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Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro differential equations

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Abstract

In this paper, we study the existence and uniqueness of solutions for the neutral Caputo fractional Volterra-Fredholm integro differential equations with fractional integral boundary conditions by means of the Arzela-Ascoli's theorem, Leray-Schauder nonlinear alternative and the Krasnoselskii fixed point theorem. New conditions on the nonlinear terms are given to pledge the equivalence. An example is provided to illustrate the results.

Keywords: Caputo fractional derivative, Neutral Volterra-Fredholm integro-differential equation, Fixed point method, Leray-Schauder nonlinear alternative.
2010 MSC: 26A33, 34A08, 34B15.

1. Introduction

The topic fractional calculus can be measured as an old as well as a new subject. Started from some speculations of Leibniz and Euler, followed by other important mathematicians like Laplace, Fourier, Abel, Liouville, Riemann and Holmgren [4, 6, 10, 13, 14, 17, 21, 22]. In the fractional calculus the various integral inequalities plays an important role in the study of qualitative and quantitative properties of solution of differential and integral equations [13, 18].

In recent years, many authors focus on the development of techniques for discussing the solutions of fractional differential and integro-differential equations. For instance, we can remember the following works:

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Ibrahim and Momani [11] studied the existence and uniqueness of solutions of a class of fractional order differential equations, Karthikeyan and Trujillo [12] proved existence and uniqueness of solutions for fractional integro-differential equations with boundary value conditions, Bahuguna and Dabas [5] applied the method of lines to establish the existence and uniqueness of a strong solution for the partial integrodifferential equations, Matar [16] deliberated the existence of solutions for nonlocal fractional semilinear integro-differential equations in Banach spaces via Banach fixed point theorem.

In [1] the authors studied a class of nonlinear differential equations with multiple fractional derivatives and Caputo type integro-differential boundary conditions

$$\begin{aligned} &D^{\alpha}[D^{\nu}u(t) - g(t,u(t))] = f(t,u(t)), \quad t \in [0,T],\\ &u(0) = 0,\ (D^{\gamma}u)(T) = \lambda I^{\rho}u(T),\ 0 < \alpha, \nu, \rho < 1. \end{aligned}$$

In [2] existence criteria are developed for the solutions of Caputo-Hadamard type fractional neutral differential equations supplemented with Dirichlet boundary conditions

$$\begin{aligned} &D^{\alpha}[D^{\nu}u(t) - g(t,u(t))] = f(t,u(t)) \ \ 0 < \alpha, \nu < 1,\\ &u(1) = 0,\ u(T) = 0,\ t \in [1,T]. \end{aligned}$$

Ntouyas [19] studied the existence results for the following fractional differential equation with fractional integral boundary condition

$$\begin{aligned} &{}^{c}D^{\nu}_{0+}u(t) = f(t,u(t)),\ 1 < \nu \le 2,\\ &u(0) = 0,\ u(1) = \alpha I^{\rho}u(\eta),\ \alpha \in \mathbb{R},\ \ 0 < \rho < 1,\ 0 < \eta < 1. \end{aligned}$$

Akiladevi et al. [3] discussed the existence and uniqueness of solutions to the nonlinear neutral fractional boundary value problem

$$\begin{aligned} &{}^{c}D^{\nu}_{0+}[u(t) - g(t,u(t))] = f(t,u(t)),\ 0 < \nu \le 1,\\ &u(0) = \alpha I^{\rho}u(\eta),\ \alpha \in \mathbb{R},\ 0 < \rho < 1,\ 0 < \eta < 1, \end{aligned}$$

Motivated by the above works, we will study a more general problem of Caputo fractional integro-differential equations which called Caputo fractional neutral Volterra-Fredholm integro-differential equations of the form

$${}^{c}D^{\nu}_{0+}[u(t) - g(t,u(t))] = f(t,u(t),Ku(t),Qu(t)),\ 0 < \nu \le 1, \tag{1}$$

with fractional integral boundary condition

$$u(0) = \alpha I^{\rho}u(\eta),\ 0 < \rho < 1,\ 0 < \eta < 1, \tag{2}$$

where ${}^{c}D^{\alpha}_{0+}$ is the Caputo's fractional derivative ν, The function $g : J\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable, $f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. I^{ρ} is the Riemann-Liouville fractional integral of order ρ and $\alpha \neq \frac{\Gamma(\rho+1)}{\eta^{\rho}} \in \mathbb{R}$, with $J := [0,1]$, $\Omega = \{(t,s) : 0 \le s < t \le 1\}$, and $Ku(t) = \int_0^t k(t,s,u(s))ds$, $Qu(t) = \int_0^1 q(t,s,u(s))ds$.

The main objective of the present paper is to study the new existence and uniqueness of solutions of fractional neutral Volterra-Fredholm integro-differential equation in Banach contraction principle.

The rest of the paper is organized as follows: In Sect. 2, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Sect. 3, the new existence and uniqueness results of the solution for Caputo fractional neutral Volterra-Fredholm integro-differential equation have been proved. In Sect. 4, an example is provided to illustrate the results. Finally, we will give a report on our paper and a brief conclusion is given in Sect. 5.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [7, 8, 9, 13, 15, 18, 20, 21].

Definition 2.1. *[13] (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in C([0,T])$ is defined as*

$$J^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

where Γ denotes the Gamma function.

Definition 2.2. *[13] (Caputo fractional derivative). The fractional derivative of $u(t)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD^{\alpha}_{0^+}u(t) &= J^{m-\alpha}_{0^+}D^m u(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\frac{\partial^m u(s)}{\partial s^m}ds, & m-1<\alpha<m, \\ \frac{\partial^m u(t)}{\partial t^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{3}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

1. $J^{\alpha}_{0^+}J^{v}u = J^{\alpha+v}_{0^+}u, \quad \alpha, v > 0.$
2. $J^{\alpha}_{0^+}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}u^{\beta+\alpha},$
3. $D^{\alpha}_{0^+}u^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}u^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1.$
4. $J^{\alpha}_{0^+}D^{\alpha}_{0^+}u(t) = u(t) - u(a),\ 0 < \alpha < 1.$
5. $J^{\alpha}_{0^+}D^{\alpha}_{0^+}u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{(t-a)^k}{k!}, \quad t > 0.$

Definition 2.3. *[13] (Riemann-Liouville fractional derivative). The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha}_{0^+}u(t) = D^{m}_{0^+}J^{m-\alpha}_{0^+}u(t), \qquad m-1<\alpha\le m, \quad m \in \mathbb{N}. \tag{4}$$

Theorem 2.1. *[23] (Banach fixed point theorem). Let $(S, \|.\|)$ be a complete normed space, and let the mapping $F : S \longrightarrow S$ be a contraction mapping. Then F has exactly one fixed point.*

Theorem 2.2. *[15] (Krasnoselskii fixed point theorem.) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that:*
1. $Ax + By \in M$ whenever $x, y \in M$.
2. A is compact and continuous.
3. B is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.3. *[15] (Leray-Schauder nonlinear alternative) Let E be a Banach space, $\mathcal{C}$ a closed, convex subset of E, U an open subset of $\mathcal{C}$ and $0 \in U$. Suppose that $F : \bar{U} \longrightarrow \mathcal{C}$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of $\mathcal{C}$) map. Then either*
(I) F has a fixed point in $\bar{U}$ or
(II) There is a $u \in \partial U$ (the boundary of U in $\mathcal{C}$) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Lemma 2.1. *Let $\alpha \neq \frac{\Gamma(\rho+1)}{\eta^\rho}$. Assume that f is continuous function. If $u \in C(J,X)$ then u satisfies the problem*

$$
\begin{aligned}
&{}^cD^\nu_{0+}[u(t)-g(t)] = f(t),\ 0<\nu\leq 1,\ t\in[0,1],\\
&u(0) = \alpha I^\rho u(\eta),
\end{aligned}
$$

if and only if u satisfies the integral equation

$$
\begin{aligned}
u(t) &= \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} f(s)ds - \frac{\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho} g(0) + g(t)\\
&+\frac{\alpha\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} g(s)ds + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)} f(s)ds\Big).
\end{aligned}
$$

3. Main Results

In this section, we shall give an existence and uniqueness results of Eq. (1), with the condition (2). Before starting and proving the main results, we introduce the following hypotheses:
(A1) The function $g: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuously differentiable, $f: J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous, and there exist constants $L_1, L_2>0$, for $t\in J,\ u_i, v_i, y_i\in\mathbb{R}$, such that

$$
\begin{aligned}
&|f(t,u_1,v_1,y_1)-f(t,u_2,v_2,y_2)| \leq L_1[|u_1-u_2|+|v_1-v_2|+|y_1-y_2|]\\
&|g(t,u_1)-g(t,u_2)|\leq L_2|u_1-u_2|.
\end{aligned}
$$

(A2) The functions $k,q: J\times J\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous and there exist constants $L_4, L_5>0$, such that

$$|k(t,s,u_1)-k(t,s,u_2)|\leq L_4|u_1-u_2|,\ t,s\in J,\ u_1,u_2\in\mathbb{R},$$

$$|q(t,s,u_1)-q(t,s,u_2)|\leq L_5|u_1-u_2|,\ t,s\in J,\ u_1,u_2\in\mathbb{R}.$$

(A3) Let $P_1 = L_1(1+L_4+L_5)\lambda_1 + L_2\lambda_2 < 1$. where

$$\lambda_1 = \frac{1}{\Gamma(\nu+1)} + \frac{\alpha\eta^{\rho+\nu}\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|\Gamma(\rho+\nu+1)} \quad\text{and}\quad \lambda_2 = \frac{|\Gamma(\rho+1)-\alpha\eta^\rho|+\alpha\eta^\rho}{|\Gamma(\rho+1)-\alpha\eta^\rho|}.$$

(A4) For $\mu_1,\mu_2\in C(J,\mathbb{R})$, we have

$$
\begin{aligned}
&|f(t,u,v,y)|\leq\mu_1(t),\ (t,u,v,y)\in J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\\
&|g(t,u)|\leq\mu_2(t),\ (t,u)\in J\times\mathbb{R}.
\end{aligned}
$$

First, we will state the following axiom lemma.

Lemma 3.1. *Let $0<\nu\leq 1$. Assume that f is continuous function. If $u\in C(J,\mathbb{R})$ then u satisfies the problem (1)-(2) if and only if u satisfies the integral equation*

$$
\begin{aligned}
u(t) &= \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} f(s,u(s),Ku(s),Qu(s))ds - \frac{\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho} g(0,u(0))\\
&+g(t,u(t)) + \frac{\alpha\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} g(s,u(s))ds\\
&+\int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)} f(s,u(s),Ku(s),Qu(s))ds\Big). \qquad (5)
\end{aligned}
$$

Theorem 3.1. *Assume f,g,k and q satisfy the assumptions (A1)-(A3). Then the problem* (1)-(2) *has a unique solution on J.*

Proof. Let $M_1 = \sup_{t\in J}|f(t,0,0,0)|,\ M_2 = \sup_{t\in J}|g(t,0)|,\ M_4 = \sup_{t,s\in J}|k(t,s,0)|,\ M_5 = \sup_{t,s\in J}|q(t,s,0)|$ and consider $B_r = \{u \in C : \|u\| \le r\}$, where $r \ge \frac{P_2}{1-P_1}$ with

$$p_2 = [(M_1 + L_1M_4 + L_1M_5)\lambda_1 + M_2\lambda_2 + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,u(0))|]$$

Let $\mathcal{C} = C(J,\mathbb{R})$ be the Banach space of all continuous functions from $J \longrightarrow \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|,\ t \in J\}$. In view of Lemma 3.1, we transform (1) as

$$u = F(u), \tag{6}$$

where $F : \mathcal{C} \longrightarrow \mathcal{C}$ is given by

$$\begin{aligned}(Fu)(t) \quad &= \quad \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} f(s,u(s),Ku(s),Qu(s))ds - \frac{\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho} g(0,u(0))\\ &\quad +g(t,u(t)) + \frac{\alpha\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} g(s,u(s))ds\\ &\quad + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)} f(s,u(s),Ku(s),Qu(s))ds\Big).\end{aligned} \tag{7}$$

For $u \in B_r$, we have

$$\begin{aligned}&\|(Fu)(t)\|\\ \le\ & \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,u(0))|\\ & +|g(t,u(t))| + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|g(s,u(s))|ds\\ & + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}|f(s,u(s),Ku(s),Qu(s))|ds\Big)\Big]\\ \le\ & \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}(|f(s,u(s),Ku(s),Qu(s)) - f(s,0,0,0)| + |f(s,0,0,0)|)ds\\ & +\frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,u(0))| + (|g(t,u(t)) - g(t,0)| + |g(t,0)|)\\ & +\frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}(|g(s,u(s)) - g(s,0)| + |g(s,0)|)ds\\ & + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}(|f(s,u(s),Ku(s),Qu(s)) - f(s,0,0,0)| + |f(s,0,0,0)|)ds\Big)\Big]\\ \le\ & (L_1r(1+L_4+L_5) + M_1 + M_4L_1 + M_5L_1)\lambda_1 + (L_2r + M_2)\lambda_2\\ & +\frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,u(0))|\\ \le\ & [L_1(1+L_4+L_5)\lambda_1 + L_2\lambda_2]r + \Big[(M_1 + M_4L_1 + M_5L_1)\lambda_1 + M_2\lambda_2\\ & +\frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,u(0))|\Big]\\ \le\ & P_1r + P_2\\ \le\ & r.\end{aligned}$$

This shows that $FB_r \subset B_r$. Next, for $u, v \in \mathcal{C}$ and $t \in J$, we obtain

$$\begin{aligned}
\|Fu - Fv\| &\leq \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s)) - f(s,v(s),Kv(s),Qv(s))|ds \\
&\quad + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(t,u(t)) - g(t,v(t))| \\
&\quad + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|g(s,u(s)) - g(s,v(s))|ds \\
&\quad + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}|f(s,u(s),Ku(s),Qu(s)) - f(s,v(s),Kv(s),Qv(s))|ds\Big)\Big] \\
&\leq [L_1(1+L_4+L_5)\lambda_1 + L_2\lambda_2]|u-v| \\
&\leq P_1|u-v|.
\end{aligned}$$

Here P_1 depends only on the parameters involved in the problem. By assumption (A3), $P_1 < 1$ and therefore F is a contraction. Hence, by the Banach contraction principle, the problem (1)-(2) has a unique solution on J. □

Now we prove the existence of solutions of the problem (1)-(2) by applying Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume f, g and h satisfy the assumptions (A1)-(A3). If*

$$L := \frac{1}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big[L_2[|\Gamma(\rho+1)-\alpha\eta^\rho| + \alpha\eta^\rho] + \frac{L_1(1+L_4+L_5)\alpha\eta^{\rho+\nu}\Gamma(\rho+1)}{\Gamma(\rho+\nu+1)}\Big] < 1. \tag{8}$$

Then the problem (1)-(2) *has at least one solution on J.*

Proof. Let $\|\mu_i\| = \sup_{t\in J}|\mu_i|,\ i = 1,2$ and $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$. Now we decompose F as $F_1 + F_2$ on B_r, where

$$\begin{aligned}
(F_1u)(t) &= \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}f(s,u(s),Ku(s),Qu(s))ds, \\
(F_2u)(t) &= -\frac{\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho}g(0,u(0)) + g(t,u(t)) \\
&\quad + \frac{\alpha\Gamma(\rho+1)}{\Gamma(\rho+1)-\alpha\eta^\rho}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}g(s,u(s))ds \\
&\quad + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}f(s,u(s),Ku(s),Qu(s))ds\Big),
\end{aligned}$$

for $t \in J$. Choose

$$\begin{aligned}
r &\geq \|\mu\|\Big[\frac{1}{\Gamma(\nu)} + \frac{\alpha\eta^{\rho+\nu}\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|\Gamma(\rho+\nu+1)} \\
&\quad + \frac{|\Gamma(\rho+1)-\alpha\eta^\rho| + \alpha\eta^\rho}{|\Gamma(\rho+1)-\alpha\eta^\rho|} + \frac{|\Gamma(\rho+1)|}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big]
\end{aligned}$$

For $u, v \in B_r$, we find that

$$\begin{aligned}
&\|F_1u - F_2v\| \\
\leq\ & \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,v(0))| \\
&+|g(t,v(t))| + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|g(s,v(s))|ds \\
&+\int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}|f(s,v(s),Kv(s),Qv(s))|ds\Big)\Big] \\
\leq\ & \frac{\|\mu_1\|}{\Gamma(\nu+1)} + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}|g(0,v(0))| + \|\mu_2\| \\
&+\frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big(\frac{\eta^\rho}{\Gamma(\rho+1)}\|\mu_2\| + \frac{\eta^{\rho+\nu}}{\Gamma(\rho+\nu+1)}\|\mu_1\|\Big).
\end{aligned}$$

Let $\mu = \max\{\mu_1, g(0,v(0)), \mu_2\}$. Then, by simplification, we have

$$\begin{aligned}
\|F_1u - F_2v\| &\leq \|\mu\|\Big[\frac{1}{\Gamma(\nu+1)} + \frac{\alpha\eta^{\rho+\nu}\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|\Gamma(\rho+\nu+1)} \\
&\quad+\frac{|\Gamma(\rho+1)-\alpha\eta^\rho|+\alpha\eta^\rho}{|\Gamma(\rho+1)-\alpha\eta^\rho|} + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|}\Big] \\
&\leq r.
\end{aligned}$$

Thus $F_1u + F_2v \in B_r$.

Next we prove that F_2 is a contraction.

$$\begin{aligned}
&\|F_2u - F_2v\| \\
\leq\ & \sup_{t\in J}\Big[|g(t,u(t)) - g(t,v(t))| + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|} \\
&\times\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|g(s,u(s)) - g(s,v(s))|ds \\
&+\int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}|f(s,u(s),Ku(s),Qu(s)) - f(s,v(s),Kv(s),Qv(s))|ds\Big)\Big] \\
\leq\ & \Big[\frac{L_2(|\Gamma(\rho+1)-\alpha\eta^\rho|+\alpha\eta^\rho)}{|\Gamma(\rho+1)-\alpha\eta^\rho|} + \frac{L_1(1+L_4+L_5)\alpha\eta^{\rho+\nu}\Gamma(\rho+1)}{|\Gamma(\rho+1)-\alpha\eta^\rho|\Gamma(\rho+\nu+1)}\Big]|u-v| \\
\leq\ & L|u-v|.
\end{aligned}$$

Hence F_2 is a contraction. Continuity of f implies that the operator F_1 is continuous. Also F_1 is uniformly bounded on B_r as

$$\begin{aligned}
\|(F_1u)(t)\| &\leq \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds\Big] \\
&\leq \frac{\|\mu_1\|}{\Gamma(\nu+1)}.
\end{aligned}$$

To prove that the operator F_1 is compact, it remains to show that F_1 is equicontinuous. For that, let

$\bar{f} = \sup|f(t,u,Ku,Qu)|$. Now, for any $t_1, t_2 \in J$, with $t_1 < t_2$ and $u \in B_r$, we have

$$
\begin{aligned}
& \|(F_1u)(t_2) - (F_1u)(t_1)\| \\
\leq\ & \sup_{t\in J}\Big[\int_0^{t_1} \frac{[(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}]}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds\Big] \\
\leq\ & \frac{\bar{f}}{\Gamma(\nu+1)}[t_2^\nu - t_1^\nu] \\
& \longrightarrow 0 \ \text{ as } \ t_2 \longrightarrow t_1.
\end{aligned}
$$

Thus F_1 is equicontinuous. By Arzela-Ascoli Theorem, F_1 is compact. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $u \in \mathcal{C}$ such that $Fu = u$ which is a solution to the boundary value problem (1)-(2).

□

The next result is based on Leray-Schauder nonlinear alternative.

Theorem 3.3. *Assume that the following hypotheses hold:*
(A4) *There exist continuous nondecreasing functions* $\psi_1, \psi_2 : [0,\infty) \longrightarrow [0,\infty)$ *and* $\phi_1, \phi_2 \in L^1(J, \mathbb{R}^+)$ *such that, for each* $(t,u,v,y) \in J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,
$|f(t,u,v,y| \leq \phi_1(t)\psi_1(\|u\|)$,
$|g(t,u)| \leq \phi_2(t)\psi_2(\|u\|)$, $(t,u) \in J \times \mathbb{R}$.
(A5) *There exists a constant* $M > 0$ *such that* $\frac{M}{N} \geq 1$, *where*

$$
N = \psi(M)\Big[I^\nu\phi_1(1) + \frac{\alpha\Gamma(\rho+1)I^\rho(I^\rho\phi_1(\eta) + \phi_2(\eta)}{|\Gamma(\rho+1) - \alpha\eta^\rho|} + \phi_2(1) + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big].
$$

Then the problem (1)-(2) *has at least one solution on* J.

Proof. Observe that the operator $F : \mathcal{C} \longrightarrow \mathcal{C}$ defined by (7) is continuous. Next we show that F maps bounded sets into bounded sets in $\mathcal{C}$. For a positive number k, let $B_k = \{u \in \mathcal{C} : \|u\| \leq k\}$ be a bounded ball in $C(J,\mathbb{R})$. Then we have

$$
\begin{aligned}
& \|(Fu)(t)\| \\
\leq\ & \sup_{t\in J}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,u(s),Ku(s),Qu(s))|ds + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}|g(0,u(0))| \\
& + |g(t,u(t))| + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big(\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|g(s,u(s))|ds \\
& + \int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}|f(s,u(s),Ku(s),Qu(s))|ds\Big)\Big] \\
\leq\ & \psi_1(\|u\|)\int_0^1 \frac{(1-s)^{\nu-1}}{\Gamma(\nu)}\phi_1(s)ds + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}|g(0,u(0))| + \phi_2(1)\psi_2(\|u\|) \\
& + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big(\psi_2(\|u\|)\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}\phi_2(s)ds + \psi_1(\|u\|)\int_0^\eta \frac{(\eta-s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)}\phi_1(s)ds\Big) \\
\leq\ & \psi_1(k)\Big[I^\nu\phi_1(1) + \frac{\alpha\Gamma(\rho+1)I^{\rho+\nu}\phi_1(\eta)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big] + \frac{\Gamma(\rho+1)|g(0,u(0))|}{|\Gamma(\rho+1) - \alpha\eta^\rho|} \\
& + \psi_2(k)\Big[\phi_2(1) + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}I^\rho\phi_2(\eta)\Big].
\end{aligned}
$$

Choosing $\psi(k) = \max\{\psi_1(k), \psi_2(k), g(0, u(0))\}$, we have

$$\begin{aligned}\|(Fu)(t)\| &\leq \psi(k)\Big[I^\nu \phi_1(1) + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|} I^\rho(I^\nu \phi_1(\eta) + \phi_2(\eta)) \\ &\quad + \phi_2(1) + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big].\end{aligned}$$

Now, we show that F maps bounded sets into equicontinuous sets in B_k. For that, let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for $u \in B_k$,

$$\begin{aligned}&\|(Fu)(t_2) - (Fu)(t_1)\| \\ \leq\ & \int_0^{t_2} \frac{(t_2 - s)^{\nu-1}}{\Gamma(\nu)} |f(s, u(s), Ku(s), Qu(s))| ds + |g(t_2, u(t_2))| \\ & - \int_0^{t_1} \frac{(t_1 - s)^{\nu-1}}{\Gamma(\nu)} |f(s, u(s), Ku(s), Qu(s))| ds - |g(t_1, u(t_1))| \\ \leq\ & \psi_1(k) \int_0^{t_1} \Big[\frac{(t_2 - s)^{\nu-1} - (t_1 - s)^{\nu-1}}{\Gamma(\nu)}\Big] \phi_1(s) ds \\ & + |g(t_2, u(t_2)) - g(t_1, u(t_1))| + \psi_1(k) \int_{t_1}^{t_2} \frac{(t_2 - s)^{\nu-1}}{\Gamma(\nu)} \phi_1(s) ds \\ & \longrightarrow 0 \ \text{ as } \ t_2 \longrightarrow t_1.\end{aligned}$$

Thus F maps bounded sets into equicontinuous sets in B_k. By Arzela-Ascoli's Theorem, F is completely continuous. Now let $u = \lambda F u$ where $\lambda \in (0, 1)$. Then, for $t \in J$, we have

$$\begin{aligned}u(t) &= \lambda \int_0^t \frac{(t - s)^{\nu-1}}{\Gamma(\nu)} f(s, u(s), Ku(s), Qu(s)) ds - \frac{\lambda\Gamma(\rho+1)}{\Gamma(\rho+1) - \alpha\eta^\rho} g(0, u(0)) \\ &\quad + \lambda g(t, u(t)) + \frac{\lambda\alpha\Gamma(\rho+1)}{\Gamma(\rho+1) - \alpha\eta^\rho}\Big(\int_0^\eta \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} g(s, u(s)) ds \\ &\quad + \int_0^\eta \frac{(\eta - s)^{\rho+\nu-1}}{\Gamma(\rho+\nu)} f(s, u(s), Ku(s), Qu(s)) ds\Big).\end{aligned}$$

Then, using the computations of the first step, we have

$$\begin{aligned}|u(t)| &\leq \psi(\|u\|)\Big[I^\nu \phi_1(1) + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|} I^\rho(I^\nu \phi_1(\eta) + \phi_2(\eta)) \\ &\quad + \phi_2(1) + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big].\end{aligned}$$

Consequently

$$\begin{aligned}\|u\| &\leq \psi(\|u\|)\Big[I^\nu \phi_1(1) + \frac{\alpha\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|} I^\rho(I^\nu \phi_1(\eta) + \phi_2(\eta)) \\ &\quad + \phi_2(1) + \frac{\Gamma(\rho+1)}{|\Gamma(\rho+1) - \alpha\eta^\rho|}\Big].\end{aligned}$$

In view of (A5), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in \mathcal{C} : \|u\| < M\}.$$

Note that the operator $F : \bar{U} \longrightarrow \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda F u$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder theorem, we deduce that F has a fixed point $u \in \bar{U}$ which is a solution to the problem (1)-(2). □

4. An Example

As an application of our results, we consider the following Caputo fractional neutral Volterra-Fredholm integro differential equation

$$
\begin{aligned}
{}^{C}D^{\frac{1}{2}}[u(t)-\frac{e^{-t}}{26+e^{t}}\frac{u(t)}{1+u(t)}] &= \frac{1}{(t+6)^2}\frac{|u(t)|}{1+|u(t)|}+\frac{1}{36}\int_0^t e^{-\frac{t}{5}u(s)}ds \\
&+\frac{1}{36}\int_0^1 \frac{e^{-s}}{9}\frac{|u(s)|}{1+|u(s)|}ds, \qquad (9)
\end{aligned}
$$

with fractional integral boundary condition

$$
u(0)=\sqrt{2}I^{\frac{1}{2}}u(\frac{1}{2}), \qquad (10)
$$

where

$$
\begin{aligned}
\nu=\tfrac{1}{2},\ g(t,u(t)) &= \tfrac{e^{-t}}{26+e^{t}}\tfrac{u(t)}{1+u(t)},\ f(t,u(t),Ku(t),Qu(t)) \\
&= \tfrac{1}{(t+6)^2}\tfrac{|u(t)|}{1+|u(t)|}+\tfrac{1}{36}\int_0^t e^{-\frac{t}{5}u(s)}ds+\tfrac{1}{36}\int_0^1\tfrac{e^{-s}}{9}\tfrac{|u(s)|}{1+|u(s)|}ds,\ \alpha=\sqrt{2},\ \rho=\eta=\tfrac{1}{2}\neq\tfrac{\Gamma(\rho+1)}{\eta^{\rho}},
\end{aligned}
$$

and

$$
\begin{aligned}
&|f(t,u_1(t),Ku_1(t),Qu_1(t))-f(t,u_2(t),Ku_2(t),Qu_2(t))| \\
&\leq \frac{1}{36}[|u_1-u_2|+|Hu_1-Hu_2|+|Qu_1-Qu_2|], \\
&|g(t,u_1)-g(t,u_2)|\leq \frac{1}{27}|u_1-u_2|, \\
&|k(t,s,u_1)-k(t,s,u_2)|\leq \frac{1}{5}|u_1-u_2|, \\
&|q(t,s,u_1)-q(t,s,u_2)|\leq \frac{1}{9}|u_1-u_2|,
\end{aligned}
$$

we get the value of $P_1=0.6042799<1$. All the conditions of the Theorem 3.1 are satisfied. Hence the problem (9)-(10) has a unique solution on [0, 1].

5. Conclusion

The main purpose of this paper was to present new existence and uniqueness of solutions by means of the Krasnoselskii fixed point theorem, Leray-Schauder nonlinear alternative and the Arzela-Ascoli's theorem for Caputo fractional neutral Volterra-Fredholm integro differential equations with fractional integral boundary conditions. New conditions on the nonlinear terms are given to pledge the equivalence. Moreover, the results of references [1, 3, 19] appear as a special case of our results.

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