le r^2 + 2\alpha_n \left[\frac{\Phi\left(\frac{r}{4}\right)}{2} - \Phi\left(\frac{r}{4}\right)\right] \\ &\le r^2 \end{aligned}$$

i.e.,$\|x_{n+1} - x^*\| \le r$, a contradiction. Therefore $x_{n+1} \in G$. Thus by induction $\{x_n\}$ is bounded. Then, $\{y_n\}$, $\{Ty_n\}$, $\{Tx_n\}$, $\{(I-T) x_n\}$ are also bounded.

Claim2:$x_n \to x^*$.

Let $A_n = \|j(x_{n+1} - x^*) - j(x_n - x^*)\|$, $B_n = \|j(x_n - x^*) - j(y_n - x^*)\|$, Note that $x_{n+1} - x_n \to 0$, $x_n - y_n \to 0$ as $n \to \infty$ and hence by the uniform continuity of j on bounded subsets of E we have that

$$A_n \to 0 , B_n \to 0 \ \text{ as } n \to \infty.$$

Let $M_1 = \max\{\sup\|x_n - x^*\| , \sup\|y_n - x^*\| , \sup\|Ty_n - x^*\| , \sup\|(I-T) x_n\|\}$, by (3.1), (3.2), we obtain that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \left[\|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|)\right] \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_n - x^*) - j(y_n - x^*)\| \\ &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n \\ &\quad + 2\alpha_n \left[\|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|)\right] \end{aligned} \tag{0.3}$$

and

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|x_n - x^*\|^2 - 2\beta_n \Phi(\|x_n - x^*\|) \\ &\quad + 2\beta_n \|(I-T) x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\ &\le \|x_n - x^*\|^2 + 2M_1 B_n. \end{aligned} \tag{0.4}$$

Taking (3.4) into (3.3),

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n M_1 A_n+2\alpha_n M_1 B_n \\ &\quad +2\alpha_n\left[\|x_n-x^*\|^2+2M_1B_n-\Phi\left(\|y_n-x^*\|\right)\right] \\ &\le \left(1+\alpha_n{}^2\right)\|x_n-x^*\|^2+2\alpha_n\left[M_1A_n+3M_1B_n-\Phi\left(\|y_n-x^*\|\right)\right] \\ &\le \|x_n-x^*\|^2+2\alpha_n\left[\frac{\alpha_n}{2}M_1{}^2+M_1A_n+3M_1B_n-\Phi\left(\|y_n-x^*\|\right)\right] \\ &\le \|x_n-x^*\|^2+2\alpha_n\left[Z_n-\Phi\left(\|y_n-x^*\|\right)\right] ,\end{aligned}$$

where $Z_n=\frac{\alpha_n}{2}M_1{}^2+M_1A_n+3M_1B_n\to 0$ as $n\to\infty$.

Set $\inf\frac{\Phi(\|y_n-x^*\|)}{\Phi(\|x_n-x^*\|)+1}=L>0$ since Φ is a strictly increasing continuous function, then L exists. Thus

$$\Phi\left(\|y_n-x^*\|\right)\ge L\Phi\left(\|x_n-x^*\|\right)+L\ge L\Phi\left(\|x_n-x^*\|\right) .$$

Then

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|x_n-x^*\|^2+2\alpha_n\left[Z_n-L\Phi\left(\|x_n-x^*\|\right)\right] \\ &\le \|x_n-x^*\|^2-2\alpha_nL\Phi\left(\|x_n-x^*\|\right)+2\alpha_nZ_n .\end{aligned} \tag{0.5}$$

Let $\lambda_n:=\|x_n-x^*\|$ and $\rho_n=2\alpha_nZ_n$, then from inequality (3.5) we obtain that $\lambda_{n+1}\le\lambda_n-2\alpha_nL\Phi\left(\lambda_n\right)+\rho_n$, where $\frac{\rho_n}{\alpha_n}\to 0$ as $n\to\infty$. Therefore, the conclusion of the theorem follows from Proposition 2.7. ■

The following corollary follow trivially,since definition 2.3 and definition 2.4.

Corollary 9. *Suppose E is a real uniformly smooth Banach space. Suppose $T:E\to E$ is a bounded generalized $\Phi-$accretive map with strictly increasing continuous function $\Phi:[0,\infty)\to[0,\infty)$ such that $\Phi(0)=0$ and the solution x^* of the equation $Tx=0$ exists. For arbitrary $x_0\in E$, the sequence $\{x_n\}$ in E is defined by*

$$\begin{cases} x_{n+1}=(1-\alpha_n)\,x_n-\alpha_nTy_n \\ y_n=(1-\beta_n)\,x_n+\beta_nTx_n,\ n\ge 0\end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}\subseteq[0,1]$, $\lim\limits_{n\to\infty}\alpha_n=\lim\limits_{n\to\infty}\beta_n=0$ and $\sum\alpha_n=\infty$. Then, there exists a constant $d_0>0$ such that if $0<\alpha_n,\beta_n\le d_0$, $\{x_n\}$ converges strongly to the unique solution of $Tx=0$.

Now, we consider to generalize to a more general case, we extend the result of Charles[1] to Ishikawa iteration process with errors as follows.

Theorem 10. *Suppose D is a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose $T:D\to D$ is a bounded generalized $\Phi-$hemi-contractive map with strictly increasing continuous function $\Phi:[0,\infty)\to[0,\infty)$ such that $\Phi(0)=0$ and $x^*\in F(T)\ne\emptyset$. For arbitrary $x_0\in D$, $\{x_n\}$ be an Ishikawa iteration process with errors defined by (1.2) ,where $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}\subseteq[0,1]$, $\lim\limits_{n\to\infty}\beta_n=\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}a_n=0$,$\gamma_n=o(\beta_n)$ and $\sum\beta_n=\infty$. Then, there exists a constant $d_0>0$ such that if $0<\beta_n$, γ_n , a_n , $b_n\le d_0$, $\{x_n\}$ converges strongly to the unique fixed point x^* of T.*

Proof. Let r be sufficiently large such that $x_1\in B_r(x^*)$. Define $G:=\overline{B_r(x^*)}\cap D$. Then, since T is bounded we have that $(I-T)(G)$ is bounded.

Let $M=\sup\{\|(I-T)\,x_n\|:x_n\in G\}+\sup\{\|x_n-v_n\|:x_n\in G\}$ $+\sup\{\|Ty_n-x^*\|+1:y_n\in G\}+\sup\{\|u_n-x^*\|\}$.

As j is uniformly continuous on bounded subsets of E, for $\varepsilon_0:=\frac{\Phi\left(\frac{r}{4}\right)}{32M}$, there exists a $\delta>0$ such that $x,y\in D(T)$,$\|x-y\|<\delta$ implies $\|j(x)-j(y)\|$ $<\varepsilon_0$.

Set $d_0 = \min\left\{1, \frac{r}{8M}, \frac{\delta}{2(2M+r)}, \frac{r}{2(2M+r)}, \frac{\delta}{4M}, \frac{\Phi\left(\frac{r}{4}\right)}{4r^2}, \frac{\Phi\left(\frac{r}{4}\right)}{20Mr}, \frac{\Phi^2\left(\frac{r}{4}\right)}{48Mr^3}\right\}$.

Claim1: $\{x_n\}$ is bounded.

Suffices to show that x_n is in G for all $n \geq 1$. The proof is by induction. By our assumption,$x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in D$, $\forall n \geq 1$, we have that $\|x_{n+1} - x^*\| > r$.

We have the following estimates:

$$\begin{aligned} \|y_n - x^*\| &= \|x_n - x^* - a_n (I-T) x_n - b_n (x_n - v_n)\| \\ &\leq \|x_n - x^*\| + a_n \|(I-T) x_n\| + b_n \|x_n - v_n\| \\ &\leq r + d_0 (M + M) \\ &\leq \frac{5}{4} r \ , \end{aligned}$$

$$\begin{aligned} \|(x_n - x^*) - (y_n - x^*)\| &\leq a_n \|(I-T) x_n\| + b_n \|x_n - v_n\| \\ &\leq d_0 (M+M) \\ &\leq \frac{\delta}{2} < \delta \ , \end{aligned}$$

now

$$\begin{aligned} \|x_n - x^*\| &\geq \|x_{n+1} - x^*\| - \beta_n \|Ty_n - x_n\| - \gamma_n \|u_n - x_n\| \\ &\geq \|x_{n+1} - x^*\| - \beta_n [\|Ty_n - x^*\| + \|x_n - x^*\|] - \gamma_n \|u_n - x_n\| \\ &\geq r - d_0 (M + r + M) \\ &\geq \frac{r}{2} \ , \end{aligned}$$

$$\begin{aligned} \|y_n - x^*\| &\geq \|x_n - x^*\| - a_n \|(I-T) x_n\| - b_n \|x_n - v_n\| \\ &\geq \|x_n - x^*\| - d_0 (M+M) \\ &\geq \frac{r}{2} - \frac{r}{4} = \frac{r}{4} \ , \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|x_n - x^*\| + \beta_n \|Ty_n - x_n\| + \gamma_n \|u_n - x_n\| \\ &\leq \|x_n - x^*\| + \beta_n (\|Ty_n - x^*\| + \|x_n - x^*\|) + \gamma_n \|u_n - x_n\| \\ &\leq r + d_0 (M + r + M) \\ &\leq \frac{3}{2} r \ , \end{aligned}$$

$$\begin{aligned} \|(x_{n+1} - x^*) - (x_n - x^*)\| &\leq \beta_n \|Ty_n - x_n\| + \gamma_n \|u_n - x_n\| \\ &\leq \beta_n [\|Ty_n - x^*\| + \|x_n - x^*\|] + \gamma_n \|u_n - x_n\| \\ &\leq d_0 (M + r + M) \\ &\leq \frac{\delta}{2} < \delta \ , \end{aligned}$$

therefore,

$$\|j (x_n - x^*) - j (y_n - x^*)\| < \varepsilon_0 \ ,$$

$$\|j(x_{n+1}-x^*)-j(x_n-x^*)\|<\varepsilon_0\ .$$

Using Proposition2.6 and the above formulas, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n-\gamma_n)x_n+\beta_nTy_n+\gamma_nu_n-x^*\|^2\\
&\le (1-\beta_n-\gamma_n)^2\|x_n-x^*\|^2+2\beta_n\langle Ty_n-x^*,j(x_{n+1}-x^*)\rangle\\
&\quad +2\gamma_n\langle u_n-x^*,j(x_{n+1}-x^*)\rangle\\
&\le (1-\beta_n)^2\|x_n-x^*\|^2+2\beta_n\langle Ty_n-x^*,j(y_n-x^*)\rangle\\
&\quad +2\beta_n\langle Ty_n-x^*,j(x_{n+1}-x^*)-j(x_n-x^*)\rangle\\
&\quad +2\beta_n\langle Ty_n-x^*,j(x_n-x^*)-j(y_n-x^*)\rangle\\
&\quad +2\gamma_n\langle u_n-x^*,j(x_{n+1}-x^*)\rangle\\
&\le (1-\beta_n)^2\|x_n-x^*\|^2+2\beta_n\left[\|y_n-x^*\|^2-\Phi(\|y_n-x^*\|)\right]\\
&\quad +2\beta_n\|Ty_n-x^*\|\,\|j(x_{n+1}-x^*)-j(x_n-x^*)\|\\
&\quad +2\beta_n\|Ty_n-x^*\|\,\|j(x_n-x^*)-j(y_n-x^*)\|\\
&\quad +2\gamma_n\|u_n-x^*\|\,\|x_{n+1}-x^*\|\\
&\le (1-\beta_n)^2r^2+4\beta_nM\varepsilon_0+2\gamma_n\cdot M\cdot\frac{3}{2}r\\
&\quad +2\beta_n\left[\|y_n-x^*\|^2-\Phi(\|y_n-x^*\|)\right]\ ,
\end{aligned}\tag{0.6}$$

$$\begin{aligned}
\|y_n-x^*\|^2 &= \|x_n-x^*-a_n(I-T)x_n-b_n(x_n-v_n)\|^2\\
&\le \|x_n-x^*\|^2-2a_n\langle (I-T)x_n,j(y_n-x^*)-j(x_n-x^*)\rangle\\
&\quad -2a_n\langle (I-T)x_n,j(x_n-x^*)\rangle-2b_n\langle x_n-v_n,j(y_n-x^*)\rangle\\
&\le \|x_n-x^*\|^2-2a_n\Phi(\|x_n-x^*\|)+2b_n\|x_n-v_n\|\,\|y_n-x^*\|\\
&\quad +2a_n\|(I-T)x_n\|\,\|j(y_n-x^*)-j(x_n-x^*)\|\\
&\le r^2+2M\varepsilon_0+2b_nM\cdot\frac{5}{4}r\ .
\end{aligned}\tag{0.7}$$

Substitute (3.7) into (3.6), since $0<\beta_n\ ,\ \gamma_n\ ,\ a_n\ ,\ b_n\le d_0\ ,\gamma_n=o(\beta_n\ )$ and $d_0=\min\left\{1,\frac{r}{8M},\frac{\delta}{2(2M+r)},\frac{r}{2(2M+r)},\frac{\delta}{4M},\ldots\right.$,we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\beta_n)^2r^2+2\beta_n\left[r^2+2M\varepsilon_0+\frac{5}{2}b_nMr-\Phi(\|y_n-x^*\|)\right]\\
&\quad +4\beta_n\cdot M\cdot\varepsilon_0+3\gamma_n\cdot M\cdot r\\
&\le r^2+2\beta_n\left[\frac{\beta_n}{2}r^2+4M\varepsilon_0+\frac{5}{2}b_nMr+\frac{3\gamma_n}{2\beta_n}Mr\right]-2\beta_n\Phi\left(\frac{r}{4}\right)\\
&\le r^2+2\beta_n\left[\frac{\Phi\left(\frac{r}{4}\right)}{2}-\Phi\left(\frac{r}{4}\right)\right]\\
&\le r^2
\end{aligned}$$

i,e.,$\|x_{n+1}-x^*\|\le r$, a contradiction. Therefore $x_{n+1}\in G$. Thus by induction $\{x_n\}$ is bounded. Then, $\{y_n\}$, $\{Ty_n\}$, $\{Tx_n\}$, $\{(I-T)x_n\}$ are also bounded.

Claim2: $x_n\to x^*$.

Let $A_n=\|j(x_{n+1}-x^*)-j(x_n-x^*)\|$, $B_n=\|j(x_n-x^*)-j(y_n-x^*)\|$, Note that $x_{n+1}-x_n\to 0$, $x_n-y_n\to 0$ as $n\to\infty$ and hence by the uniform continuity of j on bounded subsets of E we have that

$$A_n \to 0\,, B_n \to 0 \text{ as } n \to \infty.$$

Let

$$M_1 = \max\left\{ \begin{array}{l} \sup\|x_n - x^*\|\ ,\ \sup\|y_n - x^*\|\ ,\ \sup\|(I-T)\,x_n\|\ , \\ \sup\|Ty_n - x^*\|\ ,\ \sup\|u_n - x^*\|\ ,\ \sup\|x_n - v_n\| \end{array} \right\},$$

by (3.6), (3.7), we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &\quad + 2\beta_n\|Ty_n - x^*\|\,\|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\quad + 2\beta_n\|Ty_n - x^*\|\,\|j(x_n - x^*) - j(y_n - x^*)\| \\ &\quad + 2\gamma_n\|u_n - x^*\|\,\|x_{n+1} - x^*\| \\ &\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n M_1 A_n + 2\beta_n M_1 B_n + 2\gamma_n {M_1}^2 \\ &\quad + 2\beta_n\left[\|y_n - x^*\|^2 - \Phi\left(\|y_n - x^*\|\right)\right] \end{aligned} \tag{0.8}$$

and

$$\begin{aligned} \|y_n - x^*\|^2 &\le \|x_n - x^*\|^2 + 2a_n\|(I-T)\,x_n\|\,\|j(y_n - x^*) - j(x_n - x^*)\| \\ &\quad - 2a_n\Phi\left(\|x_n - x^*\|\right) + 2b_n\|x_n - v_n\|\,\|y_n - x^*\| \\ &\le \|x_n - x^*\|^2 + 2M_1B_n + 2b_n{M_1}^2\ . \end{aligned} \tag{0.9}$$

Taking (3.9) into (3.8),

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n M_1 A_n + 2\beta_n M_1 B_n + 2\gamma_n {M_1}^2 \\ &\quad + 2\beta_n\left[\|x_n - x^*\|^2 + 2M_1B_n + 2b_n{M_1}^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &\le \|x_n - x^*\|^2 + 2\beta_n\left[\frac{\beta_n}{2}{M_1}^2 + M_1A_n + 3M_1B_n + 2b_n{M_1}^2 + \frac{\gamma_n}{\beta_n}{M_1}^2 - \Phi\left(\|y_n - x^*\|\right)\right] \\ &\le \|x_n - x^*\|^2 + 2\beta_n\left[Z_n - \Phi\left(\|y_n - x^*\|\right)\right]\ , \end{aligned}$$

where $Z_n = \frac{\beta_n}{2}{M_1}^2 + M_1A_n + 3M_1B_n + 2b_n{M_1}^2 + \frac{\gamma_n}{\beta_n}{M_1}^2 \to 0$ as $n \to \infty$.
Set $\inf\frac{\Phi(\|y_n - x^*\|)}{\Phi(\|x_n - x^*\|)+1} = L > 0$, since Φ is a strictly increasing continuous function, then L exists. Thus

$$\Phi\left(\|y_n - x^*\|\right) \ge L\Phi\left(\|x_n - x^*\|\right) + L \ge L\Phi\left(\|x_n - x^*\|\right)\ .$$

Then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + 2\beta_n\left[Z_n - L\Phi\left(\|x_n - x^*\|\right)\right] \\ &\le \|x_n - x^*\|^2 - 2\beta_n L\Phi\left(\|x_n - x^*\|\right) + 2\beta_n Z_n\ . \end{aligned} \tag{0.10}$$

Let $\lambda_n := \|x_n - x^*\|$ and $\rho_n = 2\beta_n Z_n$, then from inequality (3.10) we obtain that$\lambda_{n+1} \le \lambda_n - 2\beta_n L\Phi\left(\lambda_n\right) + \rho_n$, where $\frac{\rho_n}{\beta_n} \to 0$ as $n \to \infty$. Therefore, the conclusion of the theorem follows from Proposition 2.7. ■

The following corollary follow trivially,since definition 2.3 and definition 2.4.

Corollary 11. *Suppose E is a real uniformly smooth Banach space . Suppose $T : E \to E$ is a bounded generalized $\Phi-$accretive map with strictly increasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that $\Phi(0) = 0$ and the solution x^* of the equation $Tx = 0$ exists. For arbitrary $x_0 \in E$, the sequence $\{x_n\}$ in E is defined by*

$$\begin{cases} x_{n+1} = (1 - \beta_n - \gamma_n)\, x_n - \beta_n T y_n + \gamma_n u_n \\ y_n = (1 - a_n - b_n)\, x_n + a_n T x_n + b_n v_n, \ n \geq 0, \end{cases}$$

where $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$ $\subseteq [0,1]$, $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = 0$, $\gamma_n = o(\beta_n)$ *and* $\sum \beta_n = \infty$. *Then, there exists a constant* $d_0 > 0$ *such that if* $0 < \beta_n$, γ_n , a_n , $b_n \leq d_0$, $\{x_n\}$ *converges strongly to the unique solution of* $Tx = 0$.

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