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On Some Classes of Series Representations for $1/\pi$ and π^2

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Abstract
We propose some classes of series representations for $1/\pi$ and π^2 by using a new WZ-pair. As examples, among many others. we prove that

$$\frac{3}{2}\sum_{n=1}^{\infty}\frac{n}{16^n(n+1)(2n-1)}\binom{2n}{n}^2=\frac{1}{\pi},$$

$$1-\frac{1}{4}\sum_{n=0}^{\infty}\frac{3n+2}{(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n}=\frac{1}{\pi}$$

and

$$4\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)}\frac{4^n}{\binom{2n}{n}}=\pi^2.$$

Furthermore, our results lead to new combinatorial identities and binomial sums involving harmonic numbers.

Keywords: *Ramanujan-type series; binomial sums; gamma function; digamma function; combinatorial identities; Legendre's duplication formula.*

AMS Subject Classification (2020): *Primary: 33C05; 33C20; 33C90; Secondary: 05A19.*

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1. Introduction

In 1914 in his famous paper [25] Indian genius mathematician Srinivasa Ramanujan proposed 17 extraordinary series for $1/\pi$ without giving a complete proof. The most well known two of them were as follows:

$$\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}\frac{1103+26390n}{99^{4n}}=\frac{1}{\pi}$$

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and

$$\frac{1}{16}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{(1)_n^3}\frac{42n+5}{64^n}=\frac{1}{\pi}.$$

Here $(a)_n$ stands for the Pochhammer symbol defined by

$$(a)_0=1 \quad \text{and} \quad (a)_n=a(a+1)(a+2)...(a+n-1) \quad ,n\geq 1.$$

Ramanujan's series for $1/\pi$ have not received much interest from mathematical community until 1985. In 1985 Gosper used one of Ramanujan's series to calculate 17,526,100 digits of π, which is at that time was a world record [2]. In 1987 Peter and Jon Borwein [5] provided rigorous proofs of all 17 of Ramanujan's series for $1/\pi$ for the first time and also offered many new series representations for this constant; see [3, 4, 6]. J. Guillera provided the proofs of 11 of Ramanujan series by using the WZ-method [19, Tables I,II]. At about the same time as the Borweins were devising their proofs, David and Gregory Chudnovsky [9] derived new series representations for $1/\pi$ and used the following their Ramanujan-type series

$$\frac{1}{\pi}=12\sum_{n=0}^{\infty}(-1)^n\frac{(6n)!}{(n!)^3(3n)!}\frac{13591409+545140134n}{640320^3n+3/2}$$

to calculate 2,260,331,336 digits of π, which was a world record even in 1989. It should be remarked that before Ramanujan in 1859 G. Bauer [1], and in 1905 W. L. Glaisher [10] had given series representations for $1/\pi$. The studies on Ramanujan-like series for $1/\pi$ are continuing intensively today, too and recently, many new series of this type have been published, see for example [7,8,11-23]. The aim of this paper is to derive new classes of series representations for $1/\pi$ and π^2 by using the WZ-method. Our results enable us to establish infinity many of new Ramanujan type series for the constants $1/\pi$ and π^2. Our results also lead to some new combinatorial identities involving harmonic numbers. The remainder of this paper organized as follows. In the next section, we explain how the WZ-method works briefly. In Section 3, we present our main theorems. In the final section choosing particular values for a free parameter, we offer many series representations for the constants $1/\pi$ and π^2. In this paper we shall frequently use the generalized binomial coefficient

$$\binom{s}{t}=\frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)},$$

where t and s are real numbers which are not negative integers, and the Legendre's duplication formula for the classical gamma function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\ (x>0)$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi},\ n\in\mathbb{N}\cup\{0\}. \tag{1.1}$$

2. The WZ-method (Wilf-Zeilberger Method)

In this section we want to explain the WZ-method briefly. A discrete function $A(n,k)$ is hypergeometric if both

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in both n and k. A pair (F,G) of hypergeometric functions is said to be a $WZ-$ pair (Wilf-Zeilberger pair) if for all $k\in\mathbb{Z}$ and $n=0,1,2,...$ they satisfy

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{2.1}$$

In this case Wilf and Zeilberger [24, Chapter 7] and [27] proved that there exists a rational function $C(n,k)$ such that $G(n,k)=C(n,k)F(n,k)$. Wilf and Zeilberger called $C(n,k)$ as certificate of the pair (F,G). Summing on $n\geq 0$ both sides of (2.1), one gets

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\sum_{n=0}^{\infty}\{F(n+1,k)-F(n,k)\}=\lim_{n\to\infty}F(n,k)-F(0,k). \tag{2.2}$$

In most applications it is usually very easy to evaluate $F(0,k)$ and $\lim_{n\to\infty} F(n,k)$. So, taking particular values for k in (2.2), we can obtain many identities. We can also sum both sides of (2.1) over k's and in this case we get

$$\sum_{k=0}^{\infty}\{F(n+1,k)-F(,k)\}=\sum_{k=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\lim_{k\to\infty}G(n,k)-G(n,0).$$

If $G(n,0)=0$ and $\lim_{k\to\infty} G(n,k)=0$, we get

$$\sum_{k=0}^{\infty}\{F(n+1,k)-F(n,k)\}=0\,(n=0,1,2,3,\ldots),$$

which implies that $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Let us say $\sum_{k=0}^{\infty} F(n,k)=C$. Usually, it is very easy to evaluate this constant by choosing a particular value for k (usually k=0), in other cases we evaluate it by taking the limit as $k\to\infty$. Please refer to [24] and [27] for more information about the WZ-method.

3. Main results

In this section we collect our main results.

Theorem 3.1. *Let a be any real number, which is not zero and a negative integer. Then we have*

$$\sum_{n=0}^{\infty}\frac{(3n+2a+1)\Gamma(n+1/2)\Gamma(n+a+1)}{(n+a)\Gamma(n+2)\Gamma(n+a+3/2)}=\frac{4\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)}-\frac{2}{a}. \tag{3.1}$$

Proof. Consider the following discrete function.

$$F(n,k)=\frac{1}{2\pi}\frac{(n+2a)\Gamma(k+1/2)\Gamma(n-k+1/2)\Gamma(n+a+1)\Gamma(a+1/2)}{(k+a)(n-k+a)\Gamma(a)\Gamma(k+1)\Gamma(n-k+1)\Gamma(n+a+1/2)}. \tag{3.2}$$

The package EKHAD [24] allows us to obtain the companion

$$G(n,k)=\frac{-1}{2\pi}\frac{(3n+2a-2k+3)\Gamma(n-k+3/2)\Gamma(n+a+1)\Gamma(k+1/2)\Gamma(a+1/2)}{(n-k+a+1)(n+1)\Gamma(k)\Gamma(a)\Gamma(n+a+3/2)\Gamma(n-k+2)}, \tag{3.3}$$

where $k\in\mathbb{Z}$ and $n\in\mathbb{N}\cup\{0\}$. That is, (F,G) is a WZ-pair, so that, we have

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{3.4}$$

Summing over n both sides of (3.4), we get

$$\sum_{n=0}^{\infty}\{F(n+1,k)-F(n,k)\}=\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}$$

or

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\lim_{n\to\infty}F(n,k)-F(0,k). \tag{3.5}$$

By Stirling's formula $n!\sim n^n e^{-n}\sqrt{2\pi n}$, we can easily find that

$$\lim_{n\to\infty}\frac{(n+2a)\Gamma(n-k+1/2)\Gamma(n+a+1)}{(n-k+a)\Gamma(n-k+1)\Gamma(n+a+1/2)}=1, \tag{3.6}$$

which yields

$$\lim_{n\to\infty}F(n,k)=\frac{1}{2\pi}\frac{\Gamma(k+1/2)\Gamma(a+1/2)}{(k+a)\Gamma(a)\Gamma(k+1)}.$$

We therefore have

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\frac{1}{2\pi}\frac{\Gamma(k+1/2)\Gamma(a+1/2)}{(k+a)\Gamma(a)\Gamma(k+1)}-F(0,k), \tag{3.7}$$

For $k = 0$ this immediately gives

$$\sum_{n=0}^{\infty}\{G(n,1) - G(n,0)\} = \frac{1}{2\pi}\frac{\Gamma(1/2)\Gamma(a+1/2)}{\Gamma(a+1)} - 1.$$

But since $G(n,0) = 0$ and $\Gamma(1/2) = \sqrt{\pi}$, we get

$$\sum_{n=0}^{\infty} G(n,1) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(a+1/2)}{\Gamma(a+1)} - 1.$$

or by (3.3)

$$\sum_{n=0}^{\infty}\frac{(3n+2a+1)\Gamma(n+1/2)\Gamma(n+a+1)}{(n+a)\Gamma(n+2)\Gamma(n+a+3/2)} = \frac{4\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)} - \frac{2}{a}.$$

□

If we substitute $a = m - 1/2$ ($m \in \mathbb{Z}$) in (3.1), we get

Corollary 3.2. *Let m be any integer. Then, we have*

$$\sum_{n=0}^{\infty}\frac{3n+2m}{(n+1)(2n+2m-1)}\binom{2n}{n}\binom{2n+2m}{n+m}\frac{1}{16^n} = \frac{4^{m+1}\Gamma(m+1/2)}{\sqrt{\pi}(2m-1)\Gamma(m)} - \frac{4^{m+1}}{2(2m-1)}\frac{1}{\pi}. \tag{3.8}$$

In particular, if m is zero or a negative integer, we have

$$\frac{2-4m}{4^{m+1}}\sum_{n=0}^{\infty}\frac{3n+2m}{(n+1)(2n+2m-1)}\binom{2n}{n}\binom{2n+2m}{n+m}\frac{1}{16^n} = \frac{1}{\pi}. \tag{3.9}$$

Theorem 3.3. *Let a be any real number, which is not a negative integer, then we have*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{k+a} = \frac{2^{2n}\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.10}$$

Proof. let F and G be as in (3.3) and (3.4). Summing both sides (3.4) on $k = 0, 1, 2, \ldots$, we get

$$\sum_{k=0}^{\infty}\{F(n+1,k) - F(n,k)\} = \lim_{k\to\infty} G(n,k) - G(n,0).$$

By using Stirling formula it is very easy to see that $\lim_{k\to\infty} G(n,k) = 0$. Clearly, we also have $G(n,0) = 0$. Then for all $n = 0, 1, 2, \ldots$, we get

$$\sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(n+1,k) = \sum_{k=0}^{\infty} F(n+2,k) = \cdots,$$

which implies that $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Let $\sum_{k=0}^{\infty} F(n,k) = A$. We can evaluate the constant A by setting $n = 0$, so that we obtain

$$A = \sum_{k=0}^{\infty} F(0,k) = \frac{1}{2\pi}\frac{2a\Gamma(a+1)\Gamma(a+1/2)}{\Gamma(a)\Gamma(a+1/2)}\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)\Gamma(-k+1/2)}{(k+a)(a-k)\Gamma(k+1)\Gamma(1-k)}.$$

Notice that this sum is zero for $k = 1, 2, \ldots$ except $k = 0$. Hence we get

$$A = \frac{1}{\pi}\frac{\Gamma(a+1)a}{\Gamma(a)}\frac{\Gamma(1/2)^2}{a^2} = \frac{1}{\pi}\frac{\Gamma(a+1)a}{\Gamma(a+1)}\frac{\pi}{a} = 1$$

Hence, we conclude that for all $n = 0, 1, 2, ...$

$$\sum_{k=0}^{\infty} F(n,k) = 1.$$

From this identity, by the help of (1.1), we obtain

$$\sum_{k=0}^{n}\frac{\Gamma(k+1/2)\Gamma(n-k+1/2)}{(k+a)(n-k+a)\Gamma(k+1)\Gamma(n-k+1)}=\frac{2\pi\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}.$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+a)(n-k+a)}=\frac{2^{2n+1}\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.11}$$

Since

$$\frac{1}{(k+a)(n-k+a)}=\frac{1}{n+2a}\Big(\frac{1}{k+a}+\frac{1}{n-k+a}\Big),$$

we get from (3.11)

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{k+a}=\frac{4^n\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)},$$

which is the desired result. □

Corollary 3.4. *Let a be any real number, which is not zero and a negatve integer. Then we have*

$$\sum_{n=0}^{n}\frac{\binom{2n}{n}}{4^n(n+a)}=\frac{\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)} \tag{3.12}$$

Proof. Multiplying by $\sqrt{n}4^{-n}$ both sides of (3.10) and taking infinity the upper bound of the summation, we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\sqrt{n}}{4^n}\binom{2n-2k}{n-k}\frac{1}{k+a}=\frac{\Gamma(a)}{\Gamma(a+1/2)}\frac{\sqrt{n}\Gamma(n+a+1/2)}{\Gamma(n+a+1)}. \tag{3.13}$$

Since, by Stirling's formula,

$$\lim_{n\to\infty}\frac{\sqrt{n}}{4^n}\binom{2n-2k}{n-k}=\frac{4^{-k}}{\sqrt{\pi}} \quad \text{and} \quad \lim_{n\to\infty}\frac{\sqrt{n}\Gamma(n+a+1/2)}{\Gamma(n+a+1)}=1,$$

the proof follows from (3.13) by letting taking the limit of both ides as $n\to\infty$. □

Corollary 3.5. *Let a be a non-zero real number such that $2a$ is not a negative integer. Then we have*

$$\sum_{n=0}^{\infty}\binom{2n+2a}{n+a}\frac{4^{-n}}{n+2a}=\frac{\sqrt{\pi}2^{2a-1}\Gamma(a)}{\Gamma(a+1/2)} \tag{3.14}$$

Proof. From (3.11), we have

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}}{4^k(k+a)}.\frac{\binom{2n-2k}{n-k}}{4^{n-k}(n-k+a)}=\frac{2\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.15}$$

Summing both sides (3.15) over n, it follows that

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{\binom{2k}{k}}{4^k(k+a)}\cdot\frac{\binom{2n-2k}{n-k}}{4^{n-k}(n-k+a)}\Big)=\frac{2\Gamma(a)}{\Gamma(a+1/2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)}.$$

Since the left side is a Cauchy product of two series, we conclude

$$\Big(\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(n+a)}\Big)^2=\frac{2\Gamma(a)}{\Gamma(a+1/2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)}.$$

Now the proof follows from (3.12) by the help of (1.1). the result by using (3.4). □

Remark 3.6. The identity (3.12) can also be obtained from the Gauss hypergeometric series but we want to give a proof because of the method we used can be employed in other places.

4. Applications

4.1 Series for $1/\pi$

Taking particular values for m in (3.7) and (3.8) we can obtain many series for $1/\pi$ by the help of the duplication formula (1.1).

Example 1. If we set $m = 0$ in (3.9), we get

$$\frac{3}{2}\sum_{n=1}^{\infty}\frac{n}{16^n(n+1)(2n-1)}\binom{2n}{n}^2 = \frac{1}{\pi}$$

Example 2. If we set $m = -1$ in (3.9), we get

$$\frac{3}{4}\sum_{n=0}^{\infty}\frac{n(3n-2)}{16^n(2n-3)(2n-1)(n+1)}\binom{2n}{n}^2 = \frac{1}{\pi}.$$

Example 3. If we set $m = -2$ in (3.9), we get

$$\frac{5}{8}\sum_{n=0}^{\infty}\frac{(2n+3)(3n+2)(2n+1)}{(n+1)(n+2)(n+3)(2n-1)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 4. If we set $m = -3$ in (3.9), we get

$$\frac{21}{6}\sum_{n=0}^{\infty}\frac{(2n+5)(2n+3)(2n+1)}{(2n-1)(n+4)(n+3)(n+2)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}$$

Example 5. If we set $m = -4$ in (3.9), we get

$$\frac{9}{32}\sum_{n=0}^{\infty}\frac{(3n+4)(2n+7)(2n+5)(2n+3)(2n+1)}{(2n-1)(n+5)(n+4)(n+3)(n+2)(n+1)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi},$$

Example 6. If we set $m = 1$ in (3.8), we get

$$1 - \frac{1}{4}\sum_{n=0}^{\infty}\frac{3n+2}{(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}$$

Example 7. If we set $m = 2$ in (3.8), we get

$$\frac{3}{2} - \frac{3}{8}\sum_{n=0}^{\infty}\frac{(2n+1)(3n+4)}{(n+1)^2(n+2)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 8. If we set $m = 3$ in (3.8), we get

$$\frac{15}{8} - \frac{15}{16}\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)}{(n+1)^2(n+3)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 9. If we set $m = 4$ in (3.8), we get

$$\frac{35}{16} - \frac{7}{32}\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)(2n+5)(3n+8)}{(n+2)(n+3)(n+4)(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

4.2 Series for π^2

Taking particular values for a in (3.14) we can obtain many series for π^2.

Example 1. Setting $a = 1/2$ in (3.14) we get

$$4\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)}\frac{4^n}{\binom{2n}{n}} = \pi^2.$$

Example 2. If we set $a = 3/2$ in (3.14), we get

$$16\sum_{n=0}^{\infty}\frac{n+1}{(n+3)(2n+1)(2n+3)}\frac{4^n}{\binom{2n}{n}}=\pi^2.$$

Example 3. If we set $a = 5/2$ in (3.14), we get

$$\frac{128}{3}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{(n+5)(2n+1)(2n+3)(2n+5)}\frac{4^n}{\binom{2n}{n}}=\pi^2.$$

Example 4. If we set $a = 7/2$ in (3.14), we get

$$\frac{512}{5}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{(n+7)(2n+1)(2n+3)(2n+5)(2n+7)}\frac{4^n}{\binom{2n}{n}}=\pi^2.$$

4.3 Combinatorial identities involving harmonic numbers

Differentiating w.r.t a both sides of (3.10), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+a)^2}=\frac{\binom{2n+2m}{n+m}}{m\binom{2m}{m}}\{\psi(a)+\psi(n+a+1/2)-\psi(n+a+1)-\psi(a+1/2)\}, \tag{4.1}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function. Substituting particular values for a in (4.1) and using the following duplication formula for the digamma function

$$\psi\left(n+\frac{1}{2}\right)=2\psi(2n)-2\log 2-\psi(n)=-\gamma+2H_{2n}-H_n-2\log 2,$$

where $\gamma = 0.57721...$ is the Euler constant, we obtain the following combinatorial identities involving harmonic numbers.

Example 1. If we substitute $a = 1/2$ in (4.1), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(2k+1)^2}=\frac{16^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}}. \tag{4.2}$$

Example 2. If we substitute $a = m \in \mathbb{N}$ in (4.1), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+m)^2}=\frac{2\binom{2m+2n}{m+n}}{m\binom{2m}{m}}\left(H_{m+n}+H_{2m}-H_m-H_{2m+2n}+\frac{1}{2m}\right).$$

Example 3.

$$\sum_{n=0}^{\infty}\frac{2^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}}=\frac{\pi}{4}\log 2+G,$$

where G is the Catalan constant defined by $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$. We want to give a proof of this identity. Summing both sides of (4.2), after dividing by 8^n, we get

$$\sum_{n=0}^{\infty}\frac{2^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{8^k(2k+1)^2}\frac{\binom{2n-2k}{n-k}}{8^{n-k}}=\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k(2k+1)^2}\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8k}. \tag{4.3}$$

By [3, pg. 386], we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8k}=\sqrt{2} \tag{4.4}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2k+1}x^{2k}=\frac{\arcsin(2x)}{2x}. \tag{4.5}$$

Integrating both sides of (4.5) over $(0, \sqrt{2}/4)$, and making the change of variable $\arcsin(2x) = u$, we get

$$\frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n(2n+1)^2} = \frac{1}{2} \int_0^{\pi/4} u \cot u du. \tag{4.6}$$

From [26, pg. 44,45] we have

$$\int_0^{\pi/4} u \cot u du = \frac{\pi}{8} \log 2 + \frac{G}{2}. \tag{4.7}$$

Combining the identities (4.3)-(4.7), the result is obtained.

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