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On The Connections Between Jacobsthal Numbers and Fibonacci p-Numbers

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Abstract. In this paper, we define the Fibonacci-Jacobsthal p-sequence and then we discuss the connection between of the Fibonacci-Jacobsthal p-sequence with the Jacobsthal and Fibonacci p-sequences. Also, we provide a new Binet formula and a new combinatorial representation of the Fibonacci-Jacobsthal p-numbers by the aid of the nth power of the generating matrix of the Fibonacci-Jacobsthal p-sequence. Furthermore, we derive some properties of the Fibonacci-Jacobsthal p-sequences such as the exponential, permanental, determinantal representations and the sums by using its generating matrix.

1. Introduction

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$ in which $J_0 = 0$ and $J_1 = 1$.

There are many important generalizations of the Fibonacci sequence. The Fibonacci p-sequence $\left\{F_p(n)\right\}$ (see detailed information in [21, 22]) is one of them:

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

for $n > p$ and $p = 1, 2, 3, \ldots$, in which $F_p(0) = 0$, $F_p(1) = \cdots F_p(p) = 1$. When $p = 1$, the Fibonacci p-sequence $\left\{F_p(n)\right\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$.

It is easy to see that the characteristic polynomials of Jacobsthal sequence and Fibonacci p-sequence are $g_1(x) = x^2 - x - 2$ and $g_2(x) = x^{p+1} - x^p - 1$, respectively. We will use these in the next section.

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

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Let the matrix A be defined by

$$A = \left[a_{i,j}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Several authors have used homogeneous linear recurrence relations to deduce miscellaneous properties for a plethora of sequences: see for example, [1, 4, 8–11, 19, 20]. In [5–7, 14–16, 21–23], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we discuss connections between the Jacobsthal numbers and Fibonacci p-numbers. Firstly, we define the Fibonacci-Jacobsthal p-sequence and then we study recurrence relation among this sequence, Jacobsthal sequence and Fibonacci p-sequence. Also, we give the relations between the generating matrix of the Fibonacci-Jacobsthal p-numbers and the elements of Jacobsthal sequence and Fibonacci p-sequence. Furthermore, using the generating matrix the Fibonacci-Jacobsthal p-sequence, we obtain some new structural properties of the Fibonacci p-numbers such as the Binet formula and combinatorial representations. Finally, we derive the exponential, permanental, and determinantal representations and the sums of Fibonacci-Jacobsthal p-sequences.

2. On The Connections Between Jacobsthal Numbers and Fibonacci p-Numbers

Now we define the Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(3, 4, 5, \ldots)$ and $n \geq 0$

$$F_{n+p+3}^{J,p} = 2F_{n+p+2}^{J,p} + F_{n+p+1}^{J,p} - 2F_{n+p}^{J,p} + F_{n+2}^{J,p} - F_{n+1}^{J,p} - 2F_n^{J,p} \tag{1}$$

in which $F_0^{J,p} = \cdots = F_{p+1}^{J,p} = 0$ and $F_{p+2}^{J,p} = 1$.

First, we consider the relationship between the Fibonacci-Jacobsthal p-sequence which is defined above, Jacobsthal sequence, and Fibonacci p-sequences.

Theorem 2.1. *Let J_n, $F_p(n)$ and $F_n^{J,p}$ be the nth Jacobsthal number, Fibonacci p-number, and Fibonacci-Jacobsthal p-numbers, respectively. Then,*

$$J_n + F_p(n+1) = F_{n+p+2}^{J,p} - 3F_{n+p}^{J,p} - F_n^{J,p}$$

for $n \geq 0$ and $p \geq 3$.

Proof. The assertion may be proved by induction on n. It is clear that $J_0 + F_p(1) = F_{p+2}^{J,p} - 3F_p^{J,p} - F_0^{J,p} = 0$. Suppose that the equation holds for $n \geq 1$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$, is

$$h(x) = x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2$$

and

$$h(x) = g_1(x)\, g_2(x),$$

where $g_1(x)$ and $g_2(x)$ are the characteristic polynomials of Jacobsthal sequence and Fibonacci p-sequence, respectively, we obtain the following relations:

$$J_{n+p+3} = 2J_{n+p+2} + J_{n+p+1} - 2J_{n+p} + J_{n+2} - J_{n+1} - 2J_n$$

and

$$F_p(n+p+3) = 2F_p(n+p+2) + F_p(n+p+1) - 2F_p(n+p) + F_p(n+2) - F_p(n+1) - 2F_p(n)$$

for $n \geq 1$. Thus, by a simple calculation, we have the conclusion. □

Theorem 2.2. *Let J_n and $F_n^{J,p}$ be the nth Jacobsthal number and Fibonacci-Jacobsthal p-numbers. Then,*

i.

$$J_n = F_{n+p+1}^{J,p} - F_{n+p}^{J,p} - F_n^{J,p},$$

ii.

$$J_n + J_{n+1} = F_{n+p+2}^{J,p} - F_{n+p}^{J,p} - F_{n+1}^{J,p} - F_n^{J,p}$$

for $n \geq 0$ and $p \geq 3$.

Proof. Consider the case ii. The assertion may be proved by induction on n. It is clear that $J_0 + J_1 = F_5^{J,p} - F_3^{J,p} - F_1^{J,p} - F_0^{J,p} = 1$. Now we assume that the equation holds for $n > 0$. Then we show that the equation holds for $n+1$. Since the characteristic polynomial of Jacobsthal sequence $\{J_n\}$, is

$$g_1(x) = x^2 - x - 2$$

we obtain the following relations:

$$J_{n+p+3} = 2J_{n+p+2} + J_{n+p+1} - 2J_{n+p} + J_{n+2} - J_{n+1} - 2J_n$$

for $n \geq 1$. Thus, by a simple calculation, we have the conclusion.

There is a similar proof for i. □

By the recurrence relation (1), we have

$$\begin{bmatrix} F_{n+p+2}^{J,p} \\ F_{n+p+1}^{J,p} \\ F_{n+p}^{J,p} \\ \vdots \\ F_n^{J,p} \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 0 & \cdots & 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{n+p+3}^{J,p} \\ F_{n+p+2}^{J,p} \\ F_{n+p+1}^{J,p} \\ \vdots \\ F_{n+1}^{J,p} \end{bmatrix}$$

for the Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$. Letting

$$M_p = \begin{bmatrix} 2 & 1 & -2 & 0 & \cdots & 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+3)\times(p+3)}.$$

The companion matrix $M_p = \left[m_{i,j}\right]_{(p+3)\times(p+3)}$ is said to be the Fibonacci-Jacobsthal p-matrix. For detailed information about the companion matrices, see [17, 18]. It can be readily established by mathematical induction that for $p \geq 3$ and $\alpha \geq 2p$

$$\left(M_p\right)^\alpha = \begin{bmatrix} F_{\alpha+p+2}^{J,p} & F_{\alpha+p+3}^{J,p} - 2F_{\alpha+p+2}^{J,p} & F_p(\alpha-p+2) - 2F_{\alpha+p+1}^{J,p} & F_p(\alpha-p+3) & \cdots & \\ F_{\alpha+p+1}^{J,p} & F_{\alpha+p+2}^{J,p} - 2F_{\alpha+p+1}^{J,p} & F_p(\alpha-p+1) - 2F_{\alpha+p}^{J,p} & F_p(\alpha-p+2) & \cdots & \\ F_{\alpha+p}^{J,p} & F_{\alpha+p+1}^{J,p} - 2F_{\alpha+p}^{J,p} & F_p(\alpha-p) - 2F_{\alpha+p-1}^{J,p} & F_p(\alpha-p+1) & \cdots & M_p^* \\ \vdots & \vdots & \vdots & \vdots & & \\ F_{\alpha+1}^{J,p} & F_{\alpha+2}^{J,p} - 2F_{\alpha+1}^{J,p} & F_p(\alpha-2p+1) - 2F_{\alpha}^{J,p} & F_p(\alpha-2p+2) & \cdots & \\ F_{\alpha}^{J,p} & F_{\alpha+1}^{J,p} - 2F_{\alpha}^{J,p} & F_p(\alpha-2p) - 2F_{\alpha-1}^{J,p} & F_p(\alpha-2p+1) & \cdots & \end{bmatrix},$$

where

$$M_p^* = \begin{bmatrix} F_p(\alpha) & F_p(\alpha+1) - F_{\alpha+p+2}^{J,p} & -2F_{\alpha+p+1}^{J,p} \\ F_p(\alpha-1) & F_p(\alpha) - F_{\alpha+p+1}^{J,p} & -2F_{\alpha+p}^{J,p} \\ F_p(\alpha-2) & F_p(\alpha-1) - F_{\alpha+p}^{J,p} & -2F_{\alpha+p-1}^{J,p} \\ \vdots & \vdots & \vdots \\ F_p(\alpha-p-1) & F_p(\alpha-p) - F_{\alpha+1}^{J,p} & -2F_{\alpha}^{J,p} \\ F_p(\alpha-p-2) & F_p(\alpha-p-1) - F_{\alpha}^{J,p} & -2F_{\alpha-1}^{J,p} \end{bmatrix}.$$

We easily derive that $\det M_p = (-1)^{p+1} \cdot 2$. In [21], Stakhov defined the generalized Fibonacci p-matrix Q_p and derived the *nth* power of the matrix Q_p. In [13], Kılıc gave a Binet formula for the Fibonacci p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Fibonacci-Jacobsthal p-numbers by the aid of the matrix $\left(M_p\right)^\alpha$.

Lemma 2.3. *The characteristic equation of all the Fibonacci-Jacobsthal p-numbers* $x^{p+3}-2x^{p+2}-x^{p+1}+x^p-x^2+x+2 = 0$ *does not have multiple roots for* $p \geq 3$.

Proof. It is clear that $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = \left(x^{p+1} - x^p - 1\right)\left(x^2 - x - 2\right)$. In [13], it was shown that the equation $x^{p+1} - x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation $x^2 - x - 2 = 0$ are 2 and −1. Since $(2)^{p+1} - (2)^p - 1 \neq 0$ and $(-1)^{p+1} - (-1)^p - 1 \neq 0$ for $p > 1$, the equation $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$ does not have multiple roots for $p \geq 3$. □

Let $h(x)$ be the characteristic polynomial of matrix M_p. Then we have $h(x) = x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2$, which is a well-known fact from the companion matrices. If $\lambda_1, \lambda_2, \ldots, \lambda_{p+3}$ are roots of the equation

$x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$, then by Lemma 2.3, it is known that $\lambda_1, \lambda_2, \ldots, \lambda_{p+3}$ are distinct. Define the $(p+3) \times (p+3)$ Vandermonde matrix V_p as follows:

$$V_p = \begin{bmatrix} (\lambda_1)^{p+2} & (\lambda_2)^{p+2} & \ldots & \left(\lambda_{p+3}\right)^{p+2} \\ (\lambda_1)^{p+1} & (\lambda_2)^{p+1} & \ldots & \left(\lambda_{p+3}\right)^{p+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \ldots & \lambda_{p+3} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Assume that $V_p(i, j)$ is a $(p+3) \times (p+3)$ matrix derived from the Vandermonde matrix V_p by replacing the j^{th} column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+3) \times 1$ matrix as follows:

$$W_p(i) = \begin{bmatrix} (\lambda_1)^{\alpha+p+3-i} \\ (\lambda_2)^{\alpha+p+3-i} \\ \vdots \\ \left(\lambda_{p+3}\right)^{\alpha+p+3-i} \end{bmatrix}.$$

Theorem 2.4. *Let p be a positive integer such that $p \geq 3$ and let $\left(M_p\right)^\alpha = m_{i,j}^{(p,\alpha)}$ for $\alpha \geq 1$, then*

$$m_{i,j}^{(p,\alpha)} = \frac{\det V_p(i, j)}{\det V_p}.$$

Proof. Since the equation $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$ does not have multiple roots for $p \geq 3$, the eigenvalues of the Fibonacci-Jacobsthal p-matrix M_p are distinct. Then, it is clear that M_p is diagonalizable. Let $D_p = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_{p+3}\right)$, then we may write $M_p V_p = V_p D_p$. Since the matrix V_p is invertible, we obtain the equation $\left(V_p\right)^{-1} M_p V_p = D_p$. Therefore, M_p is similar to D_p; hence, $\left(M_p\right)^\alpha V_p = V_p \left(D_p\right)^\alpha$ for $\alpha \geq 1$. So we have the following linear system of equations:

$$\begin{cases} m_{i,1}^{(p,\alpha)} (\lambda_1)^{p+2} + m_{i,2}^{(p,\alpha)} (\lambda_1)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = (\lambda_1)^{\alpha+p+3-i} \\ m_{i,1}^{(p,\alpha)} (\lambda_2)^{p+2} + m_{i,2}^{(p,\alpha)} (\lambda_2)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = (\lambda_2)^{\alpha+p+3-i} \\ \vdots \\ m_{i,1}^{(p,\alpha)} \left(\lambda_{p+3}\right)^{p+2} + m_{i,2}^{(p,\alpha)} \left(\lambda_{p+3}\right)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = \left(\lambda_{p+3}\right)^{\alpha+p+3-i} \end{cases}.$$

Then we conclude that

$$m_{i,j}^{(p,\alpha)} = \frac{\det V_p(i, j)}{\det V_p}$$

for each $i, j = 1, 2, \ldots, p+3$. □

Thus by Theorem 2.4 and the matrix $\left(M_p\right)^\alpha$, we have the following useful result for the Fibonacci-Jacobsthal p-numbers.

Corollary 2.5. *Let p be a positive integer such that $p \geq 3$ and let $F_n^{J,p}$ be the nth element of Fibonacci-Jacobsthal p-sequence, then*

$$F_n^{J,p} = \frac{\det V_p(p+3, 1)}{\det V_p}$$

and

$$F_n^{J,p} = -\frac{\det V_p\,(p+2, p+3)}{2 \cdot \det V_p}$$

for $n \geq 1$.

It is easy to see that the generating function of Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$ is as follows:

$$g(x) = \frac{x^{p+2}}{1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}},$$

where $p \geq 3$.

Then we can give an exponential representation for the Fibonacci-Jacobsthal p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.6. *The Fibonacci-Jacobsthal p-sequence* $\left\{F_n^{J,p}\right\}$ *have the following exponential representation:*

$$g(x) = x^{p+2} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+2} - \ln\left(1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}\right)$$

and

$$\begin{aligned} -\ln\left(1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}\right) \;=\;& -[-x\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right) - \\ & \frac{1}{2}x^2\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^2 - \cdots \\ & -\frac{1}{i}x^i\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+2} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.7. *(Chen and Louck [3]) The* (i, j) *entry* $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula:*

$$k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{2}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (2) are defined to be* 1 *if* $n = i - j$.

Then we can give other combinatorial representations than for the Fibonacci-Jacobsthal p-numbers by the following Corollary.

Corollary 2.8. *Let $F_n^{J,p}$ be the nth Fibonacci-Jacobsthal p-number for $n \geq 1$. Then*

i.

$$F_n^{J,p} = \sum_{(t_1,t_2,\ldots,t_{p+3})} \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\cdots,t_{p+3}} 2^{t_1}(-1)^{t_{p+2}}(-2)^{t_3+t_{p+3}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+3)\,t_{p+3} = n-p-2$.

ii.

$$F_n^{J,p} = -\sum_{(t_1,t_2,\ldots,t_{p+3})} \frac{t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\cdots,t_{p+3}} 2^{t_1}(-1)^{t_{p+2}}(-2)^{t_3+t_{p+3}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+3)\,t_{p+3} = n+1$.

Proof. If we take $i = p+3$, $j = 1$ for the case i. and $i = p+2$, $j = p+3$ for the case ii. in Theorem 2.7, then we can directly see the conclusions from $\left(M_p\right)^{\alpha}$. □

Now we consider the relationship between the Fibonacci-Jacobsthal p-numbers and the permanent of a certain matrix which is obtained using the Fibonacci-Jacobsthal p-matrix $\left(M_p\right)^{\alpha}$.

Definition 2.9. *A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Fibonacci-Jacobsthal p-numbers and the permanents of certain matrices which are obtained by using the generating matrix of Fibonacci-Jacobsthal p-numbers. Let $K_{m,p}^{F,J} = \left[k_{i,j}^{(p)}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$k_{i,j}^{(p)} = \begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1\leq\tau\leq m, \\ 1 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\leq\tau\leq m-1, \\ i=\tau \text{ and } j=\tau+p \text{ for } 1\leq\tau\leq m-p \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1\leq\tau\leq m-1,\end{array} \\ -1 & \text{if } i=\tau \text{ and } j=\tau+p+1 \text{ for } 1\leq\tau\leq m-p-1, \\ -2 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\leq\tau\leq m-2 \\ \text{and} \\ i=\tau \text{ and } j=\tau+p+2 \text{ for } 1\leq\tau\leq m-p-2,\end{array} \\ 0 & \text{otherwise.} \end{cases}, \text{ for } m \geq p+3.$$

Then we have the following Theorem.

Theorem 2.10. *For $m \geq p+3$,*

$$perK_{m,p}^{F,J} = F_{m+p+2}^{J,p}.$$

Proof. Let us consider matrix $K^{F,J}_{m,p}$ and let the equation be hold for $m \geq p+3$. Then we show that the equation holds for $m+1$. If we expand the $perK^{F,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perK^{F,J}_{m+1,p} = 2perK^{F,J}_{m,p} + perK^{F,J}_{m-1,p} - 2perK^{F,J}_{m-2,p} + perK^{F,J}_{m-p,p} - perK^{F,J}_{m-p-1,p} - 2perK^{F,J}_{m-p-2,p}.$$

Since

$$\begin{aligned} perK^{F,J}_{m,p} &= F^{J,p}_{m+p+2}, \\ perK^{F,J}_{m-1,p} &= F^{J,p}_{m+p+1}, \\ perK^{F,J}_{m-2,p} &= F^{J,p}_{m+p}, \\ perK^{F,J}_{m-p,p} &= F^{J,p}_{m+2}, \\ perK^{F,J}_{m-p-1,p} &= F^{J,p}_{m+1} \end{aligned}$$

and

$$perK^{F,J}_{m-p-2,p} = F^{J,p}_{m},$$

we easily obtain that $perK^{F,J}_{m+1,p} = F^{J,p}_{m+p+3}$. So the proof is complete. □

Let $L^{F,J}_{m,p} = \left[l^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$l^{(p)}_{i,j} = \begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-3, \\ & \text{if } i=\tau \text{ and } j=\tau \text{ for } m-2 \leq \tau \leq m, \\ & i=\tau \text{ and } j=\tau+1 \text{ for } 1 \leq \tau \leq m-1, \\ 1 & i=\tau \text{ and } j=\tau+p \text{ for } 1 \leq \tau \leq m-p-2 \\ & \text{and} \\ & i=\tau+1 \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-4, \\ -1 & \text{if } i=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \leq \tau \leq m-p-1, \\ & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1 \leq \tau \leq m-3 \\ -2 & \text{and} \\ & i=\tau \text{ and } j=\tau+p+2 \text{ for } 1 \leq \tau \leq m-p-2, \\ 0 & \text{otherwise.} \end{cases} \quad , \text{ for } m \geq p+3.$$

Then we have the following Theorem.

Theorem 2.11. *For $m \geq p+3$,*

$$perL^{F,J}_{m,p} = F^{J,p}_{m+p-1}.$$

Proof. Let us consider matrix $L^{F,J}_{m,p}$ and let the equation be hold for $m \geq p+3$. Then we show that the equation holds for $m+1$. If we expand the $perL^{F,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perL^{F,J}_{m+1,p} = 2perL^{F,J}_{m,p} + perL^{F,J}_{m-1,p} - 2perL^{F,J}_{m-2,p} + perL^{F,J}_{m-p,p} - perL^{F,J}_{m-p-1,p} - 2perL^{F,J}_{m-p-2,p}.$$

Since

$$\begin{aligned} perL^{F,J}_{m,p} &= F^{J,p}_{m+p-1}, \\ perL^{F,J}_{m-1,p} &= F^{J,p}_{m+p-2}, \end{aligned}$$

$$perL_{m-2,p}^{F,J} = F_{m+p-3}^{J,p},$$
$$perL_{m-p,p}^{F,J} = F_{m-1}^{J,p},$$
$$perL_{m-p-1,p}^{F,J} = F_{m-2}^{J,p}$$

and

$$perL_{m-p-2,p}^{F,J} = F_{m-3}^{J,p},$$

we easily obtain that $perL_{m+1,p}^{F,J} = F_{m+p}^{J,p}$. So the proof is complete. □

Assume that $N_{m,p}^{F,J} = \left[n_{i,j}^{(p)}\right]$ be the $m \times m$ matrix, defined by

$$N_{m,p}^{F,J} = \begin{bmatrix} & & (m-3)\text{ th} & & & \\ & & \downarrow & & & \\ 1 & \cdots & 1 & 0 & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & L_{m-1,p}^{F,J} & & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}, \text{ for } m > p+3,$$

then we have the following results:

Theorem 2.12. *For $m > p + 3$,*

$$perN_{m,p}^{F,J} = \sum_{i=0}^{m+p-2} F_i^{J,p}.$$

Proof. If we extend $per\, N_{m,p}^{F,J}$ with respect to the first row, we write

$$perN_{m,p}^{F,J} = perN_{m-1,p}^{F,J} + perL_{m-1,p}^{F,J}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1, -1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Fibonacci-Jacobsthal p-numbers and the determinants of certain matrices which are obtained by using the matrix $K_{m,p}^{F,J}$, $L_{m,p}^{F,J}$ and $N_{m,p}^{F,J}$. Let $m > p + 3$ and let H be the $m \times m$ matrix, defined by

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.13. *For $m > p + 3$,*

$$\det\left(K_{m,p}^{F,J} \circ H\right) = F_{m+p+2}^{J,p},$$
$$\det\left(L_{m,p}^{F,J} \circ H\right) = F_{m+p-1}^{J,p},$$

and

$$\det\left(N_{m,p}^{F,J} \circ H\right) = \sum_{i=0}^{m+p-2} F_i^{J,p}.$$

Proof. *Since* $perK_{m,p}^{F,J} = \det\left(K_{m,p}^{F,J} \circ H\right)$, $perL_{m,p}^{F,J} = \det\left(L_{m,p}^{F,J} \circ H\right)$ *and* $perN_{m,p}^{F,J} = \det\left(N_{m,p}^{F,J} \circ H\right)$ *for* $m > p+3$, *by Theorem 2.10, Theorem 2.11 and Theorem 2.12, we have the conclusion.* □

Now we consider the sums of the Fibonacci-Jacobsthal p-numbers. Let

$$S_\alpha = \sum_{u=0}^{\alpha} F_u^{J,p}$$

for $\alpha > 1$ and $p \geq 3$, and let $T_p^{F,J}$ and $\left(T_p^{F,J}\right)^\alpha$ be the $(p+4)\times(p+4)$ matrix such that

$$T_p^{F,J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & M_p & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

If we use induction on α, then we obtain

$$\left(T_p^{F,J}\right)^\alpha = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ S_{\alpha+p+1} & & & & & \\ S_{\alpha+p} & & & & & \\ \vdots & & & \left(M_p\right)^\alpha & & \\ S_\alpha & & & & & \\ S_{\alpha-1} & & & & & \end{bmatrix}.$$

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