International Journal of Educational Studies in Mathematics

IJESIM

EDUGARDEN

Case of Actualizing Geometry Knowledge in Abstraction Thinking Level for Constructing a Figure

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ABSTRACT

The purpose of studying geometry is for the development of student thinking, able to abstract and reason, against concepts in the object. Note on some study results that the combination of the student's conceptualization process and knowledge of geometry are a source of barriers to their learning. This study aims to investigate how students actualize their geometry knowledge for the development of thinking to the level of abstraction. Investigations on input, internal processing, and output. This study was conducted on high school students use questions regarding object and product of thought in geometry thinking at analysis and abstraction level. Based on study findings, students actualize their knowledge through figural aspects then developed into the conceptual aspect, and the two aspects are integrated with the relationship between properties. There is a mathematical connection to the integration process. However, because most students' understanding of geometry knowledge is still prototypical figural concepts, so there is an interaction issue between figural and conceptual aspects, namely the validity of relationship between properties. Therefore, students learn geometry in order to investigate the connections between objects and properties as a whole.

ARTICLE INFO

Article History:
Received: 21.09.2020
Received in revised form: 08.01.2020
Accepted: 04.02.2020
Available online: 08.03.2020

Article Type: Standard paper

Keywords: abstraction, actualization, conceptual, encapsulation, figural aspect.

1. Introduction

One of the fields of mathematics study is geometry. The direction for learning geometry purpose is focused on students' thinking development toward concepts in the object. Herbst, Fujita, Halverscheid and Weiss (2017) state that the origin of the development of mathematical thinking, including in geometry is one of mathematics area studies, is from perceptualization, conceptualization, organization, and axiomatization. Relevant to learning geometry that begins with the visualization phase towards deductive reasoning (Fujita, Kondo, Kumakura and Kunimune, 2017). Learning geometry as a training room for thinking ability by processing geometric concepts and facts through conceptualization and organization phases. Geometry studies concrete objects to abstract concepts (Silfverberg, 2019; Seah and Horne, 2019). Concrete objects are objects which the senses embrace through their representational visualization. Visual representations are useful as portraits of objects for introduction conceptions to geometry learning objectives like abstracting and reasoning (Seah and Horne, 2019; Abdullah and Zakaria, 2013; Fitriyani, Widodo and Hendroanto, 2018). However, the problem in learning geometry are factors that have been researched extensively (Giannakopoulos, 2017; Fujita et al., 2017). Critical issues of students' geometry thinking are noticeably global along to cross the boundaries of educational practices and curriculum (Tan, Tarmizi, Yunus and Ayub, 2015; Seah and Horne, 2019; Abdullah and Zakaria, 2013).

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DOI: https://doi.org/10.17278/ijesim.797749

In geometry context, abstraction is one level of van Hiele theory hierarchically that its position before deduction and after analysis (Silfverberg, 2019; Herbst et al., 2017; van de Walle, Karp and Bay-Williams, 2017; Yi, Flores and Wang, 2020). The hierarchical effect at the level of geometric thinking causes the organizational phase to depend on the conceptualization results. Several study results (e.g. Luneta, 2015; Maulidya, Hasanah and Retnowati, 2017; Alghadari and Herman, 2018; Utami, Kusmanto and Widodo, 2019) states that student error when solving problems is in the conceptualization process. Conceptual knowledge is authorized capital for processing in the conceptualization phase (Seah and Horne, 2019). According to the study results, most students have not been able to actualize conceptual knowledge in problem solving because the basic is memorization (Hakim and Nurlaelah, 2018), thus results in the abstraction process inoptimal (Widodo et al., 2020). The study by Cesaria and Herman (2019) found that the ontogeny obstacles arise when students' conceptual knowledge is inadequate toward given material and students aren't able to apply their knowledge because the material does not match the characteristics of their needs. Whereas, research findings by Rosilawati and Alghadari (2018) are most student conceptions lead to epistemological obstacles, and misconceptions are influenced by their factual and conceptual knowledge. Notes from some of the results of these studies are the processing students' geometry knowledge is the causes of the learning obstacles they experience and affects the student's ability to learn geometry at school.

In the context of geometry thinking, the analysis level is the object, the understanding required, the thought program, and it becomes explicit at the abstraction level (Silfverberg, 2019). Objects and products of thought in level analysis, namely the class of shape and properties of shape respectively (van de Walle et al., 2017; Abdullah and Zakaria, 2013). Study findings by Sulistiowati, Herman and Jupri (2019) states that students can identify the property of shape but the problem-solving issues in the concept actualization process. Identifying shapes, their classes, and properties are a technique for adequate conception in geometry (Yi et al., 2020), especially when the geometry elements are in a shape (Rosilawati and Alghadari, 2018) and is the point of view of the excellent student geometry thinking level in the senior high school (Rahayu and Alghadari, 2019). Based on objects and products of though of analysis level in van Hiele theory, its level is the transition from concrete object to abstract so that it becomes essential for learning and developing students' geometry thinking. If the student geometry thinking level is different from learning objectives, so the development will be stunted (Cesaria and Herman, 2019; Silfverberg, 2019; Yi et al., 2020). Because of the object and product of thought includes factual and conceptual knowledge. Radmehr and Drake (2018) defines a conceptual knowledge is knowledge of classifications and categories whereas a factual knowledge is knowledge of terminology, specific details and elements. Thus, students' factual and conceptual knowledge affects the development of the level of thinking, the actualization of concepts, and conceptions in learning or solving geometry problems.

Issues in students' geometry thinking based on the review notes above are on the relevance between actualizing and developing toward their knowledge. Herbst et al. (2017) proposed that the development of students' geometry knowledge at the intermediate level consists of the progressive sophistication students' intellectual means for modeling, predicting, and controlling geometry representations. On the other hand, it is necessary to have a conception of knowledge whose purpose is to actualize and develop students' thinking. But, conceptualizing or interpreting a concept can be biased from the real idea if the knowledge does not sufficiently support the point of identification views (Rahayu and Alghadari, 2019). Accurate measurement about students is capable to do, the knowledge and conceptual processes they need to move increase. It is important to design because students don't think based on the curriculum but through learning trajectories (Widodo et al., 2020; Seah and Horne, 2019; Tall, 2004a,b). Several issues on students' geometry ability, especially on the conceptualization process, has been detailed by the consideration for mapping learners' thinking in relevant concept with this study purpose. We design this study as an analysis of students' geometry knowledge, verify what they do specifically on some of the geometry concepts involved in the process. The purpose is to investigate how students actualize their knowledge for developing to abstraction level in geometry thinking.

2. Methodology

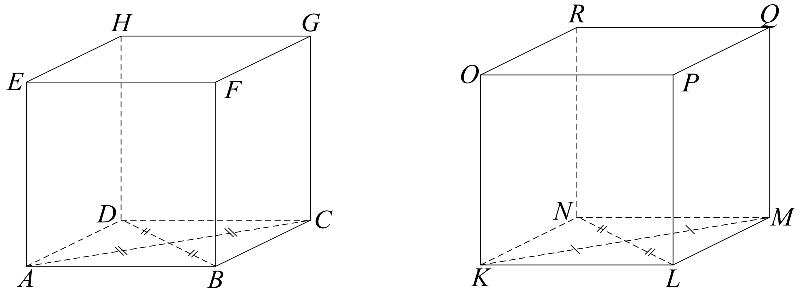
This research is a case study on students' knowledge in developing to abstraction level of geometry thinking. This study approach involves three phases, that are: analysis, design, and development. In the analysis phase, one geometry question was involved to get data students' geometry knowledge. The cases studied follow the following questions: (1) what is the student's geometry knowledge from the solid model as the object and product of analysis?; (2) how the process of model reconstruction and student techniques compare properties?; (3) how students integrate figural and conceptual aspects in the solid model?; and (4) how students abstract some geometry knowledge in the process of integrating the two aspects. Therefore, the coherence between facts, concepts, objects and product of thought should all be in sync. We investigate students' geometry knowledge in analysis to abstraction level and how are the impact for thinking and reasoning development. On the design phase, one other geometry questions are made relevant to its knowledge for underlying the development of thinking to abstraction level. All two questions are detailed in the instrument section and submitted to students. Students who are research subjects are described in the participant's section. Students answer geometry questions writing on paper accompanied by details of the reasons. The answers from students are data that will be analyzed with an investigative approach. The investigative technique is described in the data analysis section.

2.1. Instrument

This study is designed for the level of thinking geometry from analysis to abstraction. The standards for geometric questions are objects and products of thought at these two levels. The two questions are the source for obtaining research data based on: (a) two abilities for interpreting geometry representations according to Fujita et al., (2017), recognizing properties of shape and comparing objects; (b) two techniques for students identify the concept according to Rahayu and Alghadari (2019), visualization as an object of analysis or concepts as a basis for thinking. The indicators of two questions according to two identification techniques are as follows. First, students compare shapes or their properties in the two figures are for the indicator of Question 1. This indicator is relevant to the problem of students' conception of objects. Second, students design figure according to the identified concepts is an indicator of Question 2. This is relevant to the problem of the actualization concept and to develop a thinking level.

Question 1:

Figure (a) is a cube ABCD EFGH. Figure (b) is a rectangular prism KLMN OPQR with two parallel sides that are rhombus. The lengths of all edges in both figures are the same.



(a) (b)

Write differences of properties in the model of the pictures, also state your reason why.

Question 2:

The longest distance between the two points is $4\sqrt{3}$ cm. If there are diagonals of equal length and intersect perpendicularly, so the figure of cube and prism and their sizes are?

Question 1 is for the analysis thinking level because properties of shape are the product of thought and investigated from different models, but Question 2 is for abstract concepts. We involve visual objects on

Question 1 because it agrees with the statement Seah and Horne (2019) that student learn based on curriculum has been dominated by prototypical geometry shapes. The interesting things are presented here is 3D representation objects are pictured on 2D media by the oblique parallel perspective as the classical representation (Fujita et al., 2017). On Question 2, we use a defined concept from the figure where students are assigned to design a model and interact with the appropriate relationship between properties. The two Questions above have something relationship, that is on the geometry shape at solid, class and properties of shape. The van Hiele theory is the framework of these two questions.

2.2. Participants

Students participate in this study are they have studied geometry about the distance between two geometry elements in a solid. In the Indonesian educational curriculum, the learning material is for student grade 12 in senior high school. They are a student from one school in the district place. A total of 61 students from two study groups in these schools were involved in this study. They were selected by stratified random sampling based on the educational stage and major programs.

2.3. Data Analysis

We identified students' geometry knowledge in their responses toward the two Questions. Respon of the Question 1 and 2 respectively, are a description of the input and output of the constructing process figure by actualizing geometry knowledge. A subtype of factual and conceptual geometry knowledge is the identified element and it is investigated data on its relevance and interaction in actualization. Analyzing students' conceptual knowledge based on their answers is to elaborate the process for seeing how solutions emerge, what plans students make and implement, and how students understand problems. On the other hand, geometry requires the existence of many cognitive functions simultaneously (Gray and Tall, 2007; Giannakopoulos, 2017; Luneta, 2015). Therefore, the investigation we are doing is on three components of human activity by Tall (2013) as input, internal processing, and output as an alternative theory for looking at two developments, visuospatial to verbal-deductive and process-to-concept encapsulations using manipulable symbols.

Data on factual and conceptual knowledge obtained can be grouped based on these three activities. The input data group has implications for the output data based on the function of internal processing. Cases are reduced from misconnection by: (1) input and internal processing are correct but output is false, or (2) the output matches the input but the processing problem.

3. Result and Discussion

Overall, We found five categories of students' behavior towards their answers based on the relevance between the identified geometry concepts and their actualized results: (A1) no answer (3,28%); (A2) just for representing the figural aspect (11,48%); (A3) interaction issue between figural and conceptual aspect (6,59%); (A4) just true for generating the relationship between properties of shape in a cube (67,21%); (A5) true for generating a relationship between properties of shape for all (11,48%).

3.1. The Output from Analysis Thinking Level

The data analyzed as output is the product of thought in analysis thinking level, namely properties of shape (van de Walle et al., 2017). The input is square and rhombus as the class of shape in the figures. In this study, overall, the properties that the students analyzed were the side of the shape and precisely on the base. The side of solid shapes so of course students know the category class of shape. In this study, analyzed property by all students is side on the base of the figure exactly. It was the initial point when the student detailing the properties. Seah and Horne (2019) call it the initial geometric ideas are based on prototypical models of shapes where the research result reported that many students rely on it. There is a symbol on the base plane drawn to indicate the similarity or inequality of property sizes so students' views focused on it. Some of the generated product of thought by students about the base of solid as the class of shape is edges or the angle of their intersection, diagonals on the side or size of length, the angle of intersection of diagonals on the side, or the size, and the area of the base plane.

Fujita et al. (2017) have suggested focusing on 2D shapes is very important and is the main mediating means of 3D representation. Not only that, but there was also student response, who signed their view on the other plane in their figure as a consequence of the first focus. For example, is diagonal on space is also one property of the two figures as the result of analysis thinking geometrically.

3.2. Internal Process for Analyzing Property of Shape

An internal process begins when students see the figure then encode it as information in their conscious mind (Fiantika, Budayasa and Lukito, 2018). Referring to Question 1, there is no process of redrawing the two figures on all student answer papers. Through this question, students are encouraged to build geometry constructions internally. Giannakopoulos (2017) stated that this method is one of the contributors to get a better insight into the structure and strengthened the basis of conceptual knowledge. According to the interpretation of most students, there is a reconstruction process of the cube model internally to produce a prism representation because there is a change of symbol at the angular points from *ABCD.EFGH* to *KLMN.OPQR*. In addition, there are property changes in the representation figure. All of the angular points symbols on the object change but the representation figure only changes the length of the two parallel of diagonals on the side, like a Figure (a) is pressed on two facing vertical edges diagonally to become Figure (b). There is a copying process from Figure (a) to produce Figure (b), but it is accompanied by a process of reconstructing properties and revising symbols. The internal representation is not built through the recognition process. The properties of the figure other than the base and top of diagonals on the side along with the angle on the base plane are still in capsulation. The product of thought by analytical thinking described above is information that supports this process. Therefore, the encapsulation of process-to-concept occurs before the student revises the symbol of angular points. Students learning experiences in their school are often to symbolize angular points of a cube by *ABCD.EFGH*. Our identification have found it in the entire drawing of the cube students created for the answer to Question 2.

According to the results of the case analysis, There are two techniques of how students compare two objects, that is: (a) properties between two are compared then encapsulated (or reification) in geometry relationship, or (b) properties of each figure are capsulated and then compared. Of course, this internal process occurs because it is influenced by the opportunities presented in the Question. In the first context, An example of a response is that the diagonal of one solid is longer than the other. This technique transparency the properties of shape and base on a comparison where it is not done on the object of thought but the product of thought in the analysis of geometry thinking. The object of thought on each model is analyzed to produce property of shape and then compared the similarities or differences. Here, the product emerges through empirical abstraction because students focus is on objects and their properties (Gray and Tall, 2007).

While the example of student responses for the second context of the comparing technique is the size of the diagonal on the side and space of two figures is different. We counted the number of students who applied this technique and we only met one out of 61 participants where he is into A5. It is confirmed that the responses are made by students not based on the results of the analysis properties of shape one by one but it is a consequence of the focus of his analysis on the base plane. Piaget called this case an empirical pseudo-abstraction that focusing on actions on objects and the properties of the actions (Scheiner and Pinto, 2014; Fitriani, Suryadi and Darhim, 2018a,b), where the objects being compared represent a conceptualization of the process, or embodied mathematics world conceptually, because of routine factors (Tall, 2004a,b; 2013; 2020), based on perception and reflection on object properties (Gray and Tall, 2007). In this case, geometry properties are packaged in a class of shape and then comparing. The student sees all the properties of shape as an individual object which is then recursively excavated. In other words, students compare the class of shape on the two figures. Because there are differences in the class of shape, there are different abstract properties of course. The similarity or difference of geometry properties is based on shape as the object of thought so student thinking geometrically on analysis level occurs after comparing class of shape. Here, students compare the representation of the object of thought then think geometrically occurs after the mathematical connection process, and they formulate concepts to build thinkable concepts. Its internal process is not easy enough for most people and virtually impossible for the many who simply learn the rules by rote, but it is a process that seems to be performed implicitly by those who make sense of the hierarchical structure (Tall, 2013; 2020). The explanation by Silfverberg (2019) that conceptualization does require synthesis through association, abstraction, and differentiation between properties before analyzing.

The two techniques are a case of the kind of cognitive development that Tall (2004a,b) introduced to the world of mathematics. One technique of empirical pseudo-abstraction is excellent because it is more sophisticated than the sensory experience (Gray and Tall, 2007; Scheiner and Pinto, 2014; Tall, 2013; 2020). This case is an example of the progressive sophistication of students' intellectual means to control for representations of the phenomenon of geometrical knowledge development as suggested by Herbst et al. (2017), that requires a specific focus on relevant aspects of a situation to name and compress into a thinkable concept (Gray and Tall, 2007; Tall, 2020). These two techniques have drawn attention to the development of abstraction thinking in geometry. Fitriani et al. (2018a) have detailed several possible ways of bringing up abstractions, two of which are directing students to recognize commonalities across contexts and condition students so that they feel the same way to form universal concepts. We have presented students in Question 2 an opportunity to abstract the geometry properties in the process of constructing a cube and prism model to touch their awareness of the intersection of the conceptual aspect from the two solid.

3.3. Abstracting for Relationship Between Properties

Students' geometry knowledge regarding the properties of shape has been shown through their response to Question 1. Generally, Their response perspective is diagonal on the side of the base plane. This knowledge is an internal condition and relevant to participate in an abstraction process for the response of Question 2. We have examined students' responses and there are case findings in each of their categories of behavior. For example, case finding on category A2 and A3, that is, students show a problem in abstracting the properties of shape. The case investigation results on these two categories are there are similarities and differences. The difference is in the abstraction stage, the first is just for representing the figural aspect and the other is in the integration process both of figural and conceptual aspect. he similarity is that students can analyze properties of shape but their problem with conceptualizing both the size of the cube and the prism. Representations that students make and include in the behavior of category indicate that they specify a specific size of either the lengths of edge or diagonal on the side. Then, on the figure, they represent the intersecting of diagonals on the side. The same length of diagonal on the side conceptualized them by putting down the measurements or represents a specific size as in the edge. However, The conceptualized sizes do not yet fulfill the relationship geometrically so students have not produced a valid relationship between properties.

A cube has the longest distance between two points is the length of diagonal on space, while its distance as a given of defined concept. When the size of the model that students conceptualized was not relevant to the defined concept, We declare this as an error in conceptualizing size and consequently, it does not satisfy the specified geometry relationships by the defined concept. Some literature (e.g. Fitriani, et al., 2018b; Silfverberg, 2019) stated that the process of abstraction in geometry can be started from the observation and measurement of space and physical figure. We add the statement that the validity of the abstraction product is on the relationship between properties, which is an interaction between figural and conceptual aspects. Harusnya, It should be, the design of the properties is measured, synchronized, and fulfills the geometry relationship (Luneta, 2015). Mariotti (1995) stated that geometry concepts have certain properties because of their relationship to spatial properties, where between figure and concepts is not only limited to geometry concepts but which concerns the relationship between figures and concepts. However, there is an integration problem between the figural and conceptual components of the model designed by students. This finding is not different from the study results by Sulistiowati et al. (2019) and Hakim and Nurlaelah (2018) that students able to identify the property of shape but have not been able to actualize their knowledge by abstraction process in the validity of geometry relationship. As an identification center to find the concept of identity, representations of objects receive more attention than sets of conceptual definitions (Rahayu and Alghadari, 2019). Furthermore, because there are problems in the figural and conceptual aspects so it is relevant to geometric reasoning (Mariotti, 1995). While Fischbein (1993) stated that psychological constraints are the cause of not achieving the combination of concepts and figures.

There were cases of other findings and we classified as category A4. Here, most of the students, more than half the number of participants involved in this study, they are only able to abstract geometric properties of the cube, and they involve the Pythagorean theorem for the interaction between figural and conceptual aspects. The longest distance between two points they choose is along the diagonal on space. Abstraction involving the Pythagorean Theorem has confirmed the geometry relationship between diagonal on the side, diagonal on space, and edges. We did not see any interaction problems with the relationship between the properties of the defined concept of diagonals on the side are intersecting perpendicularity in a cube in Question 2, because the concept is encapsulated in a square as the shaped plane. On the behavior of category A4, there are students who show awareness of the involvement of the angle. A total of 8 students whose property analysis results were certain of size angle on the base, such as the perpendicularity between the intersecting edges (7 students) or diagonal on the side (1 student) in all two figures. The involvement of the concept of angles in the abstraction process is another direction of an investigation. Actualization concepts for constructing a model of the test to the students whether they can abstract by their geometry thinking. Our investigation found that the prism model was constructed by belonging to this category are triangular prisms with the validity issue in the relationship between properties because the interaction has not been tight between the concept of angles, size of properties, and the intersect perpendicularly of diagonal on the side. Therefore, one of the problems with this finding is because there are concepts that are not involved in the abstraction process. On the other hand, this case also sheds light on the terms that quotes by Sharma (2019) as a prototypical figural concept in students' understanding of the prism, that is students do not recognize the cube as a special representation of prism even though they are familiar with the conceptual nature of the prism.

Involving the concept of angles as geometry properties for the design of a solid figure to be potential for finding the relationship between properties. Two of the seven students were included in the behavior of category A5 show the completeness analysis the property of shape on angle concept and they can conceptualize the relationship of properties in a prism geometrically. In other words, its property is involved in the abstraction and it is an example of abstraction as a process. According to the study results by Fiantika et al. (2018), two types of abstraction, namely processes and products. As a process, abstraction is an internal process that occurs in the mind whereas abstraction as a product is an idea formed by abstraction and it is a reference from abstract ideas. Case findings on category A5 is the interesting part because on the whole that the students represent the figure of a rectangular prism. There are prism models where all the sides are squares. Also, there were also students who represented a prism model with two sides of a square and the other four sides being rectangles. A concept of perpendicularity of diagonals on the side they are shown on the square side. Look at this case, it seems that there are students who realize and agree that the cube is a special figured prism, and there are also students who agree that the cuboid is included in the prism category, although its reasoning does not explain them explicitly. Fitriani et al. (2018a) have suggested that to bring up abstraction is to condition students so that they feel the same in order to form universal concepts. Question 1 is a source of information that has the potential to lead students to think that a cube is a special figured of a prism. besides, another topic of this question is so that students understand the intersection of the conceptual aspect the figure through the geometry construction of cubes and prisms by abstracting, it can go through a process or as a product.

3.4. Abstraction Thinking for Actualizing Concept

Based on case findings, there are various geometry properties of cubes and prisms, such as edges, diagonals on the side, and angle, that students understand and they try to involve it in the abstraction. At the level of abstraction in geometry thinking, the actualization of the concept is about how students process the property of shape to produce a relationship between properties (van de Walle et al., 2017), construct a model based on geometry relationship, so emerging an external representation model (Fiantika et al., 2018). In the process, students model cubes and prisms by designing figures and continue with the concept. Generally, the construction of a does not show a different figure for each student but there are differences in conceptual aspects. Design a cube model in accordance with the defined concept is hat should fulfill both figural and conceptual aspects. The geometric figure is conceptualized in a symbiotic relationship between these two aspects, figural and conceptual (Fischbein, 1993; Mariotti, 1995; Sharma, 2019). However, in the abstraction process, there are students who are not suitable in putting down the defined concept. Some have conceptualized the size of edge or diagonals on the side of a cube, the process is not a problem if it is still within the limits of conformity with the defined concept so the relationship between properties was still valid. However, there are problems found in the abstraction process properties of the cube, namely on the interaction between figural and conceptual aspects.

Whereas in the prism construction, there are two possible models of representation based on the properties of two parallel sides it has, namely triangular and quadrilateral prisms. In the triangular prism construction model, diagonals on the side in the rectangular plane is the longest size. There are students who conceptualize the size of edges in the triangle but ignoring the property of the perpendicular intersecting diagonals. Therefore, the study findings on student responses represent a triangular prism model is the validity issue in relationship between properties. The problem is also in the interaction between figural and conceptual aspects. Its problem is a natural stage of conceptual development (Luneta, 2015). Based on the results of the investigation, most of the students who participated in this study because their understanding of the conceptual definition of solid is prototypical figural concepts. Rahayu and Alghadari (2019) calls it a subjective concept identity and not the same as universally agreed-upon ideas. In fact, the conceptual aspect guarantees the accuracy and logical consistency of the operation (Sharma, 2019). In contrast to many students who construct a rectangular prism model. They account for 11.48% of all participants, represent a prism fit to the defined concept is by encapsulating properties of diagonals on the side are equal and intersecting perpendicularity in the base of square-shaped and apply the diagonal on space as the longest size. There are different conceptual techniques when students construct a prism model based on its properties.

The findings of this study are information for the integration process of figural and conceptual aspects carried out in their way of thinking go to the level of abstraction. There are two ways of conceptualizing, through the intrinsic properties or physical environment of the object (Mariotti, 1995; Rahayu and Alghadari, 2019), and in the process of abstraction that the overall technique of conceptualizing students is through the physical environment of the object where it is relevant to understanding in prototypical figural concepts characteristics. Tall (2004a,b; 2013) stated that embodied mathematics world technique in conceptually includes not only mental perception but also internal conceptions involving spatial images. Conceptual embodiment refers to the long-term development of the properties of objects as their conception becomes more sophisticated, from making sense of the relationships between physical objects to more abstract relationships between mental objects (Tall, 2013; 2020). Furthermore, when the mental perspective of some process is understood one can see it as a whole and use it as only one step, like encapsulate properties in the class of shape, this context says Gray and Tall (2007) as compression, it is the phenomenon of storing information economically. This context is typical of gifted students where solutions are created through the process of completing a small number of powerful steps. Furthermore, the conceptual embodied world as a combination of enactive and iconic and the symbolic and formal worlds.

Implications of the results of this study for the learning process are efforts that lead students to see the connection between objects in their entirety and followed by the interpretation of the results of these connections based on the abstract properties of the object. The consideration is on the reification of the process to a concept as a form of abstraction thinking application. The importance of information on how to study geometry has been recommended by Silfverberg (2019) that students geometry depends more on the content discussed and the quality of learning, and must know the main characteristics of the development of students' geometry thinking and adjust for that stage in development. For progressing, they must understand that shapes and objects are not stand-alone entities, but rather, a connected network of concepts presented in the form of points, lines, angles, diagrams, and concrete objects (Seah and Horne, 2019). Gray and Tall (2007) suggests that there must be a connection to develop a more flexible form of knowledge with the expected goal of solving new problems. Furthermore, the development of conceptual embodiment in the long term is an attempt to shift to the formal operation thinking stage by enhancing meaningful so that formal operations are manipulated in sophisticated situations (Tall, 2013; 2020). Besides, students need to interpret representations of geometric concepts in various situations, improve and communicate with their understanding so that their geometrical knowledge is increasingly integrated and synthesized (Seah and Horne, 2019).

4. Conclusion

Based on study findings, students actualize their geometry knowledge through figural aspects and then develop its conceptual aspects, and both aspects are integrated with the relationship between properties so that there is a connection process in it. The validity of the abstraction product is on the relationship between properties by involved concepts that should be strict. When students ignore conceptual aspects, there will be geometry properties that have not been involved in the abstraction process. The result is that the conceptual aspect has not guaranteed accuracy and logical consistency in the figural component, so there is a problem of integration between the figural and conceptual components. The source of the problem is because students' understanding is still prototypical figural concepts. The conceptualization process is carried out by students with one of the two identified techniques. When students actualize the concept with the capsulation technique for properties in a class of shape in thinking geometrically for analysis and abstraction level, or the capsulation technique process-to-concept, as the conceptualization technique, so it is an excellent technique for the validity of the product of thought. Therefore, students learn geometry by looking at the connections between objects and properties as a whole is needed for reification purposes.

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