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Regional Controllability for Caputo Type Semi-Linear Time-Fractional Systems.

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Abstract

The main purpose of this paper, is to study the regional controllability concept of a semi-linear time-fractional diffusion systems involving Caputo derivative of order $\alpha \in (0,1)$. The main result is obtained by using an extension of the Hilbert Uniqueness Method (HUM) in addition to a fixed point technique and under several assumptions on the data of the considered equation. At the end, some numerical simulations are given to illustrate the efficiently of our result.

Keywords: Regional Controllability Fractional Calculus Caputo Time-Fractional Systems Fixed Point Theorems HUM Approach Compact Operators

1. Introduction

Controllability is one of the fundamental concepts in the field of control theory, it plays a central role in the analysis and control of both finite and infinite dimensional systems ([5],[2]). There are several faces to this concept, for instance, exact controllability, null controllability and approximate controllability, the most adequate one in applications is the approximate controllability which consists of steering a system into an arbitrary small neighborhood of finite state from an arbitrary initial state, several researchers studied this concept for systems which are represented by linear and nonlinear evolution equations, in particular there have been many papers on the approximate controllability of semi-linear systems, using several approach, for example the Hilbert Uniqueness Method (HUM) introduced by Lions where the fixed point theory and the semi-group theory are effectively used ([11],[12]).

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From a practical point of view, it is needed to control such systems only in a subregion of its evolution domain, this is the aim of regional controllability. This concept has been widely developed using partial differential equations and has some interesting results ([18],[19]).

In the last decades, a considerable interest has been shown in the so-called fractional calculus, which is a generalization of integer order integration and differentiation to arbitrary order . Different forms of fractional operators have been introduced by Riemann-Liouville and Caputo along time [10].

Fractional partial differential equations (FPDEs) have many applications in physics, chemistry, engineering, aerodynamics, biology, finance, control, for example the viscoelastic behavior of geological strata and of metals and glasses have been modeled by Caputo derivative ([16]). Due to the memory character of fractional derivative, that can describe many phenomena that integer derivative cannot characterize like the anamalous diffusion models. Several researchers studied the existence of mild solutions of fractional systems which is based on the probability density function ([9], [4],[20],[21] and the references therein), Ren and Mahmudov [15] investigated the fractional differential equations . Wang and Zhou [17] studied the optimal control for a class of controllability for a class of semi-linear fractional systems in Banach space. Duraisamy et al. [3] investigated the controllability problem for a class of fractional impulsive evolution systems of mixed type in an infinite dimensional Banach space by a new estimation technique of the measure of noncompactness.

Several authors have established the regional controllability (internal, boundary, gradient...) results for linear time-fractional diffusion systems ([6],[7],[8],[1]).

The motivation of this work rose from both the development of regional analysis and fractional calculus, especially for the semi-linear fractional equations.

The rest of this work is organized as follows. In section 2 we present some basic definitions of fractional operators, in section 3 we present the problem statement, some properties and the mathematical concepts of the regional controllability problem. In section 4 we study the optimal control using HUM approach for time fractional semi-linear systems and we finish by given an algorithm and a successful numerical application in the last section.

2. Some Basic Definitions

In this section, we introduce the definition of some fractional operators (fractional integrals, fractional derivatives), we also give some results which will be used throughout this paper.

Definition 2.1. *[10] The left (resp. right) sided fractional integral of a function y at a point t of order $\alpha \in]0,1]$ can be written as*

$$\mathrm{I}^{\alpha}_{0^+}y(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \qquad 0<t\leq T,$$

respectively

$$\mathrm{I}^{\alpha}_{T^-}y(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}y(s)ds, \qquad 0\leq t< T.$$

Definition 2.2. *[10] We define the left (resp. right) Riemann-Liouville fractional derivative of y at a point t of order $\alpha \in]0,1]$ with the formula*

$${}^{RL}\mathrm{D}^{\alpha}_{0^+}f(t) \quad = \quad \frac{d}{dt}\mathrm{I}^{1-\alpha}_{0^+}f(t) \qquad 0<t\leq T,$$

respectively

$${}^{RL}\mathrm{D}^{\alpha}_{T^-}f(t) \quad = \quad -\frac{d}{dt}\mathrm{I}^{1-\alpha}_{T^-}f(t) \qquad 0\leq t< T.$$

Definition 2.3. *[10] The Caputo fractional derivative (left sided) of y at a point t of order $\alpha \in]0,1]$ is defined by the following equations :*

$${}^{C}\mathrm{D}^{\alpha}_{0^+}y(t) = \mathrm{I}^{1-\alpha}_{0^+}\frac{d}{dt}(y(t)) \qquad 0\leq t< T. \tag{1}$$

We recall the following proposition.

Proposition 2.4. *([10],[6]) Let φ be a function defined on $[0,T]$. We define the reflexion operator of a function φ, denoted $\mathcal{Q}$, by*

$$(\mathcal{Q}\varphi)(t) = \varphi(T-t),$$

then we have the two following results

$$\mathcal{Q}\mathrm{I}^{\alpha}_{T^-}\varphi(t) = \mathrm{I}^{\alpha}_{0^+}\mathcal{Q}\varphi(t) \qquad \mathcal{Q}^{RL}\mathrm{D}^{\alpha}_{T^-}\varphi(t) =^{RL} \mathrm{D}^{\alpha}_{0^+}\mathcal{Q}\varphi(t). \tag{2}$$

3. Problem Statement

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. For a time $T>0$, let $Q=\Omega\times]0,T]$ and $\Sigma=\partial\Omega\times]0,T]$, then we consider the following Fractional diffusion semi-linear system of order $\alpha\in]0,1]$:

$$\left\{\begin{array}{lcllc} {}^{C}D^{\alpha}_{0^+}y(x,t) & = & Ay(x,t)+Ny(x,t)+Bu(t) & \text{in} & Q \\ y(\xi,t) & = & 0 & \text{on} & \Sigma \\ y(x,0) & = & y_0(x) & \text{in} & \Omega \end{array}\right. \tag{3}$$

Where ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative of order α defined by (1), A is the infinitesimal generator of a C_0 semi-group $\{S(t)\}_{t\geq 0}$ on the Hilbert space $\mathrm{X}=L^2(\Omega)$, N a locally Lipschitz continuous nonlinear operator, B is bounded linear operator from $\mathbb{R}^p$ into X where p is the number of actuators, u is given in $U=L^2(0,T,\mathbb{R}^p)$ and $y_0\in \mathrm{X}$.
System (3) admits a mild solution in $\mathrm{C}(0,T;\mathrm{X})$ satisfying the following integral equation [21],[4]:

$$y_u(t) = S_\alpha(t)y_0 + \int_0^t (t-\tau)^{\alpha-1}\mathrm{K}_\alpha(t-\tau)[Ny(\tau)+Bu(\tau)]d\tau \tag{4}$$

where $\quad S_\alpha(t) = \displaystyle\int_0^\infty \phi_\alpha(\theta)S(t^\alpha\theta)d\theta, \qquad \mathrm{K}_\alpha(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)S(t^\alpha\theta)d\theta$
and

$$\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}W_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0.$$

The function ϕ_α is called "the Wright function" and its given by means of a probability density, W_α defined by :

$$W_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha).$$

We associate to system (3), the following linear system :

$$\left\{\begin{array}{lcllc} {}^{C}D^{\alpha}_{0^+}y(x,t) & = & Ay(x,t)+Bu(t) & \text{in} & Q \\ y(\xi,t) & = & 0 & \text{in} & \Sigma \\ y(x,0) & = & y_0(x) & \text{in} & \Omega \end{array}\right. \tag{5}$$

We recall some lemmas

Lemma 3.1. *[1](Fractional Green's formula) Let's consider $0 < \alpha \leq 1$, then for any $\Phi \in C^{\infty}(\overline{Q})$, we have*

$$\begin{aligned}\int_0^T \int_{\Omega} [{}^C D_{0+}^{\alpha} y(x,t) - Ay(x,t)]\Phi(x,t)dxdt &= \int_0^T \int_{\Omega} [{}^{RL} D_{T-}^{\alpha} \Phi(x,t) - A^*\Phi(x,t)]y(x,t)dxdt \\ &+ \int_{\Omega} y(x,T) \mathrm{I}_{T-}^{\alpha} \Phi(x,T)dx \\ &- \int_{\Omega} y(x,0) \mathrm{I}_{T-}^{\alpha} \Phi(x,0)dx \\ &+ \int_0^T \int_{\partial\Omega} \frac{\partial y(x,t)}{\partial \nu_A} \Phi(x,t) d\nu dt \\ &- \int_0^T \int_{\partial\Omega} y(x,t) \frac{\partial \Phi(x,t)}{\partial \nu_{A^*}} d\nu dt.\end{aligned}$$

Where A^* is the adjoint operator of A.

Lemma 3.2. *[21] For any $t \geq 0$, the operators $S_{\alpha}(t)$ and $\mathrm{K}_{\alpha}(t)$ are linear and bounded, i.e., there exist $M > 0$ such that*

$$\| S_{\alpha}(t) \|_{\mathcal{L}(\mathrm{X},\mathrm{X})} \leq M \quad \text{and} \quad \| \mathrm{K}_{\alpha}(t) \|_{\mathcal{L}(\mathrm{X},\mathrm{X})} \leq \frac{M\alpha}{\Gamma(1+\alpha)}. \tag{6}$$

Lemma 3.3. *[21] The operators $\{S_{\alpha}(t)\}_{t\geq 0}$ and $\{\mathrm{K}_{\alpha}(t)\}_{t\geq 0}$ are continous.*

Lemma 3.4. *[21] Let's consider $\alpha_1 \in]0, \alpha[$ and $t > 0$, consider the mapping*

$$\begin{array}{rcccl} h & : & [0,t[& \longrightarrow & \mathbb{R}^+ \\ & & s & \longmapsto & (t-s)^{\alpha-1}. \end{array}$$

Therefore

$$h(s) \in L^{\frac{1}{1-\alpha_1}}[0,t],$$

$$\|(t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}}[0,t]} = \frac{t^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}.$$

Where $a = \dfrac{\alpha-1}{1-\alpha_1}$.

Let ω be a non empty regular subset of Ω, then the restriction operator is defined by

$$\begin{array}{rccl} \chi_{\omega} : & L^2(\Omega) & \longrightarrow & L^2(\omega) \\ & y & \longmapsto & y_{|_{\omega}}, \end{array}$$

and we denote by χ_{ω}^* its adjoint.

We give the two following definitions.

Definition 3.5. *The system (3) is said to be exactly regionally controllable in ω (ω-controllable) at time T if for all $y_d \in L^2(\omega)$, there exist a control $u \in U$ such that $\chi_{\omega} y_u(T) = y_d$.*

Definition 3.6. *The system (3) is said to be approximately regionally controllable in ω (approximately ω-controllable) at time T if for all $y_d \in L^2(\omega)$, for all $\varepsilon > 0$, there exist a control $u \in U$ such that $\| \chi_{\omega} y_u(T) - y_d \|_{L^2(\omega)} \leq \varepsilon$.*

Question: Given a desired state "y_d", can we find a control u^* which steers the studies system (3) to y_d, only in a subregion ω of Ω ?

4. HUM Approach

The purpose of this section is to explore the Hilbert Uniqueness Method for fractional semi linear system, which is an extension of HUM approach [12] developed in the case of distributed semi linear system on [19] Let's consider

$$G = \left\{ f \in L^2(\Omega) \ f = 0 \ \text{ in } \ \Omega \setminus \omega \right\}$$

and

$$C = \{ h \in \mathrm{L}^2(\Omega) \ : h = 0 \,\text{in} \quad \omega \},$$

We have $G \subseteq C^{\perp}$.
For $g \in G$ we consider the auxiliary system

$$\begin{cases} \mathrm{Q}^{RL}\mathrm{D}^{\alpha}_{T^-}\varphi(t) = \mathrm{Q}\mathrm{A}^*\varphi(t) \quad t \in [0,T] \\ \lim\limits_{t\to 0^+} \mathrm{Q}\mathrm{I}^{1-\alpha}_{T^-}\varphi(t) = g, \end{cases} \tag{7}$$

by the relation (2) of proposition (2.4), system (7) is equivalent to

$$\begin{cases} {}^{RL}\mathrm{D}^{\alpha}_{0^+}\mathrm{Q}\varphi(t) = \mathrm{A}^*\mathrm{Q}\varphi(t) \quad t \in [0,T] \\ \lim\limits_{t\to 0^+} \mathrm{I}^{1-\alpha}_{0^+}\mathrm{Q}\varphi(t) = g, \end{cases} \tag{8}$$

which has the following mild solution [20] :

$$\varphi(t) = (T-t)^{\alpha-1}\mathrm{K}^*_{\alpha}(T-t)g. \tag{9}$$

Where K^*_{α} is the adjoint of K_{α} and it can be written as follows:

$$\mathrm{K}^*_{\alpha}(t) = \alpha \int_0^{\infty} \theta\phi_{\alpha}(\theta)S^*(t^{\alpha}\theta)d\theta.$$

Consider the system (3) controlled by $u(t) = \mathrm{B}^*\varphi(t)$

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha}_{0^+}y(t) = \mathrm{A}y(t) + Ny(t) + \mathrm{B}\mathrm{B}^*\varphi(t) \quad t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{10}$$

which, we decompose to the following three systems

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha}_{0^+}\psi_0(t) &=& \mathrm{A}\psi_0(t) \quad t \in]0,T] \\ \psi_0(0) &=& y_0, \end{cases} \tag{11}$$

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha}_{0^+}\psi_1(t) &=& \mathrm{A}\psi_1(t) + \mathrm{B}\mathrm{B}^*\varphi(t) \quad t \in]0,T] \\ \psi_1(0) &=& 0, \end{cases} \tag{12}$$

$$\begin{cases} {}^{C}\mathrm{D}^{\alpha}_{0^+}\psi_2(t) &=& \mathrm{A}\psi_2(t) + N(\psi_0+\psi_1+\psi_2) \quad t \in]0,T] \\ \psi_2(0) &=& 0. \end{cases} \tag{13}$$

For a given $g \in G$ we define the mapping:

$$|| \,.\, ||_G : \quad g \quad \longmapsto \quad || \ B^*\varphi(.) \ ||_{L^2(0,T;\mathbb{R}^p)} \ .$$

We have the following lemma.

Lemma 4.1. *[8] If the linear system (5) is approximately ω-controllable then $||\,.\,||_G$ define a norm in G*

We denote the completion of G with respect to norm $||\ .\ ||_G$ again by G.
Let $\mu : G \to C^{\perp}$ be the nonlinear operator defined by

$$\mu g = \Lambda g + \mathrm{K} g,$$

where $\Lambda g = \mathrm{P}(\psi_1(T))$, $\mathrm{K}g = \mathrm{P}(\psi_2(T))$ and $\mathrm{P} = \chi_\omega^* \chi_\omega$.
The problem of regional controllability of system (3) is reduced to the equation

$$\mu g = \chi_\omega^* y_d - \mathrm{P}(\psi_0(T)),$$

which is equivalent to

$$\Lambda g = \chi_\omega^* y_d - \mathrm{P}(\psi_0(T)) - \mathrm{K}g. \tag{14}$$

If the linear system (5) is approximately ω-controllable, then Λ is isomorphism ([6]), in this case, by applying the inverse operator of Λ to equation (14), we have

$$g = \Lambda^{-1}\chi_\omega^* y_d - \Lambda^{-1}\mathrm{P}(\psi_0(T)) - \Lambda^{-1}\mathrm{K}g.$$

Now we define the operator

$$\tilde{\mathrm{K}}(g) = \Lambda^{-1}\chi_\omega^* y_d - \Lambda^{-1}\mathrm{P}(\psi_0(T)) - \Lambda^{-1}\mathrm{K}g. \tag{15}$$

Then the ω-controllability of system (3) under certain conditions becomes a problem of finding a fixed point of the operator $\tilde{\mathrm{K}}$.
Under the following conditions:
(H_1) $\alpha \in \left]\frac{1}{2}, 1\right]$.

(H_2) The linear system (5) is approximately $\omega-$ controllable.
(H_3) The nonlinear operator N satisfies the condition

$$||\ N(x)\ ||_{L^2(0,T;X)} \leq c\ ||\ x\ ||^2_{L^2(0,T;X)} \qquad 0 < c \leq 1, \tag{16}$$

We obtain the following theorem.

Theorem 4.2. *Let φ defined by (9) and g the initial state of system (8).*
If the hypotheses (H_1)-(H_3) are satisfied, then g is a unique fixed point of operator $\tilde{\mathrm{K}}$ given by formula (15).
Therefore $u^*(t) = \mathrm{B}^*\varphi(t)$ steers the system (3) to the desired regional state y_d in ω at $t = T$.

Proof. The proof of this theorem is technical, therefore it is convenient to divide it into two steps:
Step 1: We prove that $\tilde{\mathrm{K}}$ is a compact operator, so it is sufficient to prove that K is a compact operator.
Let's consider $k > 0$ and a set

$$B_k = \{f \in G \quad | \quad ||f||_G \leq k\}.$$

We have

$$\mathrm{K}(B_k) = \{\mathrm{P}(\psi_2(T) \quad | \quad g \in B_k)\},$$

where g is the initial state of system (8).
Remarque that:

$$\mathrm{K}(B_k) \subseteq \{\mathrm{P}(\psi_2(t) \quad / \quad g \in B_k)\} := \tilde{B_k}.$$

Then it is sufficient to show that $\tilde{B_k}$ is relatively compact.
Since $\psi_2(.) \in \mathrm{C}(0, T; \mathrm{X})$ is a mild solution of system (13), we have

$$\psi_2(t) = \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) N[\psi_0(s) + \psi_1(s) + \psi_2(s)] ds \quad t > 0, \tag{17}$$

also there exists $c_p > 0$ such that

$$||\mathrm{P}(\psi_2(t)||_{C^\perp} \leq c_p||\psi_2(t)||_{\mathrm{X}}.$$

$*$ We show that $\tilde{B_k}$ is uniformly bounded.

From the integral equation (17) and lemma (3.2), we obtain

$$\begin{aligned} ||\psi_2(t)||_{\mathrm{X}} &\leq \int_0^t ||(t-s)^{\alpha-1}K_\alpha(t-s)N[\psi_0(s)+\psi_1(s)+\psi_2(s)]||_{\mathrm{X}}ds \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}||N[\psi_0(s)+\psi_1(s)+\psi_2(s)]||_{\mathrm{X}}ds \end{aligned}$$

Using lemma (3.4) we get $(t-s)^{\alpha-1} \in L^2[0,t]$, also by the condition (16) and the Cauchy-Schwarz Inequality, we obtain

$$||\psi_2(t)||_{\mathrm{x}} \leq \frac{M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} \parallel \psi_0(.)+\psi_1(.)+\psi_2(.)\parallel^2_{L^2(0,T;\mathrm{X})},$$

Then by Minkowski's Inequality and Young Inequality, we obtain

$$\parallel \psi_0(.)+\psi_1(.)+\psi_2(.)\parallel^2_{L^2(0,T;\mathrm{X})} \leq 3\left[\parallel \psi_0(.)\parallel^2_{L^2(0,T;\mathrm{X})}+\parallel \psi_1(.)\parallel^2_{L^2(0,T;\mathrm{X})}+\parallel \psi_2(.)\parallel^2_{L^2(0,T;\mathrm{X})}\right].$$

Hence

$$||\psi_2(t)||_{\mathrm{x}} \leq \frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left[\parallel \psi_0(.)\parallel^2_{L^2(0,T;\mathrm{X})}+\parallel \psi_1(.)\parallel^2_{L^2(0,T;\mathrm{X})}+\parallel \psi_2(.)\parallel^2_{L^2(0,T;\mathrm{X})}\right]. \quad (18)$$

Since ψ_0 and ψ_1 are, respectively, solution of system (11) and (12), we have

$$\psi_0(s) = S_\alpha(s)y_0 \quad \text{for all} \qquad s>0$$

$$\psi_1(s) = \int_0^s (s-\tau)^{\alpha-1}K_\alpha(s-\tau)BB^*\varphi(\tau)d\tau \quad \text{for all} \quad s>0,$$

Using lemma (3.2)

$$\parallel \psi_0(.)\parallel^2_{L^2(0,T;\mathrm{X})} \leq TM^2 \parallel y_0\parallel^2_{\mathrm{X}}. \quad (19)$$

We also have

$$\parallel \psi_1(s)\parallel_{\mathrm{X}} \leq \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^s (s-\tau)^{\alpha-1}\parallel BB^*\varphi(\tau)\parallel_{\mathrm{X}} ds$$

using Cauchy-Schwartz inequality and lemma (3.4)

$$\begin{aligned} \parallel \psi_1(s)\parallel_{\mathrm{X}} &\leq \frac{M\alpha}{\Gamma(1+\alpha)}\frac{s^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}M_1\parallel B^*\varphi(\tau)\parallel_{L^2(0,T;\mathbb{R}^p)} \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}M_1\parallel g\parallel_G, \end{aligned}$$

where $M_1 = ||B||_{\mathcal{L}(\mathrm{X},\mathbb{R}^p)}$, this implies that

$$\parallel \psi_1(.)\parallel^2_{L^2(0,T;\mathrm{X})} \leq \left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2\frac{T^{2\alpha}}{(2\alpha-1)}M_1^2\parallel g\parallel^2_G. \quad (20)$$

Substituting (19) and (20) in (18), we get

$$\begin{aligned} ||\psi_2(t)||_{\mathrm{x}} &\leq \frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left[TM^2\parallel y_0\parallel^2_{\mathrm{X}}+\left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2\frac{T^{2\alpha}}{(2\alpha-1)}M_1^2\parallel g\parallel^2_G\right] \\ &+ \quad \frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\int_0^t \parallel \psi_2(s)\parallel^2_{\mathrm{x}} ds, \end{aligned}$$

and under the assumption

$$A_c(g) := T\left[\frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\right]^2\left[TM^2 \mid\mid y_0 \mid\mid_{\mathrm{X}}^2 + \left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2\frac{T^{2\alpha}}{(2\alpha-1)}M_1^2 \mid\mid g \mid\mid_G^2\right] < 1$$

by the generalization of Gronwall's lemma (theorem 2.2, [13]), we obtain

$$||\psi_2(t)||_{\mathrm{X}} \leq \frac{A_c(g)}{T(1-A_c(g))}. \tag{21}$$

Therefore,

$$\begin{aligned}\sup_{||g||_G\leq k}||\mathrm{P}(\psi_2(t))||_{C^\perp} &\leq \frac{A_c(k)c_p\Gamma(1+\alpha)(2\alpha-1)^{\frac{1}{2}}}{3M\alpha c T^{\alpha+\frac{1}{2}}(1-A_c(k))}\\ &< +\infty.\end{aligned}$$

Hence $\tilde{B_k}$ is uniformly bounded.

$*$ Let us show that $\tilde{B_k}$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq T$,for any $g \in \mathrm{B}_k$, we have

$$\begin{aligned}\psi_2(t_2)-\psi_2(t_1) &= \int_0^{t_2}(t_2-s)^{\alpha-1}\mathrm{K}_\alpha(t_2-s)N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\mathrm{K}_\alpha(t_1-s)]N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\\ &= \int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathrm{K}_\alpha(t_2-s)N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\\ &+ \int_0^{t_1}(t_1-s)^{\alpha-1}[\mathrm{K}_\alpha(t_2-s)-\mathrm{K}_\alpha(t_1-s)]N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\\ &+ \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{K}_\alpha(t_2-s)N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\\ ||\psi_2(t_2)-\psi_2(t_1)||_{\mathrm{X}} &\leq \mathrm{T}_1+\mathrm{T}_2+\mathrm{T}_3.\end{aligned}$$

Where

$$\begin{aligned}\mathrm{T}_1 &= ||\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathrm{K}_\alpha(t_2-s)N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\,||_{\mathrm{X}}\\ \mathrm{T}_2 &= ||\int_0^{t_1}(t_1-s)^{\alpha-1}[\mathrm{K}_\alpha(t_2-s)-\mathrm{K}_\alpha(t_1-s)]N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\,||_{\mathrm{X}}\\ \mathrm{T}_3 &= ||\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{K}_\alpha(t_2-s)N(\psi_0(s)+\psi_1(s)+\psi_2(s))ds\,||_{\mathrm{X}}\,.\end{aligned}$$

We have

$$\begin{aligned}\mathrm{T}_1 &\leq \frac{\alpha M}{\Gamma(1+\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|||N(\psi_0(.)+\psi_1(.)+\psi_2(.))||_{\mathrm{X}}ds\\ &\leq \frac{\alpha M}{\Gamma(1+\alpha)}||(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}||_{L^2[0,t_1]}||N(\psi_0(.)+\psi_1(.)+\psi_2(.))||_{L^2(0,t_1;\mathrm{X})}\\ &\leq \frac{\alpha Mc}{\Gamma(1+\alpha)}\left[||(t_2-s)^{\alpha-1}||_{L^2[0,t_1]}+||(t_1-s)^{\alpha-1}||_{L^2[0,t_1]}\right]||\psi_0(.)+\psi_1(.)+\psi_2(.)||^2_{L^2(0,T;\mathrm{X})}\end{aligned}$$

and

$$\begin{aligned}\mathrm{T}_3 &\leq \frac{\alpha M}{\Gamma(1+\alpha)}\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}|||N(\psi_0(.)+\psi_1(.)+\psi_2(.))||_{\mathrm{X}}ds\\ &\leq \frac{\alpha M}{\Gamma(1+\alpha)}||(t_2-s)^{\alpha-1}_{L^2[t_1,t_2]}||N(\psi_0(.)+\psi_1(.)+\psi_2(.))||_{L^2(t_1,t_2;\mathrm{X})}\\ &\leq \frac{\alpha Mc}{\Gamma(1+\alpha)}||(t_2-s)^{\alpha-1}||_{L^2[t_1,t_2]}||\psi_0(.)+\psi_1(.)+\psi_2(.)||^2_{L^2(0,T;\mathrm{X})}\end{aligned}$$

hence by inequalities (19), (20) and (21), we have

$$||\psi_0(.) + \psi_1(.) + \psi_2(.)||^2_{L^2(0,T;\mathrm{X})} \leq \mathcal{M}$$

where

$$\begin{aligned}\mathcal{M} &= 3TM^2 \parallel y_0 \parallel^2_{\mathrm{X}} + 3\left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2 \frac{T^{2\alpha}}{(2\alpha-1)} M_1^2 \parallel g \parallel^2_G \\ &+ 3T\left[\frac{\frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left[TM^2 \parallel y_0 \parallel^2_{\mathrm{X}} + \left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2 \frac{T^{2\alpha}M_1^2 \parallel g \parallel^2_G}{(2\alpha-1)}\right]}{1 - T\left[\frac{3M\alpha c}{\Gamma(1+\alpha)}\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\right]^2 \left[TM^2 \parallel y_0 \parallel^2_{\mathrm{X}} + \left[\frac{M\alpha}{\Gamma(1+\alpha)}\right]^2 \frac{T^{2\alpha}M_1^2 \parallel g \parallel^2_G}{(2\alpha-1)}\right]}\right]^2\end{aligned}$$

and by lemma (3.4), we obtain

$$\mathrm{T}_1 \leq \frac{\alpha Mc}{\Gamma(1+\alpha)} \frac{(t_2-t_1)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\mathcal{M}.$$

$$\mathrm{T}_3 \leq \frac{\alpha Mc}{\Gamma(1+\alpha)} \frac{(t_2-t_1)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\mathcal{M}.$$

Therefore

$$T_1 \underset{t_2-t_1 \longrightarrow 0}{\longrightarrow} 0 \text{ and } T_3 \underset{t_2-t_1 \longrightarrow 0}{\longrightarrow} 0.$$

For T_2

If $t_1 = 0,\ 0 < t_2 \leq T$, we have $T_2 = 0$.

For $t_1 > 0$ and $\varepsilon > 0$ small enough independent of the choose the function g, we obtain

$$\begin{aligned}\mathrm{T}_2 &\leq \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1} \parallel \mathrm{K}_\alpha(t_2-s) - \mathrm{K}_\alpha(t_1-s)||_{\mathrm{X}}||N(\psi_0(.) + \psi_1(.) + \psi_2(.))||_{\mathrm{X}} \\ &+ \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1} \parallel \mathrm{K}_\alpha(t_2-s) - \mathrm{K}_\alpha(t_1-s)||_{\mathrm{X}}||N(\psi_0(.) + \psi_1(.) + \psi_2(.))||_{\mathrm{X}} \\ &\leq \frac{c\mathcal{M}(t_1^{2(\alpha-1)} - \varepsilon^{2(\alpha-1)})^{\frac{1}{2}}}{\Gamma(1+\alpha)(2\alpha-1)^{\frac{1}{2}}} \underset{s\in[0,t_1-\varepsilon]}{sup} \parallel \mathrm{K}_\alpha(t_2-s) - \mathrm{K}_\alpha(t_1-s)||_{\mathrm{X}} + \frac{2\alpha Mc\mathcal{M}\varepsilon^{\alpha-\frac{1}{2}}}{\Gamma(1+\alpha)(2\alpha-1)^{\frac{1}{2}}}.\end{aligned}$$

By using the continuity of $\mathrm{K}_\alpha(t)$ (lemma (3.3)) we obtain

$$T_2 \underset{\substack{\varepsilon \longrightarrow 0 \\ t_2-t_1 \longrightarrow 0}}{\longrightarrow} 0.$$

Then

$$||\mathrm{P}(\psi_2(t_2) - \mathrm{P}(\psi_2(t_1))||_{C^\perp} \leq c_p[\mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3]$$

We obtain

$$||\mathrm{P}(\psi_2(t_2) - \mathrm{P}(\psi_2(t_1))||_{C^\perp} \underset{\varepsilon, t_2-t_1 \longrightarrow 0}{\longrightarrow} 0,.$$

We have proved that $\tilde{B_k}$ is uniformly bounded and equicontinuous, then by ArzÃĺla-Ascoli theorem $\tilde{B_k}$ is relatively compact, therefore $\mathrm{K}(B_k)$ is relatively compact, then the operator K is compact which gives $\tilde{\mathrm{K}}$ is compact.

Step 2. We proof that $\tilde{\mathrm{K}}(B_k) \subset B_k$.
From (15) and (21), we have

$$\begin{array}{rcl} \|\ \tilde{\mathrm{K}}(g)\ \|_G & \leq & \|\ \Lambda^{-1}\chi_\omega^* y_d - \Lambda^{-1}\mathrm{P}(\psi_0(T))\ \|_G + \|\ \Lambda^{-1}\mathrm{K}g\ \|_G \\ & \leq & \|\ \Lambda^{-1}\chi_\omega^* y_d - \Lambda^{-1}\mathrm{P}(\psi_0(T))\ \|_G + \|\ \Lambda^{-1}\ \|_{\mathcal{L}(C^\perp,G)} \\ & \times & \dfrac{\dfrac{3M\alpha c c_p}{\Gamma(1+\alpha)}\dfrac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left[TM^2\ \|\ y_0\ \|_{\mathrm{X}}^2 + \left[\dfrac{M\alpha}{\Gamma(1+\alpha)}\right]^2 \dfrac{T^{2\alpha}M_1^2\ \|\ g\ \|_G^2}{(2\alpha-1)}\right]}{1 - T\left[\dfrac{3M\alpha c}{\Gamma(1+\alpha)}\dfrac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\right]^2\left[TM^2\ \|\ y_0\ \|_{\mathrm{X}}^2 + \left[\dfrac{M\alpha}{\Gamma(1+\alpha)}\right]^2 \dfrac{T^{2\alpha}M_1^2\ \|\ g\ \|_G^2}{(2\alpha-1)}\right]}. \end{array}$$

The last inequality imply that for large enough $k > 0$ we have

$$\forall g \in G \quad \text{such that} \quad \|\ g\ \|_G \leq k \quad \Longrightarrow \quad \|\ \tilde{\mathrm{K}}(g)\ \|_G \leq k.$$

Hence by Schauder's fixed point theorem we deduce that the operator $\tilde{\mathrm{K}}$ has a fixed point. □

Then we give the following algorithm

Algorithm

- **Step 1:** Initialization
  - The fractional order of derivative α
  - Initial state and desired state z_0, z_d.
  - The region ω.
  - Actuator (D, f)
  - Error estimate ϵ
- **Step 2: Repeat**
  - Choose φ_0.
  - Resolution of (8) and obtaining φ.
  - Resolution of (11) and obtaining ψ_0.
  - Resolution of (12) and obtaining ψ_1.
  - Resolution of (13) and obtaining ψ_2.
  - Resolutation of (14) and obtaining $\tilde{\mathrm{K}}(\varphi_0)$.

  Until $||\varphi_0 - \tilde{\mathrm{K}}(\varphi_0)|| \leq \epsilon$.
- **Step 3:** The control is $u^* =< \varphi(t), f >_{L^2(D)}$.

To test the efficiency of this algorithm, we give the following application.

5. Applications

To illustrate the effectiveness of the result above, we consider two examples with different data (fractional order derivative, the considered subregion, the desired state and the actuator structure).

5.1. Example 1:

Let's consider $\Omega = [0,1]$ and the one dimensional semilinear diffusion system described by:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+} z(x,t) & = & \frac{\partial^2}{\partial x^2} z(x,t) + \chi_D u(t) + \sum_{j=1}^{\infty} (< z, \varphi_j >)^2 \varphi_j(x) & \text{in } [0,1] \times]0,1] \\ z(\xi,t) & = & 0 & \text{on } \{0,1\} \times]0,1] \\ z(x,0) & = & 0 & \text{in } [0,1] \end{cases}$$

Where $\alpha = 0.6$, $D = [0.2, 0.4]$ and the sub-region under consideration is $\omega = [0.30, 0.55]$.
The operator $\frac{\partial^2}{\partial x^2}$ has complete system of eingenfunctions $\varphi_i(x) = \sqrt{2}\sin(i\pi x)$ ([14]) corresponding to the eigenvalues $\lambda_i = -i^2\pi^2$.
Let's consider

$$z_d(x) = \begin{cases} 0 & 0 \le x < 0.30 \\ 0.99 \times (x + 0.1) \times (0.9 - x) & 0.30 \le x \le 0.55 \\ 0 & 0.55 < x \le 1. \end{cases}$$

Using the previous algorithm the simulation gives the figure 1.
In figure 1 we remark that the desired state and the reached one are very close in w=[0.30,0.55], therefore

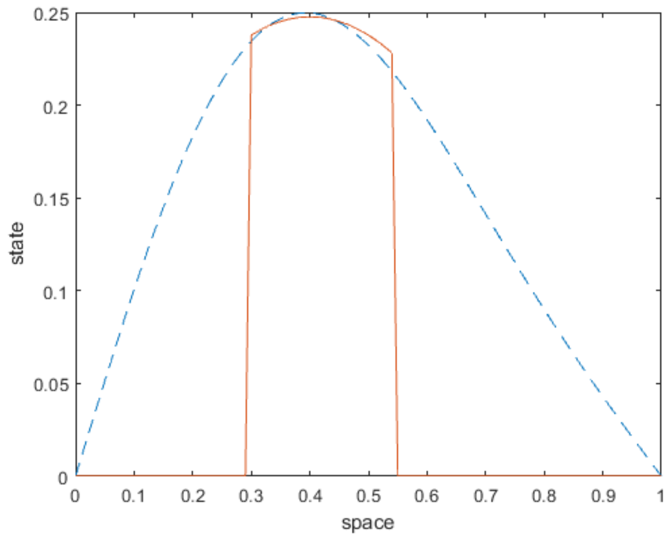

Figure 1: The desired state (continues line) and reached state (dashed line) in ω .

the regional desired state z_d is reached with error $\| \chi_\omega z_u(t) - z_d \|^2_{L^2(\omega)} = 2.0 \times 10^{-6}$.

The figure 2 shows the evolution of the control function with respect to time where the transfer cost $\| u^* \|^2_{L^2(0,T)} = 0.5$, we remark that the value of u doesn't exceed 6.
We give the following table which represent the evolution error-actuator support. We remark that the

Actuator support	**Error**
[0.1,0.3]	2×10^{-2}
[0.5,0.75]	5×10^{-2}
[0.25,0.75]	1.1×10^{-1}
[0.2,0.6]	2.1×10^{-1}

algorithm is very "affect" to the choose of the actuator support D.

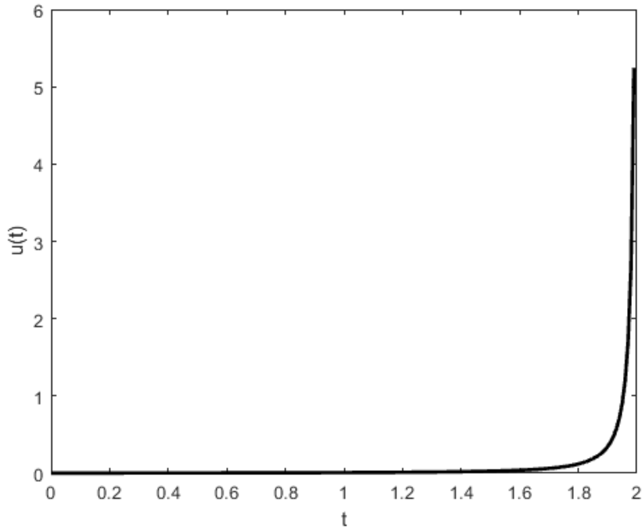

Figure 2: Control input function

Example 2:

In this example, we consider the following system excited by a zonal actuator:

$$\begin{cases} {}^C D^{0.7}_{0^+} z(x,t) &= \dfrac{\partial^2}{\partial x^2} z(x,t) + \chi_D u(t) + \displaystyle\sum_{j=1}^{\infty} (< z, \varphi_j >)^2 \varphi_j(x) & \text{in } [0,\pi] \times]0,2] \\ z(\xi,t) &= 0 & \text{on } \{0,\pi\} \times]0,2] \\ z(x,0) &= 0 & \text{in } [0,\pi] \end{cases}$$

Where $D = [0.2, 0.3]$ and the sub-region under consideration is $\omega = [1, 1.5]$.
Moreover, let

$$y_d(x) = -0.85\sqrt{x}(x-\pi)(x-2)(x-1),$$

the desired state in ω.
Using the algorithm above, we have the figure 3.

Figure 3 shows that the desired state is very close to the reached one on ω with the error 2.7×10^{-4} .

Remark 5.1. *When $\alpha = 1$, the semilinear system (5.1) is controllable in ω under the same conditions, this is the case of parabolic systems. This case demonstrate the advantage of this study in order to generalize the controllability results for ordinary systems.*

Conclusion

In this work we have extended the notion of regional controllability to Caputo time-fractional semilinear system using an extension of Hilbert Uniqueness method and we have validated the theoretical result with some numerical simulations, which are obtained with success and demonstrated the relevance of the regional approach and the fractional calculus. This result could provide some insight into the control theory analysis of fractional order system and can also be enlarged to the case of another fractional systems like Riemann-Liouville, Hadamard and Caputo-Fabrizio systems.

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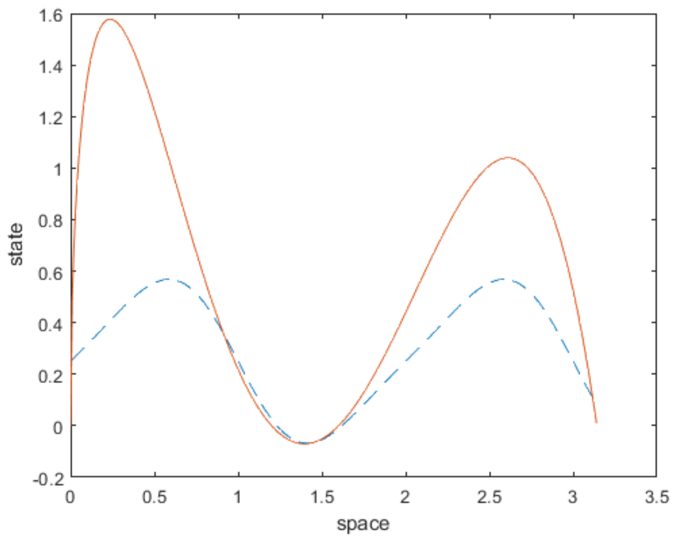

Figure 3: The desired state and reached state in ω .

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