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Relation Between Extended Stirling Numbers and q-B-splines

Genişletilmiş Stirling Sayıları ve q-B-spline Fonksiyonları Arasındaki İlişki

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Abstract

Stirling numbers of second kind *S(n,k)* denotes the number of ways partitioning a set of *n* elements into *k* nonempty sets. There are many types of Stirling numbers which are studied up to now. In this study, we use extended Stirling numbers of second kind which are defined for arbitrary reals. First, we define a relation between extended Stirling numbers and q-B-splines by using the property that divided differences have a representation with q-B-splines. In addition, we derive identities on Stirling numbers and q-integral of q-B-splines. Furthermore, we give q-generating functions of extended Stirling numbers and define a q-difference equation for this function.
Keywords: *Stirling numbers, q-B-splines, Divided Differences*

Öz

İkinci tür Stirling sayıları *S(n,k), n* elemanlı bir kümeyi *k* tane boş olmayan kümeye bölen yolların sayısını belirtir. Şimdiye kadar üzerinde çalışılan pek çok Stirling sayı çeşidi vardır. Bu çalışmada keyfi gerçel sayılar için tanımlanan genişletilmiş Stirling sayılarını kullanacağız. İlk olarak, genişletilmiş Stirling sayıları ve q-B-spline fonksiyonları arasındaki ilişkiyi bölünmüş farkların q-B-spline fonksiyonları ile gösterimini kullanarak tanımlayacağız. Buna ek olarak, Stirling sayıları ve q-B-spline fonksiyonlarının q-integralleri üzerine özdeşlikler türeteceğiz. Ayrıca genişletilmiş Stirling sayılarının q-üretici fonksiyonunu bulacağız ve bu fonksiyon için bir q-fark denklemi vereceğiz.
Anahtar Kelimeler: *Stirling sayıları, q-B-spline, Bölünmüş farklar*

1. Introduction

Stirling numbers of second kind $S(n;k)$ denotes the number of ways partitioning a set of n elements into k nonempty sets. We come across Stirling numbers in various analytic and combinatorial problems. Stirling numbers have been studied by many mathematicians up to now. There are numerous generalizations of Stirling numbers such as q-Stirling numbers (see [1]); $p;q$-Stirling numbers (see [2]); r-Stirling numbers (see [3]), $(q; r; w)$-Stirling numbers (see [4]), the degenerate truncated Stirling polynomials of the second kind (see [5]) etc..

In this study we focus on the extended Stirling numbers depending on given reals which

Nueman introduced in [6]. We see that by choosing special reals we can obtain some other forms of Stirling numbers. Nueman derived also the connection between extended Stirling numbers and classical B-splines and gave identities in [6].

Our goal is to show that there is a similar connection between extended Stirling numbers and q-B-splines, where q-B-splines are a generalization of classical B-splines. q-B-splines are first introduced in [7] and fundamental properties are studied in [8]. It is notable that as $q \to 1$, q-B-splines become classical B-splines. So, the identities given in this study can be considered as a generalization of identities in [6]. Furthermore, for a convenient value of q, we obtain the connection between Stirling numbers and moments of q-B-splines. If we choose appropriate reals, the identities become relation between classical Stirling numbers of second kind, q-Stirling numbers of the second kind and q-B-splines.

This paper is organized as follows: In Section 2, we give definitions and some properties of q-calculus, q-B-splines and extended Stirling numbers which we need to derive identities. In Section 3, we derive identities which give the relation between extended Stirling numbers and q-B-splines. Also we derive q-generating function for the extended Stirling numbers.

2. Preliminaries

In this section we give a summary for q-calculus, q-B-splines and extended Stirling numbers that we need in the next section consisting main results.

2.1. The *q*-calculus

Since this paper consists of quantum derivatives and quantum integrals as well as q-exponential functions, we give basic definitions of the q-calculus. For a fixed parameter $q \neq 1$, the q-derivatives are defined by,

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}$$

$$D_q^n f(t) = D_q(D_q^{n-1} f(t)), \quad n \geq 2$$

One may consider q-derivatives as approximations to classical derivatives, i.e., if f is a differentiable function, then

$$\lim_{q \to 1} D_q f(x) = Df(x)$$

q-derivatives of monomials can be computed as

$$D_q x^n = [n]_q x^{n-1},$$

where the q-integers $[n]_q$ are defined by

$$[n]_q = \begin{cases} (1-q)^n / (1-q), & q \neq 1, \\ n, & q = 1. \end{cases}$$

Furthermore, the q-factorial is defined by

$$[n]_q! = [1]_q [2]_q \cdots [n]_q.$$

For any constant c the q-chain rule is

$$D_q(f(cx)) = cD_q(f(y))_{y=cx}. \tag{1}$$

Now, let's give the definition of definite q-integral, for details see [9].

Definition: Let $0 < a < b$. Then the definite q-integral of a function $f(x)$ is defined by a convergent series

$$\int_a^b f(x) d_q x = (1-q) b \sum_{i=0}^{\infty} q^i f(q^i b)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Theorem 2.1. If $F(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q F(x) d_q x = F(b) - F(a)$$

where $0 \leq a < b \leq \infty$.

We also need the definition of q-exponential function. There are two types of q-exponential functions e_q^t and E_q^t. There is a relation between two q-exponential functions

$$e_{1/q}^t = E_q^t$$

where

$$e_q^t = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!},$$

$$E_q^t = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{t^n}{[n]_q!}, \qquad (2)$$

q-derivatives of q-exponential functions give theirselves;

$$D_q e_q^t = e_q^t,$$

and

$$D_{1/q} E_q^t = E_q^t.$$

Later we shall also need the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

which can be computed recursively by the q-Pascal identities

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

or

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Indeed $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q of degree $k(n-k)$, and it can be considered as the generating functions for restricted partitions of integers.

2.2. q-B-splines

q-B-splines are first introduced in [7] and fundamental properties are given in [8]. Given $k, n \in N$, and a knot sequence $t_i < t_{i+1} < \cdots < t_{i+k}$, the q-B-spline sequences $\{M_i\}_{i=1}^n$ and $\{N_i\}_{i=1}^n$ of order k are defined similar to classical B-splines in [10] with respect to

-Normalization $\int M_{i,k}(x;q) d_{1/q}x = 1$

$$M_{i,k}(x;q) = [k]_q [t_i, t_{i+1}, \ldots, t_{i+k}](t-x)_+^{k-1,q}$$

-Normalization $\sum_i N_{i,k}(x;q) = 1$

$$N_{i,k}(x;q) = (t_{i+k} - t_i)[t_i, t_{i+1}, \ldots, t_{i+k}](t-x)_+^{k-1,q}$$

-Relation

$$M_{i,k}(x;q) = \frac{[k]_q}{t_{i+k} - t_i} N_{i,k}(x;q)$$

where

$$(t-x)_+^{k-1,q} = \begin{cases} 1, & \text{if } k = 1 \\ (t - q^{k-2}x) \cdots (t - qx)(t-x)_+, & \text{if } k \neq 1 \end{cases}$$

with

$$(t-x)_+ = \begin{cases} t - x, & \text{if } t \geq x \\ 0, & \text{otherwise} \end{cases}$$

and $[t_i, t_{i+1}, ..., t_{i+k}]f$ denotes the divided difference defined by $[t_i]f = f(t_i)$ and the reccurrence relation

$$[t_i, t_{i+1}, ..., t_{i+k}]f = \frac{[t_{i+1}, t_{i+2}, ..., t_{i+k}]f - [t_i, t_{i+1}, ..., t_{i+k-1}]f}{t_{i+k} - t_i}.$$

The following property of q-B-splines will allow us to get identities between extended Stirling numbers and q-B-splines.

Proposition 2.2.

$$[t_i, \ldots, t_{i+k}]f = \frac{q^{k(k-1)/2}}{[k]_q!} \int_{-\infty}^{\infty} M_{i,k}(x;q)(D_{1/q}^k f)(q^{k-1}x) d_{1/q}x$$

Proof. See [8].

2.3. Extended Stirling numbers

Let $S_{\underline{t}}(n,k)$ be the extended Stirling numbers of the second kind defined by

$$x^n = \sum_{k=0}^{n} S_{\underline{t}}(n,k)(x-t_0)(x-t_1)\cdots(x-t_{k-1}) \quad (3)$$

and set $S_{\underline{t}}(0,0) = 1$ and $S_{\underline{t}}(n,k) = 0$ for $k > n$.

Notice that for $\underline{t} = (0, 1, \ldots, k)$ we obtain classical Stirling numbers of second kind, i.e.

$$S_{\underline{t}}(n,k) = S(n,k)$$

and for $\underline{t} = (0, [1]_q, \ldots, [k]_q)$ we obtain q-Stirling numbers of second kind, i.e.,

$$S_{\underline{t}}(n,k) = S_q(n,k)$$

where $S_q(n,k)$ denotes the q-Stirling numbers of second kind.

One can show that equation (3) is the Newton divided difference form for $f(x)=x^n$ interpolating at the points , i.e.

$$S_{\underline{t}}(n,k)=[t_0,t_1,\dots,t_k]x^n, \qquad 0\le k\le n \quad (4)$$

It is obvious that by using elementary properties of divided differences we get

$$S_{\underline{t}}(n,0)=t_0, \qquad S_{\underline{t}}(n,1)=\frac{t_1^n-t_0^n}{t_1-t_0}, \qquad S_{\underline{t}}(n,n)=1$$

The recurrence relation for extended Stirling numbers (see [6]) is as follows:

$$S_{\underline{t}}(n,k)=S_{\underline{t}}(n-1,k-1)+t_k S_{\underline{t}}(n-1,k)$$

3. Identities

In this section we give identities which establish the relation between extended Stirling numbers of the second kind $S_{\underline{t}}(.,.)$ and the q-B-splines $M_{0,k}$ by defining a new function $\lambda_l(k,t;q)$ where

$$M_{0,k}(x;q)=[k]_q[t_0,t_1,\dots,t_k](t-x)_+^{k-1,q}$$

and

$$\lambda_l(k,\underline{t};q)=\int_{-\infty}^{\infty} x^l M_{0,k}(x;q)d_{1/q}x, \qquad l=0,1,\dots$$

Notice that as $q\to 1$, $\lambda_l(k,\underline{t};q)$ becomes moments of q-B-splines.

Proposition 3.1. Let $l=0,1,\dots,$ $k=1,2,\dots$ and $\underline{t}=(t_0,t_1,\dots,t_k)$ with $t_0<t_1<\cdots t_k$. Then

$$S_{\underline{t}}(n,k)=q^{-l}\begin{bmatrix}k+l\\ l\end{bmatrix}_q \lambda_l(k,\underline{t};q) \qquad (5)$$

Proof. It is given in [7] that

$$[t_0,\dots,t_k]f=\frac{q^{k(k-1)/2}}{[k]_q!}\int_{-\infty}^{\infty} M_{0,k}(x;q)(D_{1/q}^k f)(q^{k-1}x)d_{1/q}x$$

(6)

Suppose $f(x)=x^{k+l}$, then

$$(D_{1/q}^k f)(q^{k-1}x)=\frac{[k+l]_{1/q}!}{[l]_{1/q}!}(q^{k-1}x)^l. \qquad (7)$$

By using the fact that $[r]_{1/q}=\frac{1}{q^{r-1}}[r]_q$ equation (6) becomes

$$[t_0,\dots,t_k]f=\frac{q^{k(k-1)/2}}{[k]_q!}\,\frac{\frac{1}{q^{(k+l-1)(k+l)/2}}[k+l]_q!}{\frac{1}{q^{l(l-1)/2}}[l]_q!}$$

$$*\int_{-\infty}^{\infty} M_{0,k}(x;q)(q^{k-1}x)^l d_{1/q}x$$

$$= q^{-l}\begin{bmatrix}k+l\\ l\end{bmatrix}_q \int_{-\infty}^{\infty} x^l M_{0,k}(x;q)d_{1/q}x$$

From equation (4) we obtain

$$S_{\underline{t}}(k+l,k)= q^{-l}\begin{bmatrix}k+l\\ l\end{bmatrix}_q \int_{-\infty}^{\infty} x^l M_{0,k}(x;q)d_{1/q}x \quad (8)$$

Putting $l=n-k$ gives

$$\begin{aligned} S_{\underline{t}}(n,k)&=q^{-(n-k)}\begin{bmatrix}n\\ k\end{bmatrix}_q \int_{-\infty}^{\infty} x^{n-k} M_{0,k}(x;q)d_{1/q}x \\ &= q^{-(n-k)}\begin{bmatrix}n\\ k\end{bmatrix}_q \lambda_{n-k}(k,\underline{t};q)\end{aligned} \qquad (9)$$

Remark: By properties of divided differences

$$S_{\underline{t}}(k+l,k)=\sum_{j=0}^{k}\frac{t_j^{k+1}}{w_k'(t_j)}$$

where $w_k=\prod_{i=0}^{k}(t-t_i)$. From equation (9) we have for $l=0,1,\dots$ and $k=1,2,\dots$

$$\lambda_l(k,\underline{t};q)=q^l\begin{bmatrix}k+l\\ l\end{bmatrix}_q^{-1}\sum_{j=0}^{k}\frac{t_j^{k+1}}{w_k'(t_j)}$$

In particular

$$\lambda_0(k,\underline{t};q)=1$$

$$\lambda_1(k,\underline{t};q)=\frac{q}{[k+1]_q}\sum_{j=0}^{k}t_j$$

$$\lambda_2(k,\underline{t};q)=\frac{q^2[2]_q}{[k+1]_q[k+2]_q}\sum_{0\le j\le m\le k}t_j t_m$$

Proposition 3.2. Let $l=0,1,\dots$ and $k=2,3,\dots$. Then

$$\lambda_l(k,\underline{t};q)=\begin{bmatrix}k+l\\ l\end{bmatrix}_q^{-1}\sum_{j=0}^{l}t_k^{l-j}q^{l-j}\begin{bmatrix}j+k-1\\ k-1\end{bmatrix}_q \lambda_j(k-1,\underline{t};q).$$

Furthermore, for $k=1$ we have

$$\lambda_l(1,\underline{t};q)=\frac{q^l(t_1^{l+1}-t_0^{l+1})}{[l+1]_q(t_1-t_0)}$$

Proof. Recall the recurrence relation

$$S_{\underline{t}}(n,k)=S_{\underline{t}}(n-1,k-1)+t_kS_{\underline{t}}(n-1,k).$$

One can observe that

$$\begin{aligned}S_{\underline{t}}(n,k)&=\sum_{j=k-1}^{n-1}t_j^{n-j-1}S_{\underline{t}}(j,k-1)\\&=\sum_{j=k-1}^{n-1}t_j^{n-j-1}q^{k-j-1}\begin{bmatrix}j\\k-1\end{bmatrix}_q\lambda_{j-k+1}(k-1,\underline{t};q)\\&=\sum_{j=0}^{n-k}t_j^{n-k-j}q^{-j}\begin{bmatrix}j+k-1\\k-1\end{bmatrix}_q\lambda_j(k-1,\underline{t};q)\end{aligned}$$

Hence

$$S_{\underline{t}}(k+l,k)=\sum_{j=0}^{l}t_j^{l-j}q^{-j}\begin{bmatrix}j+k-1\\k-1\end{bmatrix}_q\lambda_j(k-1,\underline{t};q)$$

Using equation (5) yields

$$\lambda_l(k,\underline{t};q)=q^l\begin{bmatrix}k+l\\l\end{bmatrix}_q^{-1}*\sum_{j=0}^{l}t_j^{l-j}q^{-j}\begin{bmatrix}j+k-1\\k-1\end{bmatrix}_q\lambda_j(k-1,\underline{t};q).$$

In equation (9) putting $n=l+1$ and $k=1$ gives

$$S_{\underline{t}}(l+1,k)=q^{-l}[l+1]_q\lambda_l(1,\underline{t};q)$$

that is

$$\lambda_l(1,\underline{t};q)=\frac{q^l}{[l+1]_q}S_{\underline{t}}(l+1,1),\qquad l=0,1,\ldots$$

Since

$$S_{\underline{t}}(n,1)=\frac{t_1^n-t_0^n}{t_1-t_0}$$

we have

$$\lambda_l(1,\underline{t};q)=\frac{q^l(t_1^{l+1}-t_0^{l+1})}{[l+1]_q(t_1-t_0)}$$

Proposition 3.3 Let $0\le\mu\le n.$ Then

$$\sum_{j=1}^{\mu}q^{j-n}\begin{bmatrix}n\\j\end{bmatrix}_q\lambda_{n-j}(j,\underline{t};q)\prod_{l=0}^{j-1}(t_\mu-t_l)=t_\mu^n-t_0^n$$

Proof. Since

$$t^n=\sum_{j=0}^{n}S_{\underline{t}}(n,j)\prod_{l=0}^{j-1}(t-t_l),$$

for $0\le\mu\le n,$ putting $t=t_\mu$ in the last equation gives

$$t_\mu^n=\underbrace{S_{\underline{t}}(n,0)}_{t_0^n}+\sum_{j=1}^{\mu}S_{\underline{t}}(n,j)\prod_{l=0}^{j-1}(t_\mu-t_l)$$

From equation (9)

$$\sum_{j=1}^{\mu}q^{j-n}\begin{bmatrix}n\\j\end{bmatrix}_q\lambda_{n-j}(j,\underline{t};q)\prod_{l=0}^{j-1}(t_\mu-t_l)=t_\mu^n-t_0^n.$$

Theorem 3.4. For arbitrary $k=1,2,\ldots$ and arbitrary $\underline{t}=(t_0,t_1,\ldots,t_k)$, the q-generating function for the numbers $S_{\underline{t}}(.,.)$ is

$$Y_k(t)=\sum_{j=k}^{\infty}S_{\underline{t}}(j,k)q^{j(j-1)/2}\frac{t^j}{[j]_q!}=\sum_{j=0}^{k}\frac{E_q^{t_jt}}{w_k'(t_j)}$$

And also it satisfies the q-difference equation

$$D_{1/q}Y_k(t)-t_kY_k(t)=Y_{k-1}(t).$$

Proof. From Taylor expansion (see [8])

$$f(x)=\sum_{j=0}^{\infty}q^{j(j-1)/2}\frac{(D_{1/q}^jf)(0)}{[j]_q!}x^j.$$

Applying divided difference of order k with nodes t_j for $j=0,1,\ldots,k$ such that $t_0<\cdots<t_k$ to the both sides gives

$$[t_0,\ldots,t_k]f=\sum_{j=0}^{\infty}S_{\underline{t}}(j,k)q^{j(j-1)/2}\frac{(D_{1/q}^jf)(0)}{[j]_q!}\tag{10}$$

Let $f(x)=E_q^{xt}$. Hence $(D_{1/q}^jf)(0)=t^j$. Putting this in equation (10) gives

$$\begin{aligned}\sum_{j=0}^{k}\frac{E_q^{t_jt}}{w_k'(t_j)}&=\sum_{j=0}^{\infty}S_{\underline{t}}(j,k)q^{j(j-1)/2}\frac{(D_{1/q}^jf)(0)}{[j]_q!}\\&=\sum_{j=k}^{\infty}S_{\underline{t}}(j,k)q^{j(j-1)/2}\frac{t^j}{[j]_q!}\end{aligned}$$

The last equation follows from $S_{\underline{t}}(j,k)=0$ for $j<k$.

Hence the q-exponential generating function for the numbers $S_{\underline{t}}(.,.)$ is given by

$$Y_k(t)=\sum_{j=k}^{\infty} S_{\underline{t}}(j,k)q^{j(j-1)/2}\frac{t^j}{[j]_q!}=\sum_{j=0}^{k}\frac{E_q^{t_j t}}{w_k^{'}(t_j)}.$$

It can be easily shown that Y_k satisfies the following q-difference equation

$$D_{1/q}Y_k(t)-t_kY_k(t)=Y_{k-1}(t).$$

To show that this equation is valid, we use

$$D_{1/q}E_q^{t_j t}=t_jE_q^{t_j t}. \tag{11}$$

Since

$$Y_k(t)=\sum_{j=0}^{k}\frac{E_q^{t_j t}}{w_k^{'}(t_j)},$$

using equation (11) yields

$$D_{1/q}Y_k(t)=\sum_{j=0}^{k}\frac{t_jE_q^{t_j t}}{w_k^{'}(t_j)}.$$

Hence

$$D_{1/q}Y_k(t)-t_kY_k(t)=\sum_{j=0}^{k}t_j\frac{E_q^{t_j t}}{w_k^{'}(t_j)}-t_k\sum_{j=0}^{k}\frac{E_q^{t_j t}}{w_k^{'}(t_j)}$$

$$=\sum_{j=0}^{k-1}(t_j-t_k)\frac{E_q^{t_j t}}{w_k^{'}(t_j)}$$

$$=\sum_{j=0}^{k-1}\frac{E_q^{t_j t}}{w_{k-1}^{'}(t_j)}$$

$$=Y_{k-1}(t)$$

4. Discussion and Conclusion

We give identities on extended Stirling numbers and q-B-splines. Finally, we derive a q-generating function for extended Stirling numbers and give a q-difference equation for this function. For future work, we continue with finding generating function for $\lambda_q(k,\underline{t};q)$. Then, we will study on the relation between Bell numbers, Eulerian numbers and q-B-splines.

References

[1] Carlitz, L. 1948. q-Bernoulli numbers and polynomials, Duke Math. J. Vol. 15 , no. 4, page 987–1000

[2] Wachs, M., White D. 1991. *p;q*-Stirling numbers and set partition statistics, Journal of Combinatorial Theory, Series A, 56 (1), 27-46

[3] Broder A. 1984. The *r*-Stirling numbers, Discrete Math., 49, 241-259

[4] Duran U., Acikgoz M., Araci S. 2018. On *(q;r;w)*-Stirling numbers of the second kind, Journal of Inequalities and Special Functions, 9 (1), 9-16

[5] Duran, U.; Acikgoz, M. 2020. On Degenerate Truncated Special Polynomials. Mathematics, 8, 144.

[6] Neuman E. 1981. Moments and Fourier transforms of B-splines, Journal of Computational and Applied Mathematics, 7(1), 51-62

[7] Simeonov P., Goldman R. 2013. Quantum B-splines, BIT Numerical Mathematics, Vol. 53, pp. 193-223

[8] Budakçı G., Dişibuyuk Ç., Goldman R., Oruc¸ H. 2015. Extending Fundamental Formulas from Classical B-Splines to Quantum B-Splines, Journal of Computational and Applied Mathematics, 282, 17-33

[9] Kac V., Cheung P. 2002. Quantum Calculus. Universitext Series, IX, Springer Verlag

[10] Curry, H.B., Schoenberg, I.J. 1966. On Polya frequency functions IV: The fundamental spline functions and their limits, J. Anal. Math. 17, 71-107