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The Duffin-Kemmer-Petiau (DKP) Equation Wavefunctions Solutions According to the Virial Theorem for a Spin - one Particle Interacting with a Potential V(r)=k r^n

Virial Teoremine Göre V(r)=kr^n Potansiyeli ile Etkileşen Bir Spin-bir Parçacığının Duffin-Kemmer-Petiau (DKP) Denkleminin Dalgafonksiyonları Çözümleri

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Abstract

The DKP Eqution is written for a spin-one particle interacting with a potential V(r)=k r^n. The written equation is solved for this potential according to the virial theorem. The ten – component wavefunctions are obtained.

Keywords: DKP equation, Virial theorem, Wave functions

Öz

DKP Eşitliği, potansiyel V(r)=kr^n ile etkileşime giren bir spin-parçacık için yazılmıştır. Yazılı denklem bu potansiyel için virial teoremine göre çözülür. On bileşenli dalga fonksiyonları elde edilir.

Anahtar Kelimeler: DKP denklemi, Virial teoremi, Dalga fonksiyonları

1. Introduction

The classical and the quantum mechanical derivation of the virial theorem are given in this section from the previous works. The virial theorem is used in many works, see [1-11, 15-23, 26-32, 34,35] and the references given therein. The DKP equation is introduced in section 2 [8,12-14, 24, 25, 33] and the virial theorem is applied to it. The wavefunction solutions of this application are found in section 3. Finally, the conclusion is given in section 4. The purpose of this study is to get the explicit wavefunction solutions of DKP equation for a potential of $V(r) = k\,r^n$.

In the references [4-8,16,19,31,32], the virial theorem is derived as follows:

$$G = \sum_i \vec{p}_i . \vec{r}_i \,, \tag{1}$$

here G is considered to be a quantity, product of the momentum and the position of the particle in a stable system. Taking the derivative of Eq. (1), it is obtained:

$$\frac{dG}{dt} = \sum_i \frac{d\vec{p}_i}{dt} . \vec{r}_i + \vec{p}_i . \frac{d\vec{r}_i}{dt} . \tag{2}$$

The second term in the right hand of the equation (2) can be written as [5]

$$\sum_i \overset{r}{p_i} . \frac{d^{r}_{r_i}}{dt} = \sum_i \left(m^{p\&}_{r_i} \right) . r^{r\&}_i = m \tag{3}$$

Here T is the kinetic energy. If the quantity G is bounded in a time interval, one can write:

$$\frac{1}{\tau} \int_0^{\tau} \frac{dG}{dt} dt = \frac{1}{\tau} (G(\tau) - G(0)) = 0. \tag{4}$$

From Eq. (2), it can be written as

$$\frac{1}{\tau} \int_0^{\tau} \frac{dG}{dt} dt = 2T + \sum \vec{F}_i . \vec{r}_i \tag{5}$$

for the periodical momentum. From Eqs. (4) and (5) one gets;

$$2T = -\sum_i \vec{F}_i . \vec{r}_i \tag{6}$$

Eq. (6) is the virial theorem in the classical case.

Now, it is the time looking for the quantum mechanical virial theorem. The time-dependent Schrödinger Equation is;

$$i\hbar \frac{d\psi}{dt} = H\psi . \tag{7}$$

The derivative of expectation value of an operator A with respect to time is;

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$$i\hbar\frac{d}{dt}\langle\psi|A|\psi\rangle=\langle\psi|[H,A]\psi\rangle. \quad (8)$$

Let'schoose A to be $A=\vec{r}.\vec{p}$ [4-8,15,16,19,31,32,]. Putting this in Eq. (7) andtaking A to be time-independent, the following equation is obtained;

$$\langle\psi|[H,A]\psi\rangle=0. \quad (9)$$

Then virial theorem is written as

$$\begin{aligned}[H,A]&=\frac{p^2}{2m}+V(r),\vec{r}.\vec{p}\\ &=i\hbar\vec{r}.\nabla V-\frac{i\hbar}{m}\vec{p}^2 \quad ,\\ &=i\hbar\vec{r}.\nabla V-2i\hbar T=0\end{aligned} \quad (10)$$

where T is the kinetic energy and V is the potential energy. Then, the virial theorem can be written as

$$2\langle T\rangle=\langle\vec{r}.\vec{\nabla}V\rangle. \quad (11)$$

The kinetic energy can be defined by

$$T=\frac{p^2}{2m}, \quad (12)$$

where p is the momentum operator and m is the mass of the particle under consideration. The momentum operator is

$$p=-i\hbar\nabla. \quad (13)$$

Using Eq. (11) for a potential of the form $V=kr^n$, the kinetic energy is obtained as

$$\langle T\rangle=\frac{n}{2}\langle V(r)\rangle. \quad (14)$$

Here it must be noted that if n = 0, the Eq. (14) cannot be valid because of the singularity, for $n\neq 0$ the equation is valid.

Due to the Eq.(12) and the Eq. (14), the following relation can be written:

$$V(r)=\frac{p^2}{nm}. \quad (15)$$

2. The Application of the Virial Theorem to the DKP Equation

One can write the DKP equation for a spin-one particle [8,12-14,24,25,33] interacting via the potential $V(r)=kr^n$ in ten-dimension matrices as:

$$(i\beta^\mu\partial_\mu+Ikr^n-\mathrm{Im})\psi(t,r)=0 \quad (16)$$

for $\mu=0,1,2,3$ where

$$\beta^0=\begin{pmatrix}0&\bar{0}&\bar{0}&\bar{0}\\ \bar{0}^T&0&I&I\\ \bar{0}^T&I&0&0\\ \bar{0}^T&0&0&0\end{pmatrix},\beta^i=\begin{pmatrix}0&\bar{0}&e_i&\bar{0}\\ \bar{0}^T&0&0&-iS_i\\ e_i^T&0&0&0\\ \bar{0}^T&-iS_i&0&0\end{pmatrix};\ i=1,2,3 \quad (17)$$

and

$$0=0,\bar{0}=(0\ 0\ 0),0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},S_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},$$
$$S_y=\frac{1}{\sqrt{2}i}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix},S_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},e_1=(1\ 0\ 0),$$
$$e_2=(0\ 1\ 0),e_3=(0\ 0\ 1),I_{3x3} \quad (18)$$

T defines the transposation of matrices, and constants are taken as $c=\hbar=1$. Hence, the β^μ matrices can be written as

$$\beta^0=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&1&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&0&0&1&0&0&1\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},$$

$$\beta^1=\begin{pmatrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\0&0&0&0&0&0&0&-i/\sqrt{2}&0&-i/\sqrt{2}\\0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&-i/\sqrt{2}&0&0&0&0&0&0&0\\0&-i/\sqrt{2}&0&-i/\sqrt{2}&0&0&0&0&0&0\\0&0&-i/\sqrt{2}&0&0&0&0&0&0&0\end{pmatrix}$$

$$\beta^2=\begin{pmatrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&-1/\sqrt{2}&0\\0&0&0&0&0&0&0&1/\sqrt{2}&0&-1/\sqrt{2}\\0&0&0&0&0&0&0&0&1/\sqrt{2}&0\\0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&-1/\sqrt{2}&0&0&0&0&0&0&0\\0&1/\sqrt{2}&0&-1/\sqrt{2}&0&0&0&0&0&0\\0&0&1/\sqrt{2}&0&0&0&0&0&0&0\end{pmatrix}$$

$$\beta^{3}=\begin{pmatrix} 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0&0&0 \end{pmatrix} \tag{19}$$

Using Eq. (15) in Eq. (16) and the matrices given by Eq. (19) one writes the DKP equation as

as:

$$i\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}\frac{\partial\psi(t;x,y,z)}{\partial t}$$

$$+\begin{pmatrix} 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\ 0&0&0&0&0&0&0&-i/\sqrt{2}&0&-i/\sqrt{2}\\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0\\ 0&-i/\sqrt{2}&0&-i/\sqrt{2}&0&0&0&0&0&0\\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}\left(i\partial_{x}-\frac{p_{x}^{2}}{nm}\right)\psi(t;x,y,z)$$

$$+\begin{pmatrix} 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1/\sqrt{2}&0\\ 0&0&0&0&0&0&0&1/\sqrt{2}&0&-1/\sqrt{2}\\ 0&0&0&0&0&0&0&0&1/\sqrt{2}&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-1/\sqrt{2}&0&0&0&0&0&0&0\\ 0&1/\sqrt{2}&0&-1/\sqrt{2}&0&0&0&0&0&0\\ 0&0&1/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}\left(i\partial_{y}-\frac{p_{y}^{2}}{nm}\right)\psi(t;x,y,z)$$

$$+\begin{pmatrix} 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0&0&0 \end{pmatrix}\times\left(i\partial_z-\frac{p_z^2}{nm}\right)\psi(t;x,y,z)-mI\psi(t;x,y,z)=0. \tag{20}$$

3. The Wave Function Solutions of the DKP Equation Written by the Virial Theorem

One can take the wave function as

$$\psi(t;x,y,z)=\phi(x,y,z)e^{-iEt},\phi(x,y,z)=\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\\ \phi_9\\ \phi_{10}\end{pmatrix} \tag{21}$$

to get a solution of the Eq. (20). Inserting Eq. (21) in Eq. (20) the following equations are obtained;

$$i\frac{\partial\phi_5}{\partial x}{}_1-\frac{1}{mn}\frac{\partial^2\phi_5}{\partial x^2}+i\frac{\partial\phi_6}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_6}{\partial y^2}+i\frac{\partial\phi_7}{\partial z}-\frac{1}{mn}\frac{\partial^2\phi_7}{\partial z^2}-m\phi_1=0, \tag{22}$$

$$i\frac{\partial\varphi_5}{\partial t}+\frac{\partial\varphi_8}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\varphi_9}{\partial x}+\frac{i}{mn}\frac{\partial^2\varphi_9}{\partial x^2}+\frac{1}{\sqrt{2}}-i\frac{\partial\varphi_9}{\partial y}+\frac{1}{mn}\frac{\partial^2\varphi_9}{\partial y^2}+\frac{\partial\varphi_8}{\partial z}+\frac{i}{mn}\frac{\partial^2\varphi_8}{\partial z^2}-m\varphi_2=0, \tag{23}$$

$$i\frac{\partial\phi_6}{\partial t}+\frac{\partial\phi_9}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\phi_8}{\partial x}+\frac{\partial\phi_{10}}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\phi_8}{\partial x^2}+\frac{\partial^2\phi_{10}}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\phi_8}{\partial y}-\frac{\partial\phi_{10}}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\phi_8}{\partial y^2}-\frac{\partial^2\phi_{10}}{\partial y^2}-m\phi_3=0, \tag{24}$$

$$i\frac{\partial\phi_7}{\partial t}+\frac{\partial\phi_{10}}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\phi_9}{\partial x}+\frac{i}{mn}\frac{\partial^2\phi_9}{\partial x^2}+\frac{1}{\sqrt{2}}i\frac{\partial\phi_9}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_9}{\partial y^2}-\frac{\partial\phi_{10}}{\partial z}+\frac{i}{mn}\frac{\partial^2\phi_{10}}{\partial z^2}-m\phi_4=0, \tag{25}$$

$$i\frac{\partial\phi_2}{\partial t}+i\frac{\partial\phi_1}{\partial x}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial x^2}-m\phi_5=0, \tag{26}$$

$$i\frac{\partial\phi_3}{\partial t}+i\frac{\partial\phi_1}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial y^2}-m\phi_6=0, \tag{27}$$

$$i\frac{\partial\phi_4}{\partial t}+i\frac{\partial\phi_1}{\partial z}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial z^2}-m\phi_7=0, \tag{28}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\phi_3}{\partial x}+\frac{i}{mn}\frac{\partial^2\phi_3}{\partial x^2}-\frac{1}{\sqrt{2}}i\frac{\partial\phi_3}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_3}{\partial y^2}+\frac{\partial\phi_2}{\partial z}+\frac{i}{mn}\frac{\partial^2\phi_2}{\partial z^2}-m\phi_8=0, \tag{29}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_4}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_4}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\varphi_2}{\partial y}-\frac{\partial\varphi_4}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\varphi_2}{\partial y^2}-\frac{\partial^2\varphi_4}{\partial y^2}-m\varphi_9=0, \tag{30}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\phi_3}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\phi_3}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\phi_3}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\phi_3}{\partial y^2}-\frac{\partial\phi_4}{\partial z}-\frac{i}{mn}\frac{\partial^2\phi_4}{\partial z^2}-m\phi_{10}=0. \tag{31}$$

By taking

$$\phi_i = e^{i\vec{k}.\vec{r}}, \tag{32}$$

The wavefunctions in terms of the wavenumbers can be written from Eqs. (22-31) as follows;

$$\phi_1 = \frac{1}{m}\left(-k_x + \frac{k_x^2}{mn} - k_y + \frac{k_y^2}{mn} - k_z + \frac{k_z^2}{mn}\right)e^{i\vec{k}.\vec{r}}, \tag{33}$$

$$\begin{aligned}\phi_2 = -\frac{1}{E^2-m^2}\Bigg[&E\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)\\ &+m\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}\right) + E\left(-k_x + \frac{ik_x^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}},\end{aligned} \tag{34}$$

$$\phi_3 = \frac{-1}{E^2-m^2}\left[E\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn} - k_y + \frac{k_y^2}{mn}\right) + m\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}}, \tag{35}$$

$$\phi_4 = \frac{-1}{E^2-m^2}\left[(E+m)\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right) + E\left(-k_z + \frac{k_z^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}}, \tag{36}$$

$$\begin{aligned}\phi_5 = \frac{-1}{E^2-m^2}\Bigg[&\frac{E^2}{m}\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)\\ &+E\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} + \frac{1}{\sqrt{2}}k_y - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}\right) + m\left(-k_x + \frac{k_x^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}},\end{aligned} \tag{37}$$

$$\phi_6 = \frac{-1}{E^2-m^2}\left[\frac{E^2}{m}\left(\sqrt[3]{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right) + E\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right) + m\left(-k_y + \frac{k_y^2}{mn\sqrt{2}}\right)\right]e^{i\vec{k}.\vec{r}}, \tag{38}$$

$$\phi_7 = \frac{-1}{E^2-m^2}\left[\left(\frac{E^2}{m} + E\right)\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right) + m\left(-k_z + \frac{k_z^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}}, \tag{39}$$

$$\phi_8 = \frac{1}{m}\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)e^{i\vec{k}.\vec{r}}, \tag{40}$$

$$\phi_9 = \frac{1}{m}\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right)e^{i\vec{k}.\vec{r}} \tag{41}$$

$$\phi_{10} = \frac{1}{m}\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{k_z^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}}. \tag{42}$$

The calculations are also given in Appendix.

An application of the virial theorem is done in [6] to find the wavefunctions for the Schrödinger Equation. Compared to the solutions of the references [14,33], the obtained wavefunctions in this study are the explicit solutions of the wavefunctions of the DKP equation for spin-one particle.

4. Discussions and Conclusions

The wavefunctions of the DKP equations for the given potential in terms of the wavenumbers, the radial powers of the given potential, mass and energy of the spin-one particle, for the particle being in a closed system are obtained. This is the explicit solutions of the DKP equation wavefunctions compared to the previous studies. The obtained solutions will be used in the applications of the DKP equation for spin-one particle, and it will give to the researchers a useful tool for applications.

5. Acknowledgments

I would like to thank to Assoc. Dr. Zafer Şiar for his helps in solving the differential equations used in this study.

6. Appendix

$$(i\beta^{\mu}\partial_{\mu} + Ikr^{n} - \mathrm{Im})\psi(t,r) = 0 \tag{1}$$

for $\mu = 0,1,2,3$ where

$$\beta^0 = \begin{matrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^T & 0 & I & I \\ \overline{0}^T & I & 0 & 0 \\ \overline{0}^T & 0 & 0 & 0 \end{matrix}, \beta^i = \begin{matrix} 0 & \overline{0} & e_i & \overline{0} \\ \overline{0}^T & 0 & 0 & -iS_i \\ e_i^T & 0 & 0 & 0 \\ \overline{0}^T & -iS_i & 0 & 0 \end{matrix};\ i = 1,2,3 \tag{2}$$

and

$$0 = 0, \overline{0} = (0\ \ 0\ \ 0), 0 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, S_x = \frac{1}{\sqrt{2}} \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}, S_y = \frac{1}{\sqrt{2}\, i} \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{matrix}, S_z = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}, e_1 = (1\ \ 0\ \ 0), \tag{3}$$
$$e_2 = (0\ \ 1\ \ 0), e_3 = (0\ \ 0\ \ 1), I_{3\times3}$$

T defines the transposation of matrices, and constants are taken as $c = \hbar = 1$. Hence, the β^μ matrices can be written as

$$\beta^0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\beta^1 = \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0 \\ 0&0&0&0&0&0&0&-i/\sqrt{2}&0&-i/\sqrt{2} \\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0 \\ 0&-i/\sqrt{2}&0&-i/\sqrt{2}&0&0&0&0&0&0 \\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\beta^2 = \begin{pmatrix} 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1/\sqrt{2}&0 \\ 0&0&0&0&0&0&0&1/\sqrt{2}&0&-1/\sqrt{2} \\ 0&0&0&0&0&0&0&0&1/\sqrt{2}&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&-1/\sqrt{2}&0&0&0&0&0&0&0 \\ 0&1/\sqrt{2}&0&-1/\sqrt{2}&0&0&0&0&0&0 \\ 0&0&1/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\beta^3=\begin{pmatrix} 0&0&0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0&0&0 \end{pmatrix} \tag{4}$$

Using Eq. $V(r)=\frac{p^2}{nm}$. in Eq. (1) and the matrices given by Eq. (4) one writes the DKP equation as:

$$i\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}\frac{\partial\psi(t;x,y,z)}{\partial t}$$

$$+\begin{pmatrix} 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\ 0&0&0&0&0&0&0&-i/\sqrt{2}&0&-i/\sqrt{2}\\ 0&0&0&0&0&0&0&0&-i/\sqrt{2}&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0\\ 0&-i/\sqrt{2}&0&-i/\sqrt{2}&0&0&0&0&0&0\\ 0&0&-i/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}\left(i\partial_x-\frac{P_x^2}{nm}\right)\psi(t;x,y,z)$$

$$+\begin{pmatrix} 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1/\sqrt{2}&0\\ 0&0&0&0&0&0&0&1/\sqrt{2}&0&-1/\sqrt{2}\\ 0&0&0&0&0&0&0&0&1/\sqrt{2}&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-1/\sqrt{2}&0&0&0&0&0&0&0\\ 0&1/\sqrt{2}&0&-1/\sqrt{2}&0&0&0&0&0&0\\ 0&0&1/\sqrt{2}&0&0&0&0&0&0&0 \end{pmatrix}\left(i\partial_y-\frac{p_y^2}{nm}\right)\psi(t;x,y,z)$$

$$+\begin{pmatrix} 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0&0&0 \end{pmatrix}\times\left(i\partial_z-\frac{p_z^2}{nm}\right)\psi(t;x,y,z)-mI(t;x,y,z)=0.$$

(5)

One can take the wave function as

$$\psi(t;x,y,z)=\phi(x,y,z)e^{-iEt},\phi(x,y,z)=\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\\ \phi_9\\ \phi_{10}\end{pmatrix} \tag{6}$$

to get a solution of the Eq. (5).Inserting Eq. (6) in Eq. (5) the following equations are obtained;

$$i\frac{\partial\phi_5}{\partial x}-\frac{1}{mn}\frac{\partial^2\phi_5}{\partial x^2}+i\frac{\partial\phi_6}{\partial y^2}+i\frac{\partial\phi_7}{\partial z}-\frac{1}{mn}\frac{\partial^2\phi_7}{\partial z^2}-m\phi_1=0, \tag{7}$$

$$i\frac{\partial\varphi_5}{\partial t}+\frac{\partial\varphi_8}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\varphi_9}{\partial x}+\frac{i}{mn}\frac{\partial^2\varphi_9}{\partial x^2}+\frac{1}{\sqrt{2}}-i\frac{\partial\varphi_9}{\partial y}+\frac{1}{mn}\frac{\partial^2\varphi_9}{\partial y^2}+\frac{\partial\varphi_8}{\partial z}+\frac{i}{mn}\frac{\partial^2\varphi_8}{\partial z^2}-m\varphi_2=0, \tag{8}$$

$$i\frac{\partial\phi_6}{\partial t}+\frac{\partial\phi_9}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\phi_8}{\partial x}+\frac{\partial\phi_{10}}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\phi_8}{\partial x^2}+\frac{\partial^2\phi_{10}}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\phi_8}{\partial y}-\frac{\partial\phi_{10}}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\phi_8}{\partial y^2}-\frac{\partial^2\phi_{10}}{\partial y^2}-m\phi_3=0\ , \tag{9}$$

$$i\frac{\partial\phi_7}{\partial t}+\frac{\partial\phi_{10}}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial\phi_9}{\partial x}+\frac{i}{mn}\frac{\partial^2\phi_9}{\partial x^2}+\frac{1}{\sqrt{2}}i\frac{\partial\phi_9}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_9}{\partial y^2}-\frac{\partial\phi_{10}}{\partial z}+\frac{i}{mn}\frac{\partial^2\phi_{10}}{\partial z^2}-m\phi_4=0, \tag{10}$$

$$i\frac{\partial\phi_2}{\partial t}+i\frac{\partial\phi_1}{\partial x}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial x^2}-m\phi_5=0, \tag{11}$$

$$i\frac{\partial\phi_3}{\partial t}+i\frac{\partial\phi_1}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial y^2}-m\phi_6=0, \tag{12}$$

$$i\frac{\partial\phi_4}{\partial t}+i\frac{\partial\phi_1}{\partial z}-\frac{1}{mn}\frac{\partial^2\phi_1}{\partial z^2}-m\phi_7=0, \tag{13}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\phi_3}{\partial x}+\frac{i}{mn}\frac{\partial^2\phi_3}{\partial x^2}-\frac{1}{\sqrt{2}}i\frac{\partial\phi_3}{\partial y}-\frac{1}{mn}\frac{\partial^2\phi_3}{\partial y}+\frac{\partial\phi_2}{\partial z}+\frac{i}{mn}\frac{\partial^2\phi_2}{\partial z^2}-m\phi_8=0, \tag{14}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_4}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_4}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\varphi_2}{\partial y}-\frac{\partial\varphi_4}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\varphi_2}{\partial y^2}-\frac{\partial^2\varphi_4}{\partial y^2}-m\varphi_9=0, \tag{15}$$

$$\frac{1}{\sqrt{2}}\frac{\partial\phi_3}{\partial x}+\frac{i}{mn\sqrt{2}}\frac{\partial^2\phi_3}{\partial x^2}+\frac{i}{\sqrt{2}}\frac{\partial\phi_3}{\partial y}-\frac{1}{mn\sqrt{2}}\frac{\partial^2\phi_3}{\partial y^2}-\frac{\partial\phi_4}{\partial z}-\frac{i}{mn}\frac{\partial^2\phi_4}{\partial z^2}-m\phi_{10}=0. \tag{16}$$

By taking

$$\phi_i = e^{i\vec{k}.\vec{r}}, \tag{17}$$

The following equations are obtained:

$$E\phi_2 + -k_x + \frac{k_x^2}{mn}e^{ikr} - m\phi_5 = 0 \tag{18}$$

$$E\phi_3 + -k_y + \frac{k_y^2}{mn}e^{ikr} - m\phi_6 = 0 \tag{19}$$

$$E\phi_4 + -k_z + \frac{k_z^2}{mn}e^{ikr} - m\phi_7 = 0 \tag{20}$$

$$-k_x + \frac{k_x^2}{mn} - k_y + \frac{k_y^2}{mn} - k_z + \frac{k_z^2}{mn}e^{ikr} - m\phi_1 = 0 \tag{21}$$

$$E\phi_5 + E\phi_8 + \frac{ik_x}{\sqrt{2}} - \frac{ik_x^2}{mn\sqrt{2}} + \frac{k_y}{\sqrt{2}} - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}e^{ikr} - m\phi_2 = 0 \tag{22}$$

$$E\phi_6 + E\phi_9 + \frac{2ik_x}{\sqrt{2}} - \frac{2ik_x^2}{mn\sqrt{2}}e^{ikr} - m\phi_3 = 0 \tag{23}$$

$$E\phi_7 + E\phi_{10} + \frac{ik_x}{\sqrt{2}} - \frac{ik_x^2}{mn\sqrt{2}} - \frac{k_y}{\sqrt{2}} + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}e^{ikr} - m\phi_4 = 0 \tag{24}$$

$$\frac{ik_x}{\sqrt{2}} - \frac{ik_x^2}{mn\sqrt{2}} + \frac{k_y}{\sqrt{2}} - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}e^{ikr} - m\phi_8 = 0 \tag{25}$$

$$\frac{2ik_x}{\sqrt{2}} - \frac{2ik_x^2}{mn\sqrt{2}}e^{ikr} - m\phi_9 = 0 \tag{26}$$

$$\frac{ik_x}{\sqrt{2}} - \frac{ik_x^2}{mn\sqrt{2}} - \frac{k_y}{\sqrt{2}} + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}e^{ikr} - m\phi_{10} = 0 \tag{27}$$

From Eq. (26)

$$\phi_9 = \frac{1}{m}\frac{2ik_1}{\sqrt{2}} - \frac{2ik_1^2}{mn\sqrt{2}}e^{ikr}. \tag{28}$$

Putting Eq. (28) in Eq. (23) and using Eq. (19), after some arrangements, two equations are obtained:

$$\phi_3 = \frac{-1}{E^2 - m^2}\left[E\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn} - k_y + \frac{k_y^2}{mn}\right) + m\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}} \tag{29}$$

$$\phi_6 = \frac{-1}{E^2 - m^2}\left[\frac{E^2}{m}\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right) + E\left(\sqrt{2}\,ik_x - \sqrt{2}\,i\frac{k_x^2}{mn}\right) + m\left(-k_y + \frac{k_y^2}{mn\sqrt{2}}\right)\right]e^{i\vec{k}.\vec{r}} \tag{30}$$

From Eq. (27) one writes;

$$\phi_{10} = \frac{1}{m}\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{k_z^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}}. \tag{31}$$

From Eq. (25) one gets;

$$\phi_8 = \frac{1}{m}\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)e^{i\vec{k}.\vec{r}}. \tag{32}$$

From Eq. (21) it is written that;

$$\phi_1 = \frac{1}{m}\left(-k_x + \frac{k_x^2}{mn} - k_y + \frac{k_y^2}{mn} - k_z + \frac{k_z^2}{mn}\right)e^{i\vec{k}.\vec{r}} \tag{33}$$

Putting Eq. (31) in Eq. (24) and using Eq. (20) , after some calculations, the following two equations are obtained;

$$\varphi_7 = \frac{-1}{E^2 - m^2}\left[\left(\frac{E^2}{m} + E\right)\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right) + m\left(-k_z + \frac{k_z^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}} \tag{34}$$

$$\varphi_4 = \frac{-1}{E^2-m^2}\left[(E+m)\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right) + E\left(-k_z + \frac{k_z^2}{mn}\right)\right]e^{i\vec{k}.\vec{r}} \quad (35)$$

Putting Eq. (32) in Eq. (22) and using Eq. (18) one gets;

$$\begin{aligned}\phi_5 = \frac{-1}{E^2-m^2}\Bigg[&\frac{E^2}{m}\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)\\ &+E\left(\frac{i}{\sqrt{2}}k_x - i\frac{k_x^2}{mn\sqrt{2}} + \frac{1}{\sqrt{2}}k_y - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}\right) + m\left(-k_x + \frac{k_x^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}}\end{aligned} \quad (36)$$

$$\begin{aligned}\phi_2 = -\frac{1}{E^2-m^2}\Bigg[&E\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y + \frac{k_y^2}{mn\sqrt{2}} - ik_z + \frac{ik_z^2}{mn}\right)\\ &+m\left(\frac{i}{\sqrt{2}}k_x - \frac{ik_x^2}{mn\sqrt{2}} - \frac{1}{\sqrt{2}}k_y - \frac{k_y^2}{mn\sqrt{2}} + ik_z - \frac{ik_z^2}{mn}\right) + E\left(-k_x + \frac{ik_x^2}{mn}\right)\Bigg]e^{i\vec{k}.\vec{r}}\end{aligned} \quad (37)$$

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