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Generalized Spherical Fuzzy Einstein Aggregation Operators: Application to Multi-Criteria Group Decision-Making Problems

Elif Güner[1,*] *Halis Aygün*[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Turkey, ORCID: 0000-0002-6969-400X*
* *Corresponding Author E-mail: elif.guner@kocaeli.edu.tr*
[2] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Turkey, ORCID: 0000-0003-3263-3884*

Abstract: The aim of this paper is to present the extension of a concept related to aggregation operators from spherical fuzzy sets to generalized spherical fuzzy sets. We first introduce Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets based on Einstein triangular norm and triangular conorm. Then we give the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators, namely generalized spherical fuzzy Einstein aggregation operators, constructed on these operations. After investigating some fundamental properties of these operators, we develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multiple attribute group decision-making problems. Finally, we give a numerical example to demonstrate that the developed method is suitable and effective for the decision process.

Keywords: Generalized spherical fuzzy number, Einstein aggregation operators, Multi-criteria group decision making.

1 Introduction

In 1965, Zadeh [32] introduced the theory of fuzzy set (FS) in which is discussed the degree of membership (positive-membership) of an element to a set. FS theory has a wide range of applications in numerous fields such as artificial intelligence, engineering, economics, computer science and etc. [7]-[32]-[33]-[34]. Also, the study of multi-criteria decision-making was started in fuzzy environment by Bellman and Zadeh [7] in 1970. While the studies and developments in the field of FS theory were continued, Atanassov [6] observed that there are some deficiencies in this theory and defined the concept of intuitionistic fuzzy set (IFS) as a generalization of FS. Since each element is expressed by a positive-membership degree and a negative-membership degree in the IFS theory, this theory is a more powerful tool to deal with vagueness than the FS theory. Then basing on the well-known weighted averaging (WA) operator [16] and the ordered weighted averaging (OWA) operator [30], Xu [29] investigated some aggregation operators in the intuitionistic fuzzy environment and studied their applications to the multi-criteria decision-making process. Another idea which is an extension of the FS theory is the soft set (SS) theory introduced by Molodtsov [22] to deal with the vagueness and uncertainties of many problems that arise in engineering, social science, medical science, economics and etc. A first practical application of soft sets to the decision-making problems was given by Maji et al. [20]-[21]. They have also initiated the concept of a fuzzy soft set (FSS) which is a combination of FS and SS and obtained some of its properties. In literature, there are many significant applications of the SS and FSS theories. (see [5]-[9]-[24]-[25]).

After, Yager [31] extended the IFS theory to Pythagorean fuzzy set (PyFS) theory by considering the condition that the sum of the squares of its positive-membership and negative-membership degrees is less than or equal to 1. IFS theory and PyFS theory have been successfully used in many distinct areas, but it is not enough to use these theories when we face human opinions involving more answers of types such as yes, no, abstain, refusal. Voting in a democratic election is a good example of such a case since the voters may be divided into four groups of those who: vote for, vote against, abstain, refusal of the voting. So, Cuong [8] proposed the notion of picture fuzzy set (PFS) which is an extension of IFS. PFS gives the positive-membership degree, the neutral-membership degree and the negative-membership degree of an element to a set. The PFS theory resolved the voting problem successfully and was applied to decision-making problems by many authors in different ways [27]-[28]-[3]-[4]-[10]. But there are some situations that PFS theory may not be handled in some uncertain and unstable data. For example, if a person said their opinions about the situation in terms of yes is 0.7, abstained is 0.3, and no is 0.5, then we obtain that $0.7 + 0.5 + 0.3 \nleq 1$. So PFS theory is not able to handle under such types of cases. To use in these types of situations, Mahmood et al. [19] initiated the concept of spherical fuzzy set (SFS) and T-spherical fuzzy set (T-SFS) as an extension of FS, IFS and PFS. While in SFS theory the sum of the square of the three membership degrees is less than or equal to 1, in T-SFS theory the nth power of the three membership degrees is less than or equal to 1. Also, Mahmood et al. [19] defined some fundamental operations of SFSs and T-SFSs along with spherical fuzzy relations and presented medical diagnostics and decision-making problems in the SFSs and T-SFSs environments as practical applications. Then, Ashraf and Abdullah [1] extended different strict Archimedean triangular norms and triangular conorms to aggregate the spherical fuzzy information and also defined some spherical aggregation operators and applied these operators to multi-criteria group decisionâĂŘmaking problems. Different types of aggregation operators for SFSs can be found in [2]-[19]-[23]. Later, Jin et al. [17] proposed a new method to solve the spherical fuzzy multi-criteria group decision-making problems by investigating logarithmic operations of spherical fuzzy sets. Also, GÃijndogdu and Kahraman [12]-[13]-[14] extended the TOPSIS, VIKOR and WASPAS methods to the spherical fuzzy environment. Recently, Haque et al. [15] presented the notion of generalized spherical fuzzy set (GSFS) as an expansion of the SFS in which the sum of the square of the three membership degrees is less than or equal to 3. They established a new exponential operational law for GSFS and investigated its various algebraic properties. They also developed a multi-criteria group decision-making method in the generalized spherical fuzzy environment by using the established exponential operational law. Peng et al. [26] introduced the Pythagorean fuzzy soft set (PyFSS) along with various binary operations and also proposed an algorithm for decision making. Then, Cuong[8] proposed the notion of picture fuzzy soft set (PFSS) as a combination of PFS and SS and also discussed various properties and operations in the theory of PFSS. Guleria and Bajaj [11] extend the concept of PFSS by proposing the T-spherical fuzzy soft set (T-SFSS) along with various aggregation operators and applications.

The main purpose of this paper is to establish the generalized spherical fuzzy Einstein aggregation operators and develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multiple attribute group decision-making problems. This paper is contained in the following sections: In Section 2, we recollect some basic notions and relevant concepts that are used in the main section. In Section 3, we introduce Einstein sum, product and scalar multiplication for GSFSs based on Einstein triangular norm and triangular conorm. Then, we give the generalized spherical fuzzy Einstein weighted averaging (GSEWA) and generalized spherical fuzzy Einstein weighted geometric (GSEWG) operators, namely generalized spherical fuzzy Einstein aggregation operators, constructed on the Einstein sum, product and scalar multiplication for GSFSs. Also, we investigate some fundamental properties of these operators. In section 4, we develop a model to solve the multiple attribute group decision-making problems in generalized spherical fuzzy environment. Then, we give a medical treatment selection problem as an example which demonstrates that the developed method is effective and suitable for the decision-making process. Finally, we give a brief summary in Section 5.

2 Preliminaries

In this section, we recall some fundamental definitions which will be used in the main sections. Throughout this paper U will denote the set of the universe.

Definition 1. *[6]-[31] Let $\mu : U \to [0,1]$ and $\nu : U \to [0,1]$ be two mappings. A set $I = \{< x, \mu(x), \nu(x) > | x \in U\}$ is called*
(i) Intuitionistic fuzzy set (IFS) if the condition $0 \le \mu(x) + \nu(x) \le 1$ hold for all $x \in U$.
(ii) Pythagorean fuzzy set (PyFS) if the condition $0 \le \mu^2(x) + \nu^2(x) \le 1$ hold for all $x \in U$.
The values $\mu(x), \nu(x) \in [0,1]$ denote the degree of positive-membership and negative-membership of x to I, respectively.
The pair $I =< \mu, \nu >$ where $\mu, \nu \in [0,1]$ and $\mu + \nu \le 1$ (or $\mu^2 + \nu^2 \le 1$), is called a intuitionistic fuzzy number (IFN) (or Pythagorean fuzzy number (PyFN)).

Remark 1. *[31] The set of intuitionistic fuzzy numbers is the subset of the set of Pythagorean fuzzy numbers.*

Definition 2. *[1]-[8]-[15] Let $\mu : U \to [0,1]$, $\iota : U \to [0,1]$ and $\nu : U \to [0,1]$ be three mappings. A set $G = \{< x, \mu(x), \iota(x), \nu(x) > | x \in U\}$ is called*
(i) Picture fuzzy set (PFS) if the condition $0 \le \mu(x) + \iota(x) + \nu(x) \le 1$ hold for all $x \in U$.
(ii) Spherical fuzzy set (SFS) if the condition $0 \le \mu^2(x) + \iota^2(x) + \nu^2(x) \le 1$ hold for all $x \in U$.
(ii) Generalized spherical fuzzy set (GSFS) if the condition $0 \le \mu^2(x) + \iota^2(x) + \nu^2(x) \le 3$ hold for all $x \in U$.
The values $\mu(x), \iota(x), \nu(x) \in [0,1]$ denote the degree of positive-membership, neutral-memberhip and negative-membership of x to G, respectively.
The triplet $G =< \mu, \iota, \nu >$ where $\mu, \iota, \nu \in [0,1]$ and $\mu^2 + \iota^2 + \nu^2 \le 3$ (or $\mu + \iota + \nu \le 1$ and $\mu^2 + \iota^2 + \nu^2 \le 1$, resp.), is called a generalized spherical fuzzy number (GSFN) (or Picture fuzzy number (PFN) and Spherical fuzzy number (SFN), resp.).

Remark 2. *[15] (1) The set of spherical fuzzy numbers is the subset of the set of generalized spherical fuzzy numbers and the set of picture fuzzy numbers is the subset of the set of spherical fuzzy numbers.*
(2) In PFN, since the sum of the three membership functions (positive, neutral and negative) is less than or equal to 1, the sum is taken as linearly and it represents a plane in space. But in the case of SFN and GSFN, it is considered the non-linear form of membership functions which represents a sphere in space.

Definition 3. *[15] Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \ge 0$. Then the operations between generalized spherical fuzzy numbers are defined as follows:*
(i) $G^c =< \nu, \iota, \mu >$,
(ii) $G_1 \le G_2$ iff $\mu_1 \le \mu_2, \iota_1 \ge \iota_2$ and $\nu_1 \ge \nu_2$,
(iii) $G_1 = G_2$ iff $G_1 \le G_2$ and $G_2 \le G_1$,
(iv) $G_1 + G_2 =< \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \iota_1\iota_2, \nu_1\nu_2 >$,
(v) $aG =< \sqrt{1 - (1 - \mu^2)^a}, \iota^a, \nu^a >$,
(vi) $G^a =< \mu^a, \iota^a, \sqrt{1 - (1 - \nu^2)^a} >$.

Lemma 1. *[15] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be two GSFNs and $a, a_1, a_2 \ge 0$. Then the following properties hold:*
(i) $G_1 + G_2 = G_2 + G_1$,
(ii) $a(G_1 + G_2) = aG_1 + aG_2$,
(iii) $(a_1 + a_2)G_1 = a_1G_1 + a_2G_2$,
(vi) $(G_1^{a_1})^{a_2} = G_1^{a_1a_2}$.

Definition 4. *[15] Let $\mathscr{G}$ be the collection of all GSFNs and $G \in \mathscr{G}$ where $G =< \mu, \iota, \nu >$.*
(i) A score function $SF : \mathscr{G} \to [-1,1]$ is defined as $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$.
(ii) An accuracy function $AF : \mathscr{G} \to [0,1]$ is defined as $AF(G) = \frac{1+3\mu^2-\nu^2}{4}$.

Definition 5. *[15] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be two GSFNs. Then the ranking method (comparison technique) as follows:*
(i) If $SF(G_1) < SF(G_2)$, then $G_1 < G_2$,
(ii) If $SF(G_1) > SF(G_2)$, then $G_1 > G_2$,
(iii) $SF(G_1) = SF(G_2)$, then
(a) $AF(G_1) < AF(G_2)$, then $G_1 < G_2$,
(b) $AF(G_1) > AF(G_2)$, then $G_1 > G_2$,
(c) $AF(G_1) = AF(G_2)$, then $G_1 = G_2$.

3 Generalized Spherical Fuzzy Einstein Aggregation Operators

In this section, we introduce the Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets based on Einstein triangular norm and triangular conorm. Then we define the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators based on these operations. Also, we investigate some fundamental properties of these operators.

Definition 6. *Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the Einstein operations are defined over the GSFNs as follow:*

(i) $G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}} \right\rangle$,

(ii) $G_1 \odot_E G_2 = \left\langle \sqrt{\frac{\mu_1^2.\mu_2^2}{1+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2}{1+\nu_1^2.\nu_2^2}} \right\rangle$,

(iii) $a \cdot_E G = \left\langle \sqrt{\frac{(1+\mu^2)^a-(1-\mu^2)^a}{(1+\mu^2)^a+(1-\mu^2)^a}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{2\nu^{2a}}{(2-\nu^2)^a+\nu^{2a}}} \right\rangle$,

(iv) $G^{\wedge_E a} = \left\langle \sqrt{\frac{2\mu^{2a}}{(2-\mu^2)^a+\mu^{2a}}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{(1+\nu^2)^a-(1-\nu^2)^a}{(1+\nu^2)^a+(1-\nu^2)^a}} \right\rangle$.

Lemma 2. *Let $G_1 =< \mu_1, \eta_1, \nu_1 >$, $G_2 =< \mu_2, \eta_2, \nu_2 >$ be two GSFNs and $a, a_1, a_2 \geq 0$. Then the following properties hold:*
(i) $G_1 \oplus_E G_2 = G_2 \oplus_E G_1$,
(ii) $a \cdot_E (G_1 \oplus_E G_2) = a \cdot_E G_1 \oplus_E a \cdot_E G_2$,
(iii) $(a_1 + a_2) \cdot_E G_1 = a_1 \cdot_E G_1 \oplus_E a_2 \cdot_E G_2$,
(iv) $G_1 \odot_E G_2 = G_2 \odot_E G_1$,
(v) $(G_1 \odot_E G_2)^{\wedge_E a} = G_1^{\wedge_E a} \odot_E G_2^{\wedge_E a}$,
(vi) $G^{\wedge_E a_1} \odot_E G^{\wedge_E a_2} = G^{\wedge_E a_1 + a_2}$,
(vii) $(G_1^{\wedge_E a_1})^{\wedge_E a_2} = G_1^{\wedge_E a_1 a_2}$.

Proof: Proofs of (i) and (iv) are trivial.

(ii) We can write the equation $G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}} \right\rangle$ in the following way:

$G_1 \oplus_E G_2 = \left\langle \sqrt{\frac{(1+\mu_1^2)(1+\mu_2^2)-(1-\mu_1^2)(1-\mu_2^2)}{(1+\mu_1^2)(1+\mu_2^2)+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{2.\iota_1^2.\iota_2^2}{(2-\iota_1^2)(2-\iota_2^2)+\iota_1^2.\iota_2^2}}, \sqrt{\frac{2.\nu_1^2.\nu_2^2}{(2-\nu_1^2)(2-\nu_2^2)+\nu_1^2.\nu_2^2}} \right\rangle$.

If we take $s = (1+\mu_1^2)(1+\mu_2^2)$, $t = (1-\mu_1^2)(1-\mu_2^2)$, $u = \iota_1^2.\iota_2^2$, $x = (2-\iota_1^2)(2-\iota_2^2)$, $y = \nu_1^2.\nu_2^2$ and $z = (2-\nu_1^2)(2-\nu_2^2)$, by the Einstein operational law (ii), we have that

$$a \cdot_E (G_1 \oplus_E G_2) = a \cdot_E \left\langle \sqrt{\frac{s-t}{s+t}}, \sqrt{\frac{2u}{x+u}}, \sqrt{\frac{2y}{z+y}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\frac{s-t}{s+t})^a - (1-\frac{s-t}{s+t})^a}{(1+\frac{s-t}{s+t})^a + (1-\frac{s-t}{s+t})^a}}, \sqrt{\frac{2(\frac{2u}{x+u})^a}{(2-\frac{2u}{x+u})^a + (\frac{2u}{x+u})^a}}, \sqrt{\frac{2(\frac{2y}{z+y})^a}{(2-\frac{2y}{z+y})^a + (\frac{2y}{z+y})^a}} \right\rangle$$

$$= \left\langle \sqrt{\frac{s^a-t^a}{s^a+t^a}}, \sqrt{\frac{2u^a}{x^a+u^a}}, \sqrt{\frac{2y^a}{z^a+y^a}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^a(1+\mu_2^2)^a - (1-\mu_1^2)^a(1-\mu_2^2)^a}{(1+\mu_1^2)^a(1+\mu_2^2)^a + (1-\mu_1^2)^a(1-\mu_2^2)^a}}, \sqrt{\frac{2\iota_1^{2a}.\iota_2^{2a}}{(2-\iota_1^2)^a(2-\iota_2^2)^a + \iota_1^{2a}.\iota_2^{2a}}}, \sqrt{\frac{2\nu_1^{2a}.\nu_2^{2a}}{(2-\nu_1^2)^a(2-\nu_2^2)^a + \nu_1^{2a}\nu_2^{2a}}} \right\rangle.$$

Also, if we take $s_1 = (1+\mu_1^2)^a$, $t_1 = (1-\mu_1^2)^a$, $u_1 = \iota_1^{2a}$, $x_1 = (2-\iota_1^2)^a$, $y_1 = \nu_1^{2a}$, $z_1 = (2-\nu_1^2)^a$, $s_2 = (1+\mu_2^2)^a$, $t_2 = (1-\mu_2^2)^a$, $u_2 = \iota_2^{2a}$, $x_2 = (2-\iota_2^2)^a$, $y_2 = \nu_2^{2a}$ and $z_2 = (2-\nu_2^2)^a$, then we have that

$a \cdot_E G_1 = \left\langle \sqrt{\frac{s_1-t_1}{s_1+t_1}}, \sqrt{\frac{2u_1}{x_1+u_1}}, \sqrt{\frac{2y_1}{z_1+u_1}} \right\rangle$ and $a \cdot_E G_2 = \left\langle \sqrt{\frac{s_2-t_2}{s_2+t_2}}, \sqrt{\frac{2u_2}{x_2+u_2}}, \sqrt{\frac{2y_2}{z_2+u_2}} \right\rangle$.

By the Einstein operational law (i), we get that

$$a\cdot_E G_1 \oplus_E a\cdot_E G_2 = \left\langle \sqrt{\frac{(\frac{s_1-t_1}{s_1+t_1})+(\frac{s_2-t_2}{s_2+t_2})}{1+(\frac{s_1-t_1}{s_1+t_1})(\frac{s_2-t_2}{s_2+t_2})}}, \sqrt{\frac{(\frac{2u_1}{x_1+u_1})(\frac{2u_2}{x_2+u_2})}{1+(1-\frac{2u_1}{x_1+u_1})(1-\frac{2u_2}{x_2+u_2})}}, \sqrt{\frac{(\frac{2y_1}{z_1+y_1})(\frac{2y_2}{z_2+y_2})}{1+(1-\frac{2y_1}{z_1+y_1})(1-\frac{2y_2}{z_2+y_2})}} \right\rangle$$

$$= \left\langle \sqrt{\frac{s_1s_2-t_1t_2}{s_1s_2+t_1t_2}}, \sqrt{\frac{2u_1u_2}{x_1x_2+u_1u_2}}, \sqrt{\frac{2y_1y_2}{z_1z_2+y_1y_2}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^a(1+\mu_2^2)^a-(1-\mu_1^2)^a(1-\mu_2^2)^a}{(1+\mu_1^2)^a(1+\mu_2^2)^a+(1-\mu_1^2)^a(1-\mu_2^2)^a}}, \sqrt{\frac{2\iota_1^{2a}\iota_2^{2a}}{(2-\iota_1^2)^a(2-\iota_2^2)^a+\iota_1^{2a}\iota_2^{2a}}}, \sqrt{\frac{2\nu_1^{2a}\nu_2^{2a}}{(2-\nu_1^2)^a(2-\nu_2^2)^a+\nu_1^{2a}\nu_2^{2a}}} \right\rangle$$

Hence, we satisfy that $a\cdot_E(G_1 \oplus_E G_2) = a\cdot_E G_1 \oplus_E a\cdot_E G_2$.

The proofs of (iii), (v), (vi) and (vii) can be completed similar to the proof of (ii). □

Definition 7. *Let $\mathscr{G}$ be a collection of all GSFNs and $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1, 2, ..., n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ be the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. A mapping $GSEWA_\alpha : \mathscr{G}^n \to \mathscr{G}$ is said to be a generalized spherical fuzzy Einstein weighted averaging (GSEWA) operator and is defined by*

$$GSEWA_\alpha(G_1, G_2, ..., G_n) = \alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2 \oplus_E ...\alpha_n\cdot_E G_n = \oplus_{i=1}^n \alpha_i\cdot_E G_i \tag{1}$$

Theorem 1. *Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$. Then the aggregated value $GSEWA_\alpha(G_1, G_2, ..., G_n)$ is also a GSFN and is calculated by*

$$GSEWA_\alpha(G_1, G_2, ..., G_n) = \left\langle \sqrt{\frac{\prod_{i=1}^n(1+\mu_i^2)^{\alpha_i}-\prod_{i=1}^n(1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^n(1+\mu_i^2)^{\alpha_i}+\prod_{i=1}^n(1-\mu_i^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n(2-\iota_i^2)^{\alpha_i}+\prod_{i=1}^n \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu_i^{2\alpha_i}}{\prod_{i=1}^n(2-\nu_i^2)^{\alpha_i}+\prod_{i=1}^n \nu_i^{2\alpha_i}}} \right\rangle \tag{2}$$

Proof The above result given in equation (2) can be proved by using induction method on n as follows:

Step I: For $n = 2$, we have

$$GSEWA_\alpha(G_1, G_2) = \alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2.$$

Since $\alpha_1\cdot_E G_1$ and $\alpha_2\cdot_E G_2$ are GSFNs, then $\alpha_1\cdot_E G_1 \oplus_E \alpha_2\cdot_E G_2$ is also a GSFN. Then, we obtain

$$GSEWA_\alpha(G_1, G_2) = \alpha_1._E G_1 \oplus_E \alpha_2\cdot_E G_2$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}}, \sqrt{\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}}, \sqrt{\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}} \right\rangle$$

$$\oplus_E \left\langle \sqrt{\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}}, \sqrt{\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}}, \sqrt{\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}+\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}}{1+\left[\frac{(1+\mu_1^2)^{\alpha_1}-(1-\mu_1^2)^{\alpha_1}}{(1+\mu_1^2)^{\alpha_1}+(1-\mu_1^2)^{\alpha_1}}\right]\left[\frac{(1+\mu_2^2)^{\alpha_2}-(1-\mu_2^2)^{\alpha_2}}{(1+\mu_2^2)^{\alpha_2}+(1-\mu_2^2)^{\alpha_2}}\right]}}, \sqrt{\frac{\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}\cdot\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}}{1+[1-\frac{2\iota_1^{2\alpha_1}}{(2-\iota_1^2)^{\alpha_1}+\iota_1^{2\alpha_1}}][1-\frac{2\iota_2^{2\alpha_2}}{(2-\iota_2^2)^{\alpha_2}+\iota_2^{2\alpha_2}}]}}, \sqrt{\frac{\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}\cdot\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}}{1+[1-\frac{2\nu_1^{2\alpha_1}}{(2-\nu_1^2)^{\alpha_1}+\nu_1^{2\alpha_1}}][1-\frac{2\nu_2^{2\alpha_2}}{(2-\nu_2^2)^{\alpha_2}+\nu_2^{2\alpha_2}}]}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1+\mu_1^2)^{\alpha_1}.(1+\mu_2^2)^{\alpha_2}-(1-\mu_1^2)^{\alpha_1}.(1-\mu_2^2)^{\alpha_2}}{(1+\mu_1^2)^{\alpha_1}.(1+\mu_2^2)^{\alpha_2}+(1-\mu_1^2)^{\alpha_1}.(1-\mu_2^2)^{\alpha_2}}}, \sqrt{\frac{2\iota_1^{2\alpha_1}.\iota_2^{2\alpha_2}}{(2-\iota_1^2)^{\alpha_1}.(2-\iota_2^2)^{\alpha_2}+\iota_1^{2\alpha_1}.\iota_2^{2\alpha_2}}}, \sqrt{\frac{2\nu_1^{2\alpha_1}.\nu_2^{2\alpha_2}}{(2-\nu_1^2)^{\alpha_1}.(2-\nu_2^2)^{\alpha_2}+\nu_1^{2\alpha_1}.\nu_2^{2\alpha_2}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\prod_{i=1}^2(1+\mu_i^2)^{\alpha_i}-\prod_{i=1}^2(1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^2(1+\mu_i^2)^{\alpha_i}+\prod_{i=1}^2(1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \iota_i^{2\alpha_i}}{\prod_{i=1}^2(2-\iota_i^2)^{\alpha_i}+\prod_{i=1}^2 \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \nu_i^{2\alpha_i}}{\prod_{i=1}^2(2-\nu_i^2)^{\alpha_i}+\prod_{i=1}^2 \nu_i^{2\alpha_i}}} \right\rangle$$

Hence, the equation (2) hold for $n = 2$.

Step II: Now, we suppose that the equation (2) hold for $n = k$, that is

$$GSEWA_\alpha(G_1, G_2, ..., G_k) = \alpha_1 \cdot_E G_1 \oplus_E \alpha_2 \cdot_E G_2 \oplus_E ... \oplus_E \alpha_k \cdot_E G_k$$
$$= \left\langle \sqrt{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}} \right\rangle$$

Similarly, we show that the equation (2) hold for $n = k + 1$. Then, we have

$$GSEWA_\alpha(G_1, G_2, ..., G_k, G_{k+1}) = GSEWA_\alpha(G_1, G_2, ..., G_k) \oplus_E \alpha_{k+1} G_{k+1}$$
$$= \left\langle \sqrt{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}} \right\rangle$$
$$\oplus_E \left\langle \sqrt{\frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}}, \sqrt{\frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}}, \sqrt{\frac{2\nu_{k+1}^{2\alpha_1}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}} \right\rangle$$
$$= \left\langle \sqrt{\frac{\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}} + \frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}}{1 + \left(\frac{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^k (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^k (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^k (1-\mu^2)^{\alpha_i}}\right)\left(\frac{(1+\mu_{k+1}^2)^{\alpha_{k+1}} - (1-\mu_{k+1}^2)^{\alpha_{k+1}}}{(1+\mu_{k+1}^2)^{\alpha_{k+1}} + (1-\mu_{k+1}^2)^{\alpha_{k+1}}}\right)}} \right.$$
$$, \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}\right)\left(\frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}\right)}{1 + \left(1 - \frac{2\prod_{i=1}^k \iota_i^{2\alpha_i}}{\prod_{i=1}^k (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^k \iota_i^{2\alpha_i}}\right)\left(1 - \frac{2\iota_{k+1}^{2\alpha_{k+1}}}{(2-\iota_{k+1}^2)^{\alpha_{k+1}} + \iota_{k+1}^{2\alpha_{k+1}}}\right)}}$$
$$\left., \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}\right)\left(\frac{2\nu_{k+1}^{2\alpha_{k+1}}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}\right)}{1 + \left(1 - \frac{2\prod_{i=1}^k \nu_i^{2\alpha_i}}{\prod_{i=1}^k (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^k \nu_i^{2\alpha_i}}\right)\left(1 - \frac{2\nu_{k+1}^{2\alpha_{k+1}}}{(2-\nu_{k+1}^2)^{\alpha_{k+1}} + \nu_{k+1}^{2\alpha_{k+1}}}\right)}} \right\rangle \quad (3)$$

Thus, the equation (2) hold for $n = k + 1$. Hence, by induction method the equation (2) hold for all $n \in \mathbb{N}$.

Lemma 3. *(Idempotency of $GSEWA_\alpha$ operator) If $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$, $G_i =< \mu_i, \iota_i, \nu_i >$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$, then $GSEWA_\alpha = G$.*

Proof Let $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$ and $G_i =< \mu_i, \iota_i, \nu_i >$. Suppose that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. SĂřnce $G_i = G$ for all $i = 1, 2, ..., n$, we have that $\mu_i = \mu, \iota_i = \iota$ and $\nu_i = \nu$ for all $i = 1, 2, ..., n$. Then

$$GSEWA_\alpha(G_1, G_2, ..., G_n) = \left\langle \sqrt{\frac{\prod_{i=1}^n (1+\mu_i^2)^{\alpha_i} - \prod_{i=1}^n (1-\mu_i^2)^{\alpha_i}}{\prod_{i=1}^n (1+\mu_i^2)^{\alpha_i} + \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^n \iota_i^{2\alpha_i}}} \right.$$
$$\left., \sqrt{\frac{2\prod_{i=1}^n \nu_i^{2\alpha_i}}{\prod_{i=1}^n (2-\nu_i^2)^{\alpha_i} + \prod_{i=1}^n \nu_i^{2\alpha_i}}} \right\rangle$$
$$= \left\langle \sqrt{\frac{\prod_{i=1}^n (1+\mu^2)^{\alpha_i} - \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}{\prod_{i=1}^n (1+\mu^2)^{\alpha_i} + \prod_{i=1}^n (1-\mu^2)^{\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota^{2\alpha_i}}{\prod_{i=1}^n (2-\iota^2)^{\alpha_i} + \prod_{i=1}^n \iota^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu^{2\alpha_i}}{\prod_{i=1}^n (2-\nu^2)^{\alpha_i} + \prod_{i=1}^n \nu^{2\alpha_i}}} \right\rangle$$
$$= \left\langle \sqrt{\frac{(1+\mu^2)^{\sum_{i=1}^n \alpha_i} - (1-\mu^2)^{\sum_{i=1}^n \alpha_i}}{(1+\mu^2)^{\sum_{i=1}^n \alpha_i} + (1-\mu^2)^{\sum_{i=1}^n \alpha_i}}}, \sqrt{\frac{2\iota^{\sum_{i=1}^n 2\alpha_i}}{(2-\iota^2)^{\sum_{i=1}^n \alpha_i} + \iota^{\sum_{i=1}^n 2\alpha_i}}}, \sqrt{\frac{2\nu^{\sum_{i=1}^n 2\alpha_i}}{(2-\nu^2)^{\sum_{i=1}^n \alpha_i} + \nu^{\sum_{i=1}^n 2\alpha_i}}} \right\rangle$$
$$= \; < \mu, \iota, \nu >= G$$

Lemma 4. *(Boundedness of $GSEWA_\alpha$ operator) Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ and $i \in \{1, 2, ..., n\}$. Then,*

$$\min_i G_i \leq GSEWA_\alpha(G_1, G_2, ..., G_n) \leq \max_i G_i$$

where $\min_i G_i =< \min_i \mu_i, \max_i \iota_i, \max_i \nu_i >$ and $\max_i G_i =< \max_i \mu_i, \min_i \iota_i, \min_i \nu_i >$.

Proof The proof is easily obtained from Definition 6 and equation (2).

Lemma 5. *(Monotonicity of $GSEWA_\alpha$ operator) Let $(G_1, G_2, ..., G_n), (G'_1, G'_2, ..., G'_n) \in \mathscr{G}^n$. If $G_i \leq G'_i$ for all $i = 1, 2, .., n$, then*

$$GSEWA_\alpha(G_1, G_2, ..., G_n) \leq GSEWA_\alpha(G'_1, G'_2, ..., G'_n).$$

Proof The proof is easily obtained from Definition 6 and equation (2).

Definition 8. *Let $\mathscr{G}$ be a collection of all GSFNs and $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1, 2, ..., n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ be the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. A mapping $GSEWG_\alpha : \mathscr{G}^n \to \mathscr{G}$ is said to be a generalized spherical fuzzy Einstein weighted geometric (GSEWG) operator and is defined by*

$$GSEWG_\alpha(G_1, G_2, ..., G_n) = G_1^{\wedge_E \alpha_1} \odot_E G_2^{\wedge_E \alpha_2} \odot_E ... \odot_E G_n^{\wedge_E \alpha_n} = \odot_{i=1}^n G_i^{\wedge_E \alpha_i} \tag{4}$$

Theorem 2. *Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$. Then the aggregated value $GSEWG_\alpha(G_1, G_2, ..., G_n)$ is also a GSFN and is calculated by*

$$\begin{aligned} GSEWG_\alpha(G_1, G_2, ..., G_n) \quad = \quad & \left\langle \sqrt{\frac{2\prod_{i=1}^n \mu_i^{2\alpha_i}}{\prod_{i=1}^n (2-\mu_i^2)^{\alpha_i} + \prod_{i=1}^n \mu_i^{2\alpha_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_i^{2\alpha_i}}{\prod_{i=1}^n (2-\iota_i^2)^{\alpha_i} + \prod_{i=1}^n \iota_i^{2\alpha_i}}} \right. \\ & \left. , \sqrt{\frac{\prod_{i=1}^n (1+\nu_i^2)^{\alpha_i} - \prod_{i=1}^n (1-\nu_i^2)^{\alpha_i}}{\prod_{i=1}^n (1+\nu_i^2)^{\alpha_i} + \prod_{i=1}^n (1-\nu^2)^{\alpha_i}}} \right\rangle \end{aligned} \tag{5}$$

Proof The proof is obtained similar to the proof of Theorem 1.

Lemma 6. *(Idempotency of $GSEWG_\alpha$ operator) If $G_i = G$ for all $i = 1, 2, ..., n$ where $G =< \mu, \iota, \nu >$, $G_i =< \mu_i, \iota_i, \nu_i >$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the weight vector corresponding to $(G_i)_{i=1}^n$ such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$, then $GSEWG_\alpha = G$.*

Proof The proof is easily obtained from equation (5).

Lemma 7. *(Boundedness of $GSEWG_\alpha$ operator) Let $(G_1, G_2, ..., G_n) \in \mathscr{G}^n$ and $i \in \{1, 2, ..., n\}$. Then,*

$$\min_i G_i \leq GSEWG_\alpha(G_1, G_2, ..., G_n) \leq \max_i G_i$$

where $\min_i G_i =< \min_i \mu_i, \max_i \iota_i, \max_i \nu_i >$ *and* $\max_i G_i =< \max_i \mu_i, \min_i \iota_i, \min_i \nu_i >$.

Proof The proof is easily obtained from Definition 6 and equation (5).

Lemma 8. *(Monotonicity of $GSEWG_\alpha$ operator) Let $(G_1, G_2, ..., G_n), (G'_1, G'_2, ..., G'_n) \in \mathscr{G}^n$. If $G_i \leq G'_i$ for all $i = 1, 2, .., n$, then*

$$GSEWG_\alpha(G_1, G_2, ..., G_n) \leq GSEWG_\alpha(G'_1, G'_2, ..., G'_n).$$

Proof The proof is easily obtained from Definition 6 and equation (5).

4 An Application of Generalized Spherical Fuzzy Einstein Aggregation Operators to Multi-Criteria Group Decision-Making Problems

In this section, we develop a method for multi-criteria group decision-making problems under the generalized spherical fuzzy environment using the defined GSEWA and GSEWG operators and then we give a numerical example to explain this method.

4.1 Methodology

Let $A = \{A_1, A_2, ..., A_m\}$ be the set of m different options and $E = \{E_1, E_2, ..., E_n\}$ be the set of n different attributes. Assume that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is the weight vector of the attribute E_i $(i = 1, 2, ..., n)$ where $\alpha_i \geq 0$ for all $i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$. Also suppose that $D = \{D_1, D_2, ..., D_k\}$ is the set of n distinct decision-makers with the options whose weight vector is expressed as $\delta = (\delta_1, \delta_2, ..., \delta_k)$ where $\delta_i \geq 0$ for all $i = 1, 2, ..., k$ and $\sum_{i=1}^k \delta_i = 1$. This vector ($\delta$) has been handled according to the age, experience, education, thinking ability, and knowledge power of the decision-maker. Actually, as a first step decision matrices associated with options to attribute values are built on considering the preference of the decision-makers. But here, we consider the entity of the decision matrices as GSFNs and are given by $B_{ij}^r =< \mu_{ij}^r, \iota_{ij}^r, \nu_{ij}^r >$ $(i = 1, 2, ..., m), (j = 1, 2, ..., n), (r = 1, 2, ..., k)$ and the associated decision matrix is given as follows:

$$\mathbf{D^r} = \begin{pmatrix} B_{11}^r & B_{12}^r & \dots & B_{1n}^r \\ B_{21}^r & B_{22}^r & \dots & B_{2n}^r \\ \vdots & \vdots & \dots & \vdots \\ B_{m1}^r & B_{m2}^r & \dots & B_{mn}^r \end{pmatrix}.$$

Now, we develop the multi-criteria group decision-making procesure under the generalized spherical fuzzy environment as the following steps:

Step I: Use the either GSEWA or GSEWG operator on each decision matrix D^r to get the following matrix:

$$\mathbf{F^r_{m\times 1}} = \begin{pmatrix} C^r_{11} \\ C^r_{21} \\ \vdots \\ C^r_{m1} \end{pmatrix}$$

where $C^r_{i1} = GSEWA_\alpha(B^r_{i1}, B^r_{i2}, ..., B^r_{in})$ (or $C^r_{i1} = GSEWG_\alpha(B^r_{i1}, B^r_{i2}, ..., B^r_{in})$) for $i = 1, 2, ..., m$ and $r = 1, 2, ..., k$.

Step II: Apply the decision-maker's weight vector (δ) under the scalar multiplication, addition and power of GSFNs to evolute the final matrix $D = \sum_{i=1}^k \delta_i F^i_{m\times 1}$ when GSEWA operator is used and $D = \sum_{i=1}^k (F^i_{m\times 1})^{\delta_i}$ where $(F^i_{m\times 1})^{\delta_i} = \begin{pmatrix} (C^i_{11})^{\delta_i} \\ (C^i_{21})^{\delta_i} \\ \vdots \\ (C^i_{m1})^{\delta_i} \end{pmatrix}$ when GSEWG operator is used. Denote this matrix as follows:

$$\mathbf{D} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \vdots \\ \tilde{A}_m \end{pmatrix}.$$

Step III: Calculate the score values $SF(\tilde{A}_i)$ $(i = 1, 2, ..., m)$ of the cumulative overall preference value. If two score values $SF(\tilde{A}_i)$ and $SF(\tilde{A}_j)$ are same for any $i, j = 1, 2, ..., m$, then it is found the accuracy values $AF(\tilde{A}_i)$ $(i = 1, 2, ..., m)$.

Step IV: Rank the options A_i $(i = 1, 2, ..., m)$ and choose the best option which has the maximum score value.

4.2 A numerical Example

The following medical treatment choice problem is given to demonstrate the suitability, validity and efficiency of the observed multi-criteria group decision making method and is handled from [18].

There is a 50-year-old man patient who was diagnosed with the acute inflammatory demyelinating polyneuropathy disease by his specialist doctor. Acute inflammatory demyelinating polyneuropathy disease is an autoimmune process that is characterized by progressive areflexic weakness and mild sensory changes. The first symptoms of this disease usually include varying degrees of weakness or tingling sensations in the legs. In many instances, the weakness and abnormal sensations ascend and spread to the arms and upper body. And also it can cause life-threatening complications by affecting the peripheral nervous system. Most patients can rescue from this disease with convenient treatment within a few months to a year. But minor by-effects may continue such as areflexia. The doctor chosen four treatment options, including steroid therapy (A_1), plasmapheresis (A_2), intravenous immunoglobulin (A_3) and immunosuppressive medicines (A_4), based on his current physical conditions and medical history.

To satisfy the patient and his family's understanding of the advantages and disadvantages of each treatment choice, the doctor described the treatment options using three criteria that including the side effects (E1), the probability of a Cure (E2), and cost (E3). A prioritization relationship among the criteria E_i $(i = 1, 2, 3)$ which satisfies $E_2 > E_1 > E_3$ was determined according to the patientâĂŹs preferences and his current financial situation. So, assume that $\alpha = (0.3, 0.45, 0.25)$ is the weight vector of the attribute $\{E_1, E_2, E_3\}$. In order to choose the optimum treatment, the patient (D1), the doctor (D2) and the patientâĂŹs family (D3), with a prioritization relationship among the decision-makers D_i $(i = 1, 2, 3)$ satisfying $D_2 > D_1 > D_3$, evaluated the four treatment options based on these criteria considering the generalized spherical fuzzy Einstein aggregation operators. Take the decision-makers weight vector as $\delta = (0.35, 0.45, 0.2)$. The decision matrices are shown as follows:

$$\mathbf{D^1} = \begin{pmatrix} <0.6, 0, 8, 0, 2> & <0.4, 0.3, 0.7> & <0.2, 0.7, 0.4> \\ <0.55, 0.2, 0.8> & <0.8, 0.75, 0.65> & <0.9, 0.8, 0.2> \\ <0.7, 0.4, 0.4> & <0.55, 0.2, 0.45> & <0.5, 0.7, 0.8> \\ <0.35, 0.6, 0.5> & <0.7, 0.8, 0.55> & <0.8, 0.6, 0.5> \end{pmatrix}$$

$$\mathbf{D^2} = \begin{pmatrix} <0.85, 0.7, 0.8> & <0.4, 0.75, 0.8> & <0.6, 0.8, 0.5> \\ <0.3, 0.4, 0.4> & <0.8, 0.2, 0.45> & <0.5, 0.6, 0.8> \\ <0.9, 0, 8, 0, 2> & <0.4, 0.8, 0.7> & <0.8, 0.7, 0.4> \\ <0.75, 0.3, 0.5> & <0.8, 0.5, 0.45> & <0.5, 0.6, 0.8> \end{pmatrix}$$

$$\mathbf{D^3} = \begin{pmatrix} <0.75, 0.4, 0.5> & <0.8, 0.8, 0.45> & <0.8, 0.6, 0.8> \\ <0.9, 0, 6, 0, 4> & <0.4, 0.6, 0.9> & <0.2, 0.7, 0.4> \\ <0.55, 0.5, 0.8> & <0.8, 0.75, 0.85> & <0.6, 0.8, 0.2> \\ <0.75, 0.4, 0.8> & <0.4, 0.8, 0.45> & <0.8, 0.6, 0.6> \end{pmatrix}$$

Step I: Using the GSEWA operator on each decision matrix D^r with the weight vector $\alpha = (0.3, 0.45, 0.25)$, we get the following matrices:

$$\mathbf{F^1_{4\times 1}} = \begin{pmatrix} <0.4396, 0.5141, 0.4299> \\ <0.7841, 0.5376, 0.5345> \\ <0.5914, 0.3442, 0.5093> \\ <0.6606, 0.6874, 0.5221> \end{pmatrix}$$

$$\mathbf{F^2_{4\times 1}} = \begin{pmatrix} <0.6436, 0.7473, 0.7196> \\ <0.6380, 0.3287, 0.5093> \\ <0.7330, 0.7748, 0.4299> \\ <0.7309, 0.4525, 0.5431> \end{pmatrix}$$

$$\mathbf{F^3_{4\times 1}} = \begin{pmatrix} <0.7861, 0.6162, 0.5431> \\ <0.6305, 0.6243, 0.6> \\ <0.6962, 0.6819, 0.6151> \\ <0.6515, 0.6162, 0.5826> \end{pmatrix}$$

Step II: Applying the decision-maker's weight vector $\delta = (0.35, 0.45, 0.2)$, we get the following matrix:

$$\mathbf{D} = \begin{pmatrix} <0.6312, 0.6308, 0.5680> \\ <0.6989, 0.4439, 0.5352> \\ <0.6837, 0.5685, 0.49> \\ <0.6932, 0.5572, 0.5432> \end{pmatrix}$$

Step III: Now, we calculate the score values $SF(\tilde{A_i})\ (i = 1, 2, 3, 4)$. Here, we have that $SF(\tilde{A_1}) = 0.03$, $SF(\tilde{A_2}) = 0.26$, $SF(\tilde{A_3}) = 0.1719$ and $SF(\tilde{A_4}) = 0.1752$.

Step IV: The ranking order of score values is that $SF(\tilde{A_2}) > SF(\tilde{A_4}) > SF(\tilde{A_3}) > SF(\tilde{A_1})$. Hence, according to the Definition 5, the ranking order of the options is that $A_2 > A_4 > A_3 > A_1$. Therefore, the best option is A_2.

4.3 Sensitivity analysis of the numerical example

The aim of sensitivity analysis is to observe the weights of the decision-makers keeping the rest of the other terms are fixed in the problem. So, the sensitivity analysis is given to understand how decision-makers' weight affects the final matrix and its ranking. The sensitivity analysis result for the problem given in the above section is shown respect to the GSEWA and GSEWG operators in Table 1 and Table 2, respectively.

Weights of the decision-makers	**Final decision matrix**	**Ranking order**
$<0.35, 0.45, 0.2>$	$\begin{pmatrix} <0.6312, 0.6308, 0.5680> \\ <0.6989, 0.4439, 0.5352> \\ <0.6837, 0.5685, 0.49> \\ <0.6932, 0.5572, 0.5432> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.35, 0.4, 0.25>$	$\begin{pmatrix} <0.6412, 0.62471, 0.5600> \\ <0.6987, 0.4584, 0.5396> \\ <0.6816, 0.5649, 0.4989> \\ <0.6892, 0.5658, 0.5451> \end{pmatrix}$	$A_2 > A_3 > A_4 > A_1$
$<0.35, 0.36, 0.29>$	$\begin{pmatrix} <0.6490, 0.6199, 0.5538> \\ <0.6984, 0.4703, 0.5432> \\ <0.6799, 0.5620, 0.5061> \\ <0.6860, 0.5729, 0.5466> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.3, 0.5, 0.2>$	$\begin{pmatrix} <0.6388, 0.6427, 0.5828> \\ <0.6909, 0.4331, 0.5339> \\ <0.6902, 0.5920, 0.4859> \\ <0.6967, 0.5456, 0.5443> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.3, 0.55, 0.15>$	$\begin{pmatrix} <0.6287, 0.6489, 0.5910> \\ <0.6912, 0.4194, 0.5296> \\ <0.6922, 0.5958, 0.4773> \\ <0.7006, 0.5372, 0.5424> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$

Table 1 Sensitivity analysis under GSEWA operator

Weights of the decision-makers	Final decision matrix	Ranking order
$<0.35, 0.45, 0.2>$	$\begin{pmatrix} <0.9910, 0.6308, 0.05263> \\ <0.9893, 0.4439, 0.06192> \\ <0.9920, 0.5685, 0.05885> \\ <0.9922, 0.5572, 0.03185> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.35, 0.4, 0.25>$	$\begin{pmatrix} <0.9898, 0.6247, 0.0557> \\ <0.9882, 0.4584, 0.0650> \\ <0.9910, 0.5649, 0.0584> \\ <0.9914, 0.5658, 0.0373> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.35, 0.36, 0.29>$	$\begin{pmatrix} <0.9893, 0.6199, 0.0571> \\ <0.9878, 0.4703, 0.0661> \\ <0.9906, 0.5620, 0.0565> \\ <0.9911, 0.5729, 0.0409> \end{pmatrix}$	$A_2 > A_3 > A_4 > A_1$
$<0.3, 0.5, 0.2>$	$\begin{pmatrix} <0.9913, 0.6427, 0.0511> \\ <0.9899, 0.4331, 0.0605> \\ <0.9923, 0.5920, 0.0648> \\ <0.9925, 0.5456, 0.0334> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$
$<0.3, 0.55, 0.15>$	$\begin{pmatrix} <0.9929, 0.6489, 0.0462> \\ <0.9916, 0.4194, 0.0552> \\ <0.9937, 0.5958, 0.0618> \\ <0.9938, 0.5373, 0.0264> \end{pmatrix}$	$A_2 > A_4 > A_3 > A_1$

Table 2 Sensitivity analysis under GSEWG operator

5 Conclusion and Future Work

In this paper, we give the generalized spherical fuzzy Einstein weighted averaging and generalized spherical fuzzy Einstein weighted geometric operators constructed on Einstein sum, product and scalar multiplication for generalized spherical fuzzy sets which are based on Einstein triangular norm and triangular conorm. We also investigate some fundamental properties of these operators and develop a model for generalized spherical fuzzy Einstein aggregation operators to solve the multi-criteria group decision-making problems. Further, we give a numerical example related to the medical treatment choosing to demonstrate that the developed method is suitable and effective for the decision process. For future work, we propose to develop the methods by considering different types of operators to solve the multi-criteria group decision-making problems under the generalized spherical fuzzy environment and also we aim to compare all obtained operators in terms of their results.

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