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Research Article

Existence and Multiplicity of Periodic Solutions for Nonautonomous Second-Order Discrete Hamiltonian Systems

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ABSTRACT. In this paper, we consider the periodic solutions of the following non-autonomous second order discrete system

$$\Delta^2 u(n-1) = \nabla F(n, u(n)), \quad n \in \mathbb{Z}.$$

When the nonlinear function $F(n, x)$ is like-quadratic for x, we obtain some existence and multiplicity results under twisting conditions by using the least action principle and a multiple critical point theorem.

Keywords: Periodic solution, second-order discrete Hamiltonian system, the least action principle, critical point theory.

2020 Mathematics Subject Classification: 35R02, 58J05.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the periodic solutions of the following nonautonomous second order discrete Hamiltonian system

$$\Delta^2 u(n-1) = \nabla F(n, u(n)),\ u(n) \in \mathbb{R}^N,\ n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta u(n) = u(n+1) - u(n), \Delta^2 u(n) = \Delta(\Delta u(n))$ and $\nabla F(n, x)$ denotes the gradient of the function F with respect to the second variable x. F satisfies the following condition:

$$(A) \quad \begin{array}{l} F(n, \cdot) \in C^1(\mathbb{R}^N, \mathbb{R}), \forall n \in \mathbb{Z}; \\ F(n+T, x) = F(n, x), \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^N, T \in \mathbb{Z} \text{ and } T \geq 2. \end{array}$$

Historically, in 2003, Guo and Yu, first considered the existence of periodic solutions of difference equations as (1.1) via variational method and critical point theory in three papers [2, 3, 4]. In 2004, Zhou, Yu and Guo [11], further studied the existence and multiplicity of periodic solutions of the discrete Hamiltonian system (1.1). After that, the existence and multiplicity of periodic solutions for system (1.1) have been extensively studied and many interesting results were obtained. We refer the readers to [5, 6, 8, 9, 10] and the references therein for these topics. Among them, we should mention some work which have relation with our work of this paper. For the condition on F, Guo and Yu in [3], first required the nonlinearity $\nabla F(n, x)$ is sub-linear included the bounded case. We say that the nonlinearity $\nabla F(n, x)$ is growing sublinearly if there exist $M_1 > 0$, $M_2 > 0$ and $\alpha \in [0, 1)$ such that

$$|\nabla F(t, x)| \leq M_1 |x|^\alpha + M_2, \quad \forall (n, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N, \tag{1.2}$$

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where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for all $a,b \in \mathbb{Z}$ with $a \le b$. Xue and Tang in [9] used the least action principle to verified that system (1.1) possesses at least one T-periodic solution with the assumption of

$$\lim_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = +\infty. \tag{1.3}$$

We also refer the paper [3] for this topic. In the case of $\alpha = 1$, assumption (1.2) becomes to the following assumption in which the nonlinearity $\nabla F(n,x)$ does not exceed linear growth, that is, there are $M_3 > 0$ and $M_4 \ge 0$, such that

$$|\nabla F(n,x)| \le M_3|x| + M_4, \quad \forall (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N. \tag{1.4}$$

The case $\nabla F(n,x) = Ax + B$ satisfies the condition (1.4). It is well known that in this case the system (1.1) in general does not possess a solution. A twisting condition is required to avoid this case. Considering the nonlinearity $\nabla F(n,x)$ which is the sum of assumption (1.2) and (1.4), Hu [6] also used the least action principle to verify that system (1.1) possesses at least one T-periodic solution under a twisting condition which is included in the following case

$$\lim_{|x|\to+\infty} |x|^{-2} \sum_{n=1}^{T} F(n,x) > -\infty. \tag{1.5}$$

When the nonlinearity $\nabla F(n,x)$ meets the following assumption that there are $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ and $\alpha \in (0,1)$, such that

$$|\nabla F(t,x)| \le f(n)|x|^\alpha + g(n), \quad \forall\, (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \tag{1.6}$$

or

$$|\nabla F(t,x)| \le f(n)|x| + g(n), \quad \forall\, (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N. \tag{1.7}$$

Tang and Zhang [8] obtained some existence results for the T-periodic solutions of system (1.1) under some different twisting conditions.

In this paper, we will further study the existence and multiplicity of T-periodic solutions of (1.5) with some different twisting conditions. The following are our main results.

Theorem 1.1. *Suppose that $F(n,x) = F_1(n,x) + F_2(n,x)$ with F_1 and F_2 satisfying the conditions (A) and the following three growing conditions:*
(B_1) There exist $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ and $\alpha \in [0,1)$, such that

$$|\nabla F_1(n,x)| \le f(n)|x|^\alpha + g(n).$$

(B_2) $F_2(n,x)$ satisfies condition (1.4), i.e., there exist constants $M_3, M_4 \in \mathbb{R}^+$ such that

$$|\nabla F_2(n,x)| \le M_3|x| + M_4,\ M_3 < \lambda_1 := \lambda_1 = 2 - 2\cos\frac{2\pi}{T}.$$

(B_3) F satisfies that

$$\lim_{|x|\to+\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) > \frac{M_3^2 T}{2(\lambda_1 - M_3)}.$$

Then, system (1.1) possesses at least one T-periodic solution that minimizes the functional φ given by

$$\varphi(u) = \frac{1}{2}\sum_{n=1}^{T} |\Delta u(n)|^2 + \sum_{n=1}^{T} F(n,u(n)) \tag{1.8}$$

in the Hilbert space H_T defined by

$$H_T = \{u : \mathbb{Z} \to \mathbb{R}^N | u(n+T) = u(n),\ n \in \mathbb{Z}\}.$$

Remark 1.1. *The condition (B_3) is twisted with the conditions (B_1) and (B_2). Our condition (B_3) in Theorem 1.1 is different form the condition (A_4) in Theorem 1 of [6]. In Theorem 1.1, when $F_1(n,x) \equiv 0$, comparing with Theorem 1.3 of [8], the condition (B_3) in some sense is loose for some choices of T, one can check it for $T = 2,3,4,5,6$ and so on.*

Theorem 1.2. *Suppose $F(n,x) = F_1(n,x) + F_2(n,x)$ satisfying condition (A) with F_1 and F_2 satisfying the following three growing conditions:*
(B_4) F_1 satisfies the condition (1.7), i.e., there exist $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ such that

$$|\nabla F_1(n,x)| \le f(n)|x| + g(n),\ \sum_{n=1}^{T} f(n) < \lambda_1.$$

(B_5) F_2 satisfies the condition (1.2), i.e., there are some constants $M_1, M_2 \in \mathbb{R}^+$ and $\alpha \in [0,1)$ such that

$$|\nabla F_2(n,x)| \le M_1|x|^\alpha + M_2.$$

(B_6) F satisfies that

$$\lim_{|x|\to+\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) > \frac{1}{2(\lambda_1 - \sum_{n=1}^{T} f(n))} \Big(\sum_{n=1}^{T} f(n)\Big)^2.$$

Then, system (1.1) possesses at least one T-periodic solution.

Remark 1.2. *Conditions of (B_4) and (B_6) in Theorem 1.2 are different form conditions of (A_3) and (A_4) in Theorem 1 of [6], respectively. Condition (B_6) in Theorem 1.2 is different form condition of (1.14) in Theorem 1.3 of [8].*

Theorem 1.3. *Suppose that $F(n,x)$ satisfies (A), (B_1), (B_2), (B_3) and*
(A_1) there are some constants $\delta > 0$, $k \in \mathbb{Z}[0, [\frac{T}{2} - 1]]$ such that

$$-\frac{1}{2}\lambda_{k+1}|x|^2 \le F(n,x) \le -\frac{1}{2}\lambda_k|x|^2,$$

$\forall\, x \in \mathbb{R}^N$ with $|x| < \delta$ and $\forall n \in [1,T]$, where $\lambda_k = 2 - 2\cos k\omega$, $\omega = \frac{2\pi}{T}$, $T > 2$, $[a] = \max\{k \in \mathbb{Z} | k \le a\}$ denotes the Gauss Function. Then, system (1.1) has at least two T-periodic solutions.

Parallelly, we have the following result.

Theorem 1.4. *Suppose that $F(n,x)$ satisfies (A), (B_4), (B_5), (B_6) and (A_1). Then, system (1.1) has at least two T-periodic solutions.*

2. Some important lemmas

$H_T := \{u : \mathbb{Z} \to \mathbb{R}^N | u(n+T) = u(n),\ n \in \mathbb{Z}\}$ can be equipped with the inner product

$$\langle u, v\rangle = \sum_{n=1}^{T} \big(u(n), v(n)\big), \quad \forall u, v \in H_T, \tag{2.9}$$

so the norm $\|\cdot\|$ is

$$\|u\| = \Big(\sum_{n=1}^{T} |u(n)|^2\Big)^{\frac{1}{2}}, \quad \forall u \in H_T, \tag{2.10}$$

where $(\cdot,\cdot)$ and $|.|$ denotes the usual inner product and the usual norm in $\mathbb{R}^N$, respectively. It is easy to verify that $(H_T, \langle\cdot,\cdot\rangle)$ is a finite dimensional Hilbert space and linear homeomorphic to $\mathbb{R}^{NT}$.

For every positive number $r > 1$, we can equip H_T with another norm $\|u\|_r$, where

$$\|u\|_r = \Big(\sum_{n=1}^{T} |u(n)|^r\Big)^{\frac{1}{r}}, \qquad \forall u \in H_T.$$

Distinctly, $\|u\|_2 = \|u\|$ and $(H_T, \|u\|_2)$ is equivalent to $(H_T, \|u\|_r)$ for $r > 1$. Thus, there are two constants $C_2 \geq C_1 > 0$, such that $\forall u \in H_T$

$$C_1\|u\|_r \leq \|u\| \leq C_2\|u\|_r. \tag{2.11}$$

For system (1.1), Xue and Tang [10] verify that the problem of seeking T-periodic solutions is equal to that of finding the critical points of $\varphi(u)$ defined in (1.8) on H_T.

To prove our results, we now give four useful lemmas.

Lemma 2.1. ([10]) *As a subspace of H_T, N_k is defined by:*

$$N_k := \{u \in H_T | -\Delta^2 u(n-1) = \lambda_k u(n)\},$$

where $\lambda_k = 2 - 2\cos k\omega$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{Z}[0, [\frac{T}{2}]]$. The following statements hold:
(i) $N_k \perp N_j$, $k \neq j$, $k, j \in \mathbb{Z}[0, [\frac{T}{2}]]$,
(ii) $H_T = \bigoplus_{k=0}^{[\frac{T}{2}]} N_k$.

Lemma 2.2 ([10]). *Define $H_k := \bigoplus_{j=0}^{k} N_j$, $H_k^{\perp} := \bigoplus_{j=k+1}^{[T/2]} N_j$, $k \in \mathbb{Z}[0, [T/2]-1]$, then one has*

$$\sum_{n=1}^{T} |\Delta u(n)|^2 \leq \lambda_k \|u\|^2, \qquad \forall u \in H_k;$$
$$\sum_{n=1}^{T} |\Delta u(n)|^2 \geq \lambda_{k+1} \|u\|^2, \qquad \forall u \in H_k^{\perp}.$$

Lemma 2.3 ([7]). *If φ is weakly lower semi continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 2.4 ([1]). *Let φ be a C^1 function on $X = X_1 \bigoplus X_2$ with $\varphi(0) = 0$, satisfying (PS) condition and for some $\delta > 0$,*

$$\varphi(u) \geq 0 \quad for\, u \in X_1, \|u\| \leq \delta,$$

$$\varphi(u) \leq 0 \quad for\, u \in X_2, \|u\| \leq \delta.$$

Assume also that φ is bounded below and $\inf_X \varphi < 0$, then φ has at least two nonzero critical points.

By Lemma 2.1, one rewrites u as

$$u = \bar{u} + \tilde{u} \in N_0 \bigoplus N_0^{\perp}, \tag{2.12}$$

where $\bar{u} = (1/T)\sum_{n=1}^{T} u(n)$.

By (2.9), (2.10) and (2.12), one has

$$\begin{aligned}\|u\| &= \Big(\sum_{n=1}^{T}|u(n)|^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{T}|\bar{u}+\tilde{u}(n)|^2\Big)^{\frac{1}{2}}\\ &= \Big(\sum_{n=1}^{T}\big(\bar{u}+\tilde{u}(n),\bar{u}+\tilde{u}(n)\big)\Big)^{\frac{1}{2}}\\ &= \Big(\sum_{n=1}^{T}\big(|\bar{u}|^2+|\tilde{u}(n)|^2\big)\Big)^{\frac{1}{2}} = \big(T|\bar{u}|^2+\|\tilde{u}\|^2\big)^{\frac{1}{2}}.\end{aligned}$$

Then, one has

$$\|u\| \leq \sqrt{T+1}(|\bar{u}|^2+\|\tilde{u}\|^2)^{\frac{1}{2}} \quad and \quad \|u\| \geq (|\bar{u}|^2+\|\tilde{u}\|^2)^{\frac{1}{2}}.$$

Therefore, one has that $\|u\| \to \infty$ if and only if $\big(|\bar{u}|+\|\tilde{u}\|^2\big)^{\frac{1}{2}} \to \infty$.

3. Proof of main results

Since the proof of Theorem 1.4 is similar to that of Theorem 1.3, we only prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in this section.

For convenience, we denote

$$R_1 = \sum_{n=1}^{T} f(n), \quad R_2 = \sum_{n=1}^{T} g(n).$$

Proof of Theorem 1.1. According to (B_3), we can choose a positive constant a_1, such that

$$a_1 > \frac{\varepsilon + M_3}{\lambda_1 - M_3} > 0 \tag{3.13}$$

for a small number $\varepsilon > 0$ and

$$\lim_{|x|\to+\infty} \inf |x|^{-2}\sum_{n=1}^{T} F(n,x) > \frac{a_1}{2}M_3 T. \tag{3.14}$$

By (B_1) , we obtain

$$\begin{aligned}&\Big|\sum_{n=1}^{T}\big[F_1(n,u(n)) - F_1(n,\bar{u})\big]\Big|\\ =&\Big|\sum_{n=1}^{T}\int_0^1 \big(\nabla F_1(n,\bar{u}+s\tilde{u}(n)),\tilde{u}(n)\big)ds\Big|\\ \leq&\sum_{n=1}^{T}\int_0^1 f(n)|\bar{u}+s\tilde{u}(n)|^{\alpha}|\tilde{u}(n)|ds + \sum_{n=1}^{T}\int_0^1 g(n)|\tilde{u}(n)|ds\\ \leq&\sum_{n=1}^{T} f(n)(|\bar{u}|^{\alpha}+|\tilde{u}(n)|^{\alpha})|\tilde{u}(n)| + \sum_{n=1}^{T} g(n)|\tilde{u}(n)|\end{aligned}$$

$$
\begin{aligned}
&\le R_1|\bar{u}|^{\alpha}\|\tilde{u}\|_\infty + R_1\|\tilde{u}\|_\infty^{\alpha+1} + R_2\|\tilde{u}\|_\infty \\
&\le \frac{\varepsilon}{2a_1}\|\tilde{u}\|_\infty^2 + \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha} + R_1\|\tilde{u}\|_\infty^{\alpha+1} + R_2\|\tilde{u}\|_\infty \\
&\le \frac{\varepsilon}{2a_1}\|\tilde{u}\|^2 + \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha} + R_1\|\tilde{u}\|^{\alpha+1} + R_2\|\tilde{u}\| \qquad (3.15)
\end{aligned}
$$

for any $u \in H_T$ with $\|u\|_\infty := \max\limits_{n\in\mathbb{Z}\cap[1,T]} |u(n)|$.

By (B_2), we have

$$
\begin{aligned}
&\left|\sum_{n=1}^{T} [F_2(u(n)) - F_2(\bar{u})]\right| \\
&= \left|\sum_{n=1}^{T} \int_0^1 (\nabla F_2(\bar{u} + s\tilde{u}(n)), \tilde{u}(n))\, ds\right| \\
&\le \sum_{n=1}^{T} \int_0^1 M_3\big(|\bar{u} + s\tilde{u}(n)|\big)|\tilde{u}(n)|ds + \sum_{n=1}^{T} \int_0^1 M_4|\tilde{u}(n)|ds \\
&\le M_3 \sum_{n=1}^{T} \Big(|\bar{u}| + \frac{1}{2}|\tilde{u}(n)|\Big)|\tilde{u}(n)| + \sum_{n=1}^{T} M_4|\tilde{u}(n)| \\
&\le M_3 \sum_{n=1}^{T} |\bar{u}||\tilde{u}(n)| + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
&\le M_3 \Big(\sum_{n=1}^{T}|\bar{u}|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{T}|\tilde{u}(n)|^2\Big)^{\frac{1}{2}} + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
&\le \frac{a_1}{2}M_3\sum_{n=1}^{T}|\bar{u}|^2 + \frac{M_3}{2a_1}\sum_{n=1}^{T}|\tilde{u}|^2 + \frac{M_3}{2}\sum_{n=1}^{T}|\tilde{u}|^2 + M_4\sum_{n=1}^{T}|\tilde{u}| \\
&\le \frac{a_1}{2}M_3T|\bar{u}|^2 + \Big(\frac{M_3}{2a_1} + \frac{M_3}{2}\Big)\|\tilde{u}\|^2 + M_4\sqrt{T}\|\tilde{u}\| \qquad (3.16)
\end{aligned}
$$

for any $u \in H_T$.

Hence, by (1.8), (3.15), (3.16) and Lemma 2.2, we have

$$
\begin{aligned}
\varphi(u) \ge& \Big(\frac{\lambda_1}{2} - \frac{\varepsilon}{2a_1} - \frac{M_3}{2a_1} - \frac{M_3}{2}\Big)\|\tilde{u}\|^2 - (R_2 + M_4\sqrt{T})\|\tilde{u}\| - R_1\|\tilde{u}\|^{\alpha+1} \\
&+ |\bar{u}|^2\left(|\bar{u}|^{-2}\sum_{n=1}^{T}F(n,\bar{u}) - \frac{a_1}{2}M_3T\right) - \frac{a_1}{2\varepsilon}R_1^2|\bar{u}|^{2\alpha}. \qquad (3.17)
\end{aligned}
$$

Since $u = \bar{u} + \tilde{u} \in N_0 \bigoplus N_0^{\perp}$, (3.13), (3.14) and (3.17) imply that

$$
\varphi(u) \to +\infty, \qquad \|u\| \to \infty.
$$

Thus, φ is coercive. Since φ is continuous, it possesses a bounded minimizing sequence in the finite dimensional Hilbert space H_T. Therefore, by Lemma 2.3, we obtain a critical point u which is a T-periodic solution of system (1.1) and the minimizer of the function φ. The proof is complete. □

By (3.15) in which the number $\frac{\varepsilon}{a_1}$ should be replaced by $\lambda_1 - \varepsilon$ and $M_3 = 0$ in (3.16), we have the following result.

Theorem 3.5. *Suppose that $F(n,x)$ with $F_2 = 0$ satisfying (A), (B_1) and*
(B_3') $\quad \lim_{|x|\to+\infty} \inf |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > \frac{M_1^2}{2\lambda_1}$.
Then, system (1.1) has at least one T-periodic solution that minimizes the functional φ in the Hilbert space H_T.

Comparing with Theorem 1.1 of [8], we see that the condition (B_3') is loose for some choices of T, for example $T = 2,3,4,5$ and so on. In Theorem 1.1, when $F_1(n,x) \equiv 0$, comparing with Theorem 1.3 of [8], the condition (B_3) in some sense is loose for some choices of T, one can check it for $T = 2,3,4,5,6$ and so on.

Now, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By (B_6), we can choose a positive constant a_2 and $\varepsilon > 0$, such that

$$a_2 > \frac{1}{\lambda_1 - R_1 - \varepsilon} \tag{3.18}$$

and

$$\lim_{|x|\to\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n) > \frac{a_2}{2} R_1^2. \tag{3.19}$$

By (B_4), $\forall u \in H_T$, we have

$$\begin{aligned}
&\Big| \sum_{n=1}^{T} [F_1(n,u(n)) - F_1(n,\bar{u})] \Big| \\
=&\Big| \sum_{n=1}^{T} \int_0^1 (\nabla F_1(n, \bar{u} + s\tilde{u}(n)), \tilde{u}(n)) ds \Big| \\
\le&\sum_{n=1}^{T} \int_0^1 f(n)|\bar{u} + s\tilde{u}(n)||\tilde{u}(n)| ds + \sum_{n=1}^{T} \int_0^1 g(n)|\tilde{u}(n)| ds \\
\le&\sum_{n=1}^{T} f(n)(|\bar{u}| + \frac{1}{2}|\tilde{u}(n)|)|\tilde{u}(n)| + \sum_{n=1}^{T} g(n)|\tilde{u}(n)| \\
\le& R_1|\bar{u}|\|\tilde{u}\|_\infty + \frac{R_1}{2}\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty \\
\le&\frac{1}{2a_2}\|\tilde{u}\|_\infty^2 + \frac{a_2}{2}R_1^2|\bar{u}|^2 + \frac{R_1}{2}\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty \\
=&(\frac{1}{2a_2} + \frac{R_1}{2})\|\tilde{u}\|_\infty^2 + R_2\|\tilde{u}\|_\infty + \frac{a_2}{2}R_1^2|\bar{u}|^2 \\
\le&(\frac{1}{2a_2} + \frac{R_1}{2})\|\tilde{u}\|^2 + R_2\|\tilde{u}\| + \frac{a_2}{2}R_1^2|\bar{u}|^2. \qquad (3.20)
\end{aligned}$$

By (B_5) , we have

$$\begin{aligned}
&\Big| \sum_{n=1}^{T} [F_2(n,u(n)) - F_2(n,\bar{u})] \Big| \\
=&\Big| \sum_{n=1}^{T} \int_0^1 (\nabla F_2(n, \bar{u} + s\tilde{u}(n)), \tilde{u}(n)) ds \Big|
\end{aligned}$$

$$\leq \sum_{n=1}^{T} \int_0^1 M_1|\bar{u}+s\tilde{u}(n)|^{\alpha}|\tilde{u}(n)|ds + \sum_{n=1}^{T} \int_0^1 M_2|\tilde{u}(n)|ds$$
$$\leq \sum_{n=1}^{T} M_1(|\bar{u}|^{\alpha}+|\tilde{u}(n)|^{\alpha})|\tilde{u}(n)| + \sum_{n=1}^{T} M_2|\tilde{u}(n)|$$
$$\leq M_1\sqrt{T}|\bar{u}|^{\alpha}\|\tilde{u}\| + M_1\sum_{n=1}^{T}|\tilde{u}|^{\alpha+1} + M_2\sum_{n=1}^{T}|\tilde{u}(n)|$$
$$\leq \frac{TM_1^2}{2\varepsilon}|\bar{u}|^{2\alpha} + \frac{\varepsilon}{2}\|\tilde{u}\|^2 + C_1\|\tilde{u}\|^{\alpha+1} + C_2\|\tilde{u}\|. \tag{3.21}$$

Hence, by (1.8), (3.20), (3.21) and Lemma 2.2, we have

$$\varphi(u) \geq (\frac{\lambda_1}{2} - \frac{1}{2a_2} - \frac{R_1}{2} - \frac{\varepsilon}{2})\|\tilde{u}\|^2 - C_1\|\tilde{u}\|^{\alpha+1} - (C_2+R_2)\|\tilde{u}(n)\|$$
$$+|\bar{u}|^2\left(|\bar{u}|^{-2}\sum_{n=1}^{T}F(n,\bar{u}) - \frac{a_2}{2}R_1^2\right) - \frac{TM_1^2}{2\varepsilon}|\bar{u}|^{2\alpha}. \tag{3.22}$$

Since $u=\bar{u}+\tilde{u}\in N_0\bigoplus N_0^{\perp}$, (3.18), (3.19) and (3.22) imply that

$$\varphi(u)\to+\infty, \qquad \|u\|\to\infty,$$

that is, φ is coercive. It is easy to verify that there exists a bounded minimizing sequence which insures that φ possesses a minimal point in the finite dimensional Hilbert space H_T by Lemma 2.3. The proof is complete. □

Proof of Theorem 1.3. According to the proof of Theorem 1.1, we can implies that φ is bounded below and satisfies the (PS) condition. By (A_1) and Lemma 2.2, one has

$$\varphi(u) \leq \frac{1}{2}\lambda_k\|u\|^2 + \sum_{n=1}^{T}\Big(-\frac{1}{2}\lambda_k|u|^2\Big) = 0 \tag{3.23}$$

for any $u\in H_k$ with $\|u\|\leq\delta$ and

$$\varphi(u) \geq \frac{1}{2}\lambda_{k+1}\|u\|^2 + \sum_{n=1}^{T}\Big(-\frac{1}{2}\lambda_{k+1}|u|^2\Big) = 0 \tag{3.24}$$

for any $u\in H_k^{\perp}$ with $\|u\|\leq\delta$.
If $\inf_{u\in H_T}\varphi(u)<0$, we completed our proof of Theorem 1.3 by Lemma 2.4.
If $\inf_{u\in H_T}\varphi(u)\geq 0$, by (3.23) and (3.24), we have $\varphi(u)=\inf_{u\in H_T}\varphi(u)=0$ for any $u\in H_k$ with $\|u\|\leq\delta$, which implies that any $u\in H_k$ with $\|u\|\leq\delta$ are minimum points of φ. Thus, any $u\in H_k$ with $\|u\|\leq\delta$ are T-periodic solutions of systems (1.1) , and systems (1.1) has infinite T-periodic solutions in H_T. Hence, we complete the proof of our main results. □

The proof of Theorem 1.4 is almost the same as that in the proof of Theorem 1.3, so we omit it.

4. Examples

In this section, we will give two examples to illustrate our theorems.

Example 4.1. *Let $F(n+T,x)=F(n,x)$ for any $(n,x)\in(\mathbb{Z},\mathbb{R}^N)$ and*

$$F(n,x)=\frac{\lambda_1}{16}|x|^2+(\frac{T+1}{2}-n)|x|^{7/4}+(\frac{4}{3}T-n)|x|^{3/2},\ n\in\mathbb{Z}\cap[1,T], \tag{4.25}$$

where

$$F_1(n,x)=(\frac{T+1}{2}-n)|x|^{7/4}+(2T-n)|x|^{3/2},\ \forall(n,x)\in(\mathbb{Z}\cap[1,T],\mathbb{R}^N) \tag{4.26}$$

and

$$F_2(x)=\frac{\lambda_1}{16}|x|^2-\frac{2}{3}T|x|^{3/2},\ \forall x\in\mathbb{R}^N. \tag{4.27}$$

According to (4.26), one has

$$\begin{aligned}|\nabla F_1(n,x)|\le&\frac{7}{8}|T+1-2n||x|^{3/4}+\frac{3}{2}|2n-T||x|^{1/2}\\ \le&\frac{7}{8}(|T+1-2n|+\varepsilon)|x|^{3/4}+\frac{9T^3}{\varepsilon^2},\ \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^N,\end{aligned} \tag{4.28}$$

where $\varepsilon>0$. Then, we obtained that (B_1) holds with $\alpha=3/4$ and

$$f(n)=\frac{7}{8}(|T+1-2n|+\varepsilon),\quad g(n)=\frac{9T^3}{\varepsilon^2}. \tag{4.29}$$

According to (4.27), we have

$$\begin{aligned}|\nabla F_2(x)|\le&\frac{\lambda_1}{8}|x|+T|x|^{1/2}\\ \le&(\frac{\lambda_1}{8}+\varepsilon)|x|+\frac{T}{\varepsilon},\ \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^N,\end{aligned} \tag{4.30}$$

where $\varepsilon>0$. Then, we obtained that (B_2) holds with

$$R_1=\frac{\lambda_1}{8}+\varepsilon,\quad R_2=\frac{T}{\varepsilon}. \tag{4.31}$$

Now, we verify that $F(n,x)$ satisfies (B_3). In fact,

$$\begin{aligned}&\lim_{|x|\to+\infty}\inf|x|^{-2}\sum_{n=1}^{T}F(n,x)\\ =&\lim_{|x|\to+\infty}\inf|x|^{-2}\sum_{n=1}^{T}[\frac{\lambda_1}{16}|x|^2+(\frac{T+1}{2}-n)|x|^{7/4}+(\frac{4}{3}T-n)|x|^{3/2}]\\ =&\frac{\lambda_1}{16}T.\end{aligned}$$

It is easy to verify that $M_3<\lambda_1$. If $T\in\{2,3,4,5,6,7,8,9,10\}$, we can choose $\varepsilon>0$, such that

$$\lim_{|x|\to+\infty}\inf|x|^{-2}\sum_{n=1}^{T}F(n,x)=\frac{\lambda_1}{16}T>\frac{(\frac{\lambda_1}{8}+\varepsilon)^2T}{2(\lambda_1-\frac{\lambda_1}{8}-\varepsilon)}=\frac{M_3^2T}{2(\lambda_1-M_3)}.$$

Thus, the system (1.1) has at least one T-periodic solution by Theorem 1.1.

Example 4.2. *Let $F(n+T,x)=F(n,x)$, for any $(n,x)\in\mathbb{Z}\times\mathbb{R}^N$ and*

$$F(n,x)=\frac{T-n}{20}|x|^2+\frac{4}{7}|x|^{7/4}-n|x|^{3/2}-|x|+(h(n),x). \tag{4.32}$$

Let

$$F_1(n,x) = \frac{T-n}{20}|x|^2 - n|x|^{3/2} + (h(n),x), \tag{4.33}$$

$h : \mathbb{Z} \cap [1,T] \to \mathbb{R}^N$, $h(n+T) = h(n)$, *for* $n \in \mathbb{Z} \cap [1,T]$ *and*

$$F_2(x) = \frac{4}{7}|x|^{7/4} - |x|. \tag{4.34}$$

According to (4.33), one has

$$\begin{aligned}|\nabla F_2(n,x)| \leq & \frac{T-n}{10}|x| + \frac{3n}{2} + |h(n)| \\ \leq & (\frac{T-n}{10} + \varepsilon)|x| + \frac{T^2}{\varepsilon} + |h(n)|, \ \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N,\end{aligned}$$

where $\varepsilon > 0$. *Then, we obtained that* (B_4) *holds with*

$$f(n) = \frac{T-n}{10} + \varepsilon, \quad g(n) = \frac{T^2}{\varepsilon} + |h(n)|. \tag{4.35}$$

It is easy to verify that (B_5) *holds with* $\alpha = 3/4$ *and* $M_3 = M_4 = 1$.
Now, we verify that $F(n,x)$ *satisfies* (B_6). *In fact, according to (4.32) and (4.35), we have*

$$\sum_{n=1}^{T} f(n) = \sum_{n=1}^{T} (\frac{T-n}{10} + \varepsilon) = T(\frac{T-1}{20} + \varepsilon)$$

and

$$\begin{aligned}&\lim_{|x|\to\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) \\ =&\lim_{|x|\to\infty} \inf |x|^{-2} \sum_{n=1}^{T} \Big[\frac{T-n}{20}|x|^2 + \frac{4}{7}|x|^{7/4} - n|x|^{3/2} - |x| + (h(n),x)\Big] \\ =&\frac{T(T-1)}{40}.\end{aligned}$$

When $T \in \{2,3,4\}$, *we can choose* $\varepsilon > 0$, *such that*

$$\sum_{n=1}^{T} f(n) = T(\frac{T-1}{20} + \varepsilon) < \lambda_1$$

and

$$\begin{aligned}\lim_{|x|\to\infty} \inf |x|^{-2} \sum_{n=1}^{T} F(n,x) =& \frac{T(T-1)}{40} \\ >& \frac{1}{2(\lambda_1 - \frac{T-1}{20} - \varepsilon)} \Big(\frac{T-1}{20} + \varepsilon\Big)^2 \\ =& \frac{1}{2(\lambda_1 - \sum_{n=1}^{T} f(n))} \Big(\sum_{n=1}^{T} f(n)\Big)^2.\end{aligned}$$

Thus, the system (1.1) has at least one T*-periodic solution by Theorem 1.2.*

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