Uşak Üniversitesi Fen ve Doğa Bilimleri Dergisi
Usak University Journal of Science and Natural Sciences

http://dergipark.gov.tr/usufedbid
https://doi.org/10.47137/usufedbid.798189

Araştırma Makalesi / Research Article

Adjoint Curve According to Modified Orthogonal Frame with Torsion in 3-Space

Murat ARIKAN[1*], Semra KAYA NURKAN[2]

[1]*Uşak University, Science Instutite, Department of Mathematics, TURKEY*

[2] *Uşak University, Science and Art Faculty, Department of Mathematics, TURKEY*

Geliş: 22 Eylül 2020 Kabul: 19 Ekim 2020 / Received: 22 September 2020 Accepted: 19 October 2020

Abstract

The main purpose in this study is to create the adjoint curve of a curve according to the modified orthogonal frame created by torsion in Euclidean and Minkowski 3-space. Besides, it is another purpose to reveal the relationships between the modified orthogonal frame of these curves.

Keywords: *Adjoint curve, Modified orthogonal frame, Euclidean space, Minkowski space.*

1. Introduction

Creating another curve from one curve with the help of frenet vectors is an interesting topic in differential geometry. An example is adjoint curve (or conjugate mate) which is defined in [4, 5, 9] as the integral of binormal vector with any parameter s of a curve. On the other hand, using different frames instead of the Frenet frame creates new ideas for mathematicians. Sasai decribed new frames, called modified orthogonal frame, with help of the frenet frame. These frames are created both curvature and torsion of a space curve in E^3 [10]. Moreover Bukcu and Karacan studied space curves and spherical curves with respect to the modified orthogonal frame in Minkowski and Euclidean space [2, 3]. Recently, Lone et al. obtained some characterization results for helices, Bertrand curves and Mannheim curves with respect to the modified orthogonal frame [7, 8].
Considering all these studies, the idea of constructing the adjoint curve of a curve according to the modified orthogonal frame formed by torsion was born in this study.

*Corresponding author:
E-mail: muratarikan3564@gmail.com
ORCID ID: 0000-0002-7199-8627

2. Preliminaries

We recall some basic notions about classical differential geometry of space curves in E^3.

Let $\beta: I \to E^3$ be a curve and $\{t, n, b\}$ denote the Frenet frame of β. $t(s) = \beta'(s)$ is called the unit tangent vector of β at s. β is a unit speed curve (or parametrized by arc-length s) if and only if $\|\beta'(s)\| = 1$. The curvature of β is given by $\kappa(s) = \|\beta''(s)\|$. The unit principal normal vector $n(s)$ of β at s is given by $\beta''(s) = \kappa(s)n(s)$. Also the unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of β at s. Then the famous Frenet formula holds as;

$$t'(s) = \kappa(s)n(s)$$

$$n'(s) = -\kappa(s)t(s) + \tau(s)b(s)$$

$$b'(s) = -\tau(s)n(s)$$

where $\tau(s)$ is the torsion of β at s and calculated as $\tau(s) = \langle n'(s), b(s)\rangle$.

Also the Frenet vectors of a curve β, which is by arc-length parameter s, can be calculated as;

$$t(s) \ = \ \beta'(s)$$

$$n(s) = \frac{\beta''(s)}{\|\beta''(s)\|}$$

$$b(s) = t(s) \times n(s).$$

Now, we would like to give a brief summary of basic definitions, facts and equations in the theory of curves in Minkowski 3-space.

Let E_1^3 denote the Minkowski 3-space with canonical Lorentzian metric tensor given by

$$\langle .,. \rangle = -dx_1^2 + dx_2^2 + +dx_3^2$$

where (x_1, x_2, x_3) are rectangular coordinates of the points of E_1^3 .

The causality of a vector in Minkowski space is defined as follows: A non-zero vector v in E_1^3 is said to be space-like, time-like and light-like (null) regarding to $\langle v\, , v\rangle > 0$, $\langle v\, , v\rangle < 0$ and$\langle v\, , v\rangle = 0$, respectively. We consider the zero vector as a spacelike vector. Note that v is said to be causal if it is not space-like. Two non-zero vectors u and v in E_1^3 are said to be orthogonal if $\langle v\, , v\rangle = 0$. A set of $\{e_1, e_2, e_3\}$of vectors in E_1^3 is called as an orthonormal frame if it satisfies that

$$< e_1, e_1 > = \text{-}1, \ < e_2, e_2 >=< e_3, e_3 >=1,$$

$$< e_i, e_j >= \ -0, \qquad i \neq j.$$

For two non-zero vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in E_1^3 , we define the Lorentzian product of u and v as in the following:

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

One can check that the vector product is skewsymmetric, i.e. $u \times v = -v \times u$.

A curve $\beta = \beta(s)$ in E_1^3 is said to be space-like, timelike or light-like (null) if its tangent vector field $\beta'(s)$ is space-like, time-like or light-like (null),respectively, for all s.

Let β be a non-null curve in $E_1{}^3$ parametrized by arclength, i.e., $|<\beta', \beta'>| = 1$, and we suppose that $|<\beta", \beta">| \neq 0$. Then this curve induces a Frenet frame $\left\{ t = \beta', \; n = \frac{\beta"}{\sqrt{(|<\beta",\beta">|)}}, \quad b = t \times n \right\}$ satisfying the following Frenet equations:

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa\varepsilon_1 & 0 \\ -\kappa\varepsilon_0 & 0 & -\tau\varepsilon_0\varepsilon_1 \\ 0 & -\tau\varepsilon_1 & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}$$

where $<t,t>= \varepsilon_0$, $<n,n>= \varepsilon_1$ $<b,b>= -\varepsilon_0.\varepsilon_1$, $<t',n>= \kappa$ and $<n',b>= \tau$. The vector fields t,n,b and the functions κ,τ are called the tangent, principal normal, binormal and curvature and torsion of β, respectively. Accordingly, the Frenet frame of β satisfies

$$t \times n = b, \qquad n \times b = -\varepsilon_1.t, \quad b \times t = -\varepsilon_0.n$$

In the frenet equations, if $\varepsilon_0 = 1$ or $\varepsilon_0 = -1$, then β is space-like or time-like, respectively. A space-like curve β is said to be type1 or type2 if $\varepsilon_1 = 1$ or $\varepsilon_1 = -1$.

Let α be a unit speed curve in E^3 with torsion $\tau \neq 0$ and the Frenet frame of α be $\{t, n, b\}$ Now we define an orthonogal frame $\{T, N, B\}$ as follows:

$$T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \times N$$

where $T \times N$ is the vector product of T and N. The relations between those and the classical Frenet frame $\{t, n, b\}$ at non-zero points of κ are

$$T = t$$

$$N = \kappa n$$

$$B = \kappa b$$

By the definition of $\{T, N, B\}$, a simple calculation shows that

$$\begin{bmatrix} \mathrm{T}'(\mathrm{s}) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa\prime}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa\prime}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where a dash denotes the differentiation with respect to the arc length s and

$$\tau = \tau(s) = \frac{det(\alpha', \alpha'', \alpha''')}{\kappa^2}$$

is the torsion of α. From the Frenet-Serret equation, we know that any zero point of κ^2 is a removable singularity of τ. The Eq. corresponds to the Frenet-Serret equation in the classical case. Moreover, $\{T, N, B\}$ satisfies:

$$< T,T > = 1, < N,N > = < B,B > = \kappa^2$$

$$< T,N > = < T,B > = < N,B > = 0$$

where <,> denotes the inner product of E^3 . The orthogonal frame is called as a modified orthogonal frame (10). We see that for κ=1, the Frenet-Serret frame coincides with the modified orthogonal frame. The modified orthogonal frame is studied by Bukcu and Karacan in Minkowski 3-space (see for details(2,3)).

Now let's talk about the modified orthogonal modified frame with torsion.

Let α and $\{t, n, b\}$ be a unit speed curve in Euclidean 3-space and the Frenet frame along the curve, respectively . The relations between orthogonal frame $\{T, N, B\}$ and the classical Frenet frame $\{t, n, b\}$ at non-zero points of τ are

$$T = t \ , N = \tau n \ , B = \tau b,$$

where

$$< T,T > = 1, < N,N > = < B,B > = \tau^2$$

$$< T,N > = < T,B > = < N,B > = 0$$

In this case, the following modified orthogonal frames hold:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa}{\tau} & 0 \\ -\kappa\tau & \frac{\tau'}{\tau} & \tau \\ 0 & -\tau & \frac{\tau'}{\tau} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Let α and $\{t, n, b\}$ be a unit speed curve in Minkowski 3-space and the Frenet frame along the curve, respectively . The relations between orthogonal frame $\{T, N, B\}$ and the classical Frenet frame $\{t, n, b\}$ at non-zero points of τ are

$$T = t \ , N = \tau n \ , B = \tau b.$$

In this case, the following modified orthogonal frames hold:

If α is timelike curve, then the orthogonal frame is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa}{\tau} & 0 \\ \kappa\tau & \frac{\tau'}{\tau} & \tau \\ 0 & -\tau & \frac{\tau'}{\tau} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

If α is a spacelike curve with a spacelike principal normal n, then the the orthogonal frame is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa}{\tau} & 0 \\ -\kappa\tau & \frac{\tau'}{\tau} & \tau \\ 0 & \tau & \frac{\tau'}{\tau} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

If α is a spacelike curve with a spacelike binormal b , then the orthogonal frame is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa}{\tau} & 0 \\ \kappa\tau & \frac{\tau'}{\tau} & \tau \\ 0 & \tau & \frac{\tau'}{\tau} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

3. Adjoint Curve According to Modified Orthogonal Frame with Torsion in Euclidean 3-space

Definition 3.1: Let α be a unit speed regular curve with curvature $\kappa \neq 0$ and torsion $\tau \neq 0$. If the modified orthogonal frame with torsion of α is $\{T, N, B\}$ then the adjoint curve of α is defined as

$$\beta(s) = \int B(s)\, ds.$$

Theorem 3.2: Let α be a unit speed regular curve in E^3 and β be the adjoint curve of α according to modified orthogonal frame with torsion. If the modified orthogonal frames of α and β are $\{T, N, B\}$ and $\{\overline{T}, \overline{N}, \overline{B}\}$, the curvature and torsion are κ, τ and $\overline{\kappa}$, $\overline{\tau}$ respectively, then the following relations hold:

$$\overline{T} = \left(\frac{1}{\tau}\right) B$$

$$\overline{N} = -\left(\frac{\kappa}{\tau^2}\right) N$$

$$\overline{B} = \left(\frac{\kappa}{\tau}\right) T$$

$$\overline{\kappa} = 1$$

$$\bar{\tau} = \frac{\kappa}{\tau}.$$

Proof: Let α is a curve with arc length parameter s and β is a curve with arc length parameter s, then the Frenet apparatus of α and β are $\{t, n, b, \kappa, \tau\}$ and $\{\bar{t}, \bar{n}, \bar{b}, \bar{\kappa}, \bar{\tau}\}$.

By the definition of adjoint curve we know that

$$\beta(s) = \int B(s))ds.$$

Differentiating this equation with respect to s and using modified orthogonal frame with torsion rules, we get

$$\frac{d\beta(s)}{ds} = B(s)$$

$$\left\langle \frac{d\beta(s)}{ds}, \frac{d\beta(s)}{ds} \right\rangle = \langle B,B \rangle = \tau^2.$$

Now, we rewrite this equation with respect to s, we have

$$\frac{d}{d\bar{s}}\beta(s)\frac{d\bar{s}}{ds} = B(s)$$

$$\bar{t}\frac{\mathrm{d}\bar{s}}{\mathrm{ds}} = B(s)$$

$$\frac{\mathrm{d}\bar{s}}{\mathrm{ds}} = \tau$$

$$\bar{t} = \frac{1}{\tau} B.$$

From the relation between frenet frame and modified orthogonal frame we see

$$\bar{T} = \frac{1}{\tau} B.$$

In order to determine the first curvature and principal normal of the curve β, we formalize

$$\frac{d\bar{t}}{d\bar{s}}\frac{\mathrm{d}\bar{s}}{\mathrm{ds}} = \frac{d}{ds}\left(\frac{1}{\tau}B\right)$$

$$\frac{d^2}{d\bar{s}^2}\beta(s) = -\left(\frac{1}{\tau}\right)N.$$

If we take the norm from both sides of this equation

$$\left\|\frac{d^2}{d\overline{s}^2}\beta(s)\right\| = \sqrt{\left\langle\left(\frac{1}{\tau}\right)N,\left(\frac{1}{\tau}\right)N\right\rangle}$$

$$= 1$$

and we obtain

$$\overline{n} = -\left(\frac{1}{\tau}\right)N$$

$$\overline{\kappa} = 1.$$

So, we have

$$\overline{N} = -\left(\frac{\overline{\tau}}{\tau}\right)N.$$

Here we have to find the torsion of β and write it down.

On the other hand, we express

$$\overline{b} = \overline{t}\times\overline{n}$$

$$= \left(\frac{1}{\tau}B\right)\times\left(-\frac{1}{\tau}N\right)$$

$$= T.$$

Hence we get

$$\overline{B} = \overline{\tau}\,T.$$

Similarly, here we have to find the torsion of β and write it down.

For the torsion of β; from the definition of torsion, we can find the following calculations simply;

$$\overline{\tau} = \left\langle\frac{d}{d\overline{s}}\overline{n}\ ,\overline{b}\right\rangle$$

$$= \left\langle\frac{\kappa}{\tau}T - \frac{1}{\tau}B\ ,\ T\right\rangle$$

$$= \frac{\kappa}{\tau}$$

which completes the proof.

Using the above theorem, the following results are obtained.

Corollary 3.3: If α is a general helix, parametrized by arc-length parameter s, then the adjoint curve of α according to modified orthogonal frame with torsion is a circular helix.

Proof: From the Lancret's theorem we know that α is a general helix, then the ratio of its torsion and curvature is constant. From the relation between modified orthogonal frames of α and β (the adjoint curve of α) ,we know

$$\overline{\kappa} = 1$$

$$\overline{\tau} = \frac{\kappa}{\tau}$$

which completes the proof.

Corollary 3.4: If β is the adjoint curve of α (not a general helix) according to modified orthogonal frame, then β is a Salkowski curve.

Proof: From the equations in theorem 3.2, we see that curvature of β is constant and torsion of β is $\frac{\kappa}{\tau}$. If the curve α is not a general helix, then $\frac{\kappa}{\tau}$ is not constant. So the proof is complete.

4. Adjoint Curve According to Modified Orthogonal Frame with Torsion in Minkowski 3-space

Definition 4.1: Let α be a unit speed regular curve with curvature $\kappa \neq 0$ and torsion $\tau \neq 0$ in E_1^3 . If the modified orthogonal frame with torsion of α is $\{T, N, B\}$ then the adjoint curve of α is defined as

$$\beta(s) = \int B(s)\, ds.$$

Theorem 4.2: Let α be a unit speed time-like Frenet curve in E_1^3 and β be the adjoint curve of α according to modified orthogonal frame with torsion. If the modified orthogonal frames of α and β are $\{T, N, B\}$ and $\{\overline{T}, \overline{N}, \overline{B}\}$, the curvature and torsion are κ, τ and $\overline{\kappa}$, $\overline{\tau}$ respectively, then the following relations hold:

$$\overline{T} = \frac{1}{\tau} B$$

$$\overline{N} = \frac{\kappa}{\tau^2} N$$

$$\overline{B} = \frac{\kappa}{\tau} T$$

$$\overline{\kappa} = 1$$

$$\overline{\tau} = -\frac{\kappa}{\tau}$$

and β is a unit speed space-like Frenet curve of type1.

Proof: Let the Frenet apparatus of α and β are $\{t, n, b, \kappa, \tau\}$ and $\{\overline{t}, \overline{n}, \overline{b}, \overline{\kappa}, \overline{\tau}\}$.

By the definition of adjoint curve we know that

$$\beta(s) = \int B(s))ds$$

Differentiating this equation with respect to s , we get

$$\left(\frac{d\beta(s)}{d(s)}\right) = B(s)$$

$$\left\langle\frac{d\beta(s)}{ds}, \frac{d\beta(s)}{ds}\right\rangle = \langle B, B\rangle = \tau^2.$$

From the last equation we see that the tangent vector of β is a spacelike vector. So the adjoint curve β is a spacelike curve.

Now, in order to determine the modified orthogonal frame with torsion of β, we make some appropriate calculations. Then we have

$$\overline{t} = \frac{1}{\tau}B \Rightarrow \overline{T} = \frac{1}{\tau}B$$

$$\overline{n} = -\frac{1}{\tau}N \Rightarrow \overline{N} = \frac{\kappa}{\tau^2}N$$

$$\overline{b} = -T \Rightarrow \overline{B} = \frac{\kappa}{\tau}T$$

$$\overline{\kappa} = \left\langle\left(\frac{d}{d\overline{s}}\right)\overline{t}, \overline{n}\right\rangle \Rightarrow \overline{\kappa} = 1$$

$$\tau = \left\langle\left(\frac{d}{d\overline{s}}\right)\overline{n}, b\right\rangle \Rightarrow \overline{\tau} = -\frac{\kappa}{\tau}$$

which completes the proof.

Theorem 4.3: Let α be a unit speed space-like Frenet curve of type1 in E_1^3 and β be the adjoint curve of α according to modified orthogonal framewith torsion. If the modified orthogonal frames of α and β are $\{T, N, B\}$ and $\{\overline{T}, \overline{N}, \overline{B}\}$, the curvature and torsion are κ, τ and $\overline{\kappa}$, $\overline{\tau}$ respectively, then the following relations hold:

$$\overline{T} = \frac{1}{\tau}B$$

$$\overline{N} = -\frac{\kappa}{\tau^2}N$$

$$\overline{B} = -\frac{\kappa}{\tau}T$$

$$\overline{\kappa} = 1$$

$$\overline{\tau} = -\frac{\kappa}{\tau}$$

and β is a unit speed time-like Frenet curve.

Proof: The proof can be made in exactly the same way as the previous theorem. It is easily seen that the adjoint of the space-like Frenet curve of type1 is a time-like curve.

Theorem 4.4: Let α be a unit speed space-like Frenet curve of type2 in E_1^3 and β be the adjoint curve of α according to modified orthogonal frame with torsion. If the modified orthogonal frames of α and β are $\{T, N, B\}$ and $\{T, N, B\}$, the curvature and torsion are κ, τ and κ, τ respectively, then the following relations hold:

$$\overline{T} = \frac{1}{\kappa} B$$

$$\overline{N} = \frac{\tau}{\kappa} N$$

$$\overline{B} = -\frac{\tau}{\kappa} T$$

$$\overline{\kappa} = \frac{\tau}{\kappa}$$

$$\overline{\tau} = -1.$$

and β is a unit speed space-like Frenet curve of type2.

Proof: The proof can be made in exactly the same way as the theorem 4.2. It is easily seen that the adjoint of the space-like Frenet curve of type2 is a space-like curve of type2.

5. References

1. Kelleci A. Conjugate mates for non-null Frenet curves. Sakarya University Journal of Science, 2009; 23(4), 600-604, DOI: 10.16984/saufenbilder.494471.
2. Bukcu B, Karacan MK. On the modified orthogonal frame with curvature and torsion in 3-space. Mathematical Sciences and Applications E-Notes 2016; 4.
3. Bukcu B, Karacan MK. Spherical curves with modified orthogonal frame. Journal of New Results in Science 2016;10: 60-68.
4. Choi J, Kim YH. Associated curves of a Frenet curve and their applications, Appl. Math.Comput, 2012; 218: 9116-9124.
5. Desmukh S, Chen BY. and Alghanemi A. Natural mates of Frenet curves in Euclidean 3-space, Turk. J. Math, 2018; 42: 2826-2840.
6. Do C, Manfredo P. Differential geometry of curves and surfaces. Prentice-Hall, 1976.
7. Lone MS, Es H, Karacan MK, Bukcu B. On some curves with Modified orthogonal frame in Euclidean 3-space. Iranian Journal of Science and Technology, Transactions A, Science (in press). doi.org/10.1007/s40995-018-0661-2.
8. Lone MS, Es H, Karacan MK, Bukcu B. Mannheim curves with modified orthogonal frame in Euclidean 3-space. Turkish Journal of Mathematics. 2019; 43(2): 648-663.

9. Nurkan SK, Güven IA, KaracanMK. Characterizations of adjoint curves in Euclidean 3-space. Proc Natl Acad Sci. India Sect A Phys Sci. https://doi.org/10.1007/s40010-017-0425-y.
10. Sasai T. The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations. Tohoku Math Journal 1984; 36: 17-24.