Karaelmas Science and Engineering Journal
Journal home page: http://fbd.beun.edu.tr
DOI: 10.7212/zkufbd.v9i2.1335

Research Article
Received / Geliş tarihi : 30.04.2018
Accepted / Kabul tarihi : 15.05.2018

Numerical Computation of the Linearized Lorenz System with Variable Step

Lineer Lorenz Sisteminin Değişken Adım Genişliği ile Nümerik Hesaplanması

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Abstract

Numerical solutions of linearized Lorenz system and diffusionless Lorenz system are aimed using variable step size strategy which consider the error analysis. Phase portraits are obtained for these strange attractors.

Keywords: Diffusionless Lorenz systems, Lorenz systems, Numerical solution, Variable step size

Öz

Hata analizini dikkate alan değişken adım genişliği kullanarak lineer Lorenz Sistemi ve difüzyonsuz Lorenz sistemlerinin nümerik çözümleri amaçlanmıştır. Bu tuhaf çekicilerin faz portreleri elde edilmiştir.

Anahtar Kelimeler: Difüzyonsuz Lorenz sistemi, Lorenz sistemi, Nümerik çözüm, Değişken adım genişliği

1. Introduction

Chaos has been intensively studied within the science, mathematics and engineering communities in recent years. Because, chaos has been found to be very useful and has a great potential in many technological disciplines such as in information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications (Lü et al. 2004). Studying of chaos has been based on the work of Edward N. Lorenz in 1963. Lorenz's equations have given the idea that systems with infinite number of effects affecting the behavior of the system, such as the atmosphere, can be modeled with simple deterministic finite-dimensional systems (Sparrow 1982).

Lorenz system has given by the following equations by Lorenz in 1963 (Lorenz 1963):

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= -xz + rx - y \\ z' &= xy - \beta z \end{aligned} \qquad (1)$$

Here, all of the parameters σ, r and β are positive real numbers and if $\sigma = 10, r = 28$ and β =8/3, then the system (1) becomes chaotic. Lorenz system concerns the motion of fluid, such as the earth's atmosphere, which is warmer at the bottom than at the top (Boyce et al. 2012). It is an important problem in meteorology and other fields related to fluid dynamics.

The analytical solution of system (1) cannot be found. So, there are many studies in the literature about numerical solution of (1) (for example; (Montoya et al. 2016), (Guran and Ahmedi 2012)) and in many universities there are courses on numerical solution of (1) (for example; (Dynamical Systems 2011), (Lorenz simulations 2014), (Ogliore 2016)). However there are some difficulties of numerical calculations, too. For the accuracy of the solution, either small step sizes or methods required long and complicated calculations are generally used in these studies. For example, let's look at Figure 1 given in (Dynamical Systems 2011). In this figure, the behavior of x in system (1) is seen for the values of $\sigma = 10, r = 20$ and β =2.67 with step sizes h = 0.01 and h = 0.02 respectively. The difference between the two shapes is interesting. Which one is closer to the truth? It is well known that small step size increases computation errors, although accuracy is thought to be higher when

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the step size is small. Therefore, the error analysis of the numerical method should be considered and the step size should be selected accordingly. And also, it is important to use variable step size for the accuracy and efficiency of solution in numerical integration of initial value problems, usually (El-Zahar 2012).

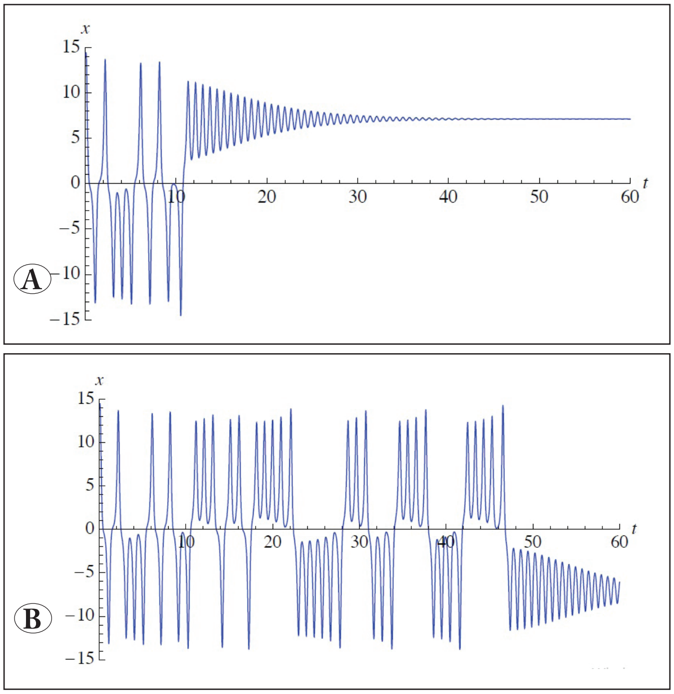

Figure 1. Trajectory of x for $\sigma = 10, r = 20$ and β =2.67
(A) h = 0.02 **(B)** h = 0.01.

Using a transformation defined as

$$(x,y,z) \to (\sigma x, \sigma y, \sigma z + r), t \to \frac{t}{\sigma}$$

and taking $R = \frac{\beta r}{\sigma^2}$, the diffusionless Lorenz system is obtained as follows:

$$\begin{aligned} x' &= -x + y \\ y' &= -xz \\ z' &= xy - R \end{aligned} \tag{2}$$

Diffusionless Lorenz system is also chaotic if $R = 1$ (Sprott 2010).

In this study, we proposed to obtain numerical solutions of linearized Lorenz system and diffusionless Lorenz system, which have been given by (Li et al. 2015), with variable step size strategy in (Çelik Kızılkan and Aydın 2011). In section 2, we have mentioned about linearization of Lorenz system. In section 3, we have given variable step size strategy for the linear systems (SSSLS). In section 4, we have obtained the numerical solution of linearized Lorenz System and diffusionless Lorenz system with variable step size strategy in (Çelik Kızılkan and Aydın 2011).

2. Linearization of the Lorenz System and Diffusionless Lorenz System

The Lorenz system was linearized using the signum function by (Li et al. 2015), because this process is successful in retaining dynamics in nonlinear systems. Using the linear transformation defined as

$$(x,y,z) \to (\sigma x, \sigma y, \sigma z), t \to \frac{t}{\sigma}$$

system (1) has been transformed into the following system

$$\begin{aligned} x' &= y - x \\ y' &= -z\, sgn(x) + c\, sgn(x) - ay \\ z' &= x\, sgn(y) - bz \end{aligned} \tag{3}$$

where $a = \frac{1}{\sigma}, b = \frac{\beta}{\sigma}$ and $c = \frac{r}{\sigma}$.

In a similar way, diffusionless Lorenz system given by (2) has reduced as

$$\begin{aligned} x' &= y - x \\ y' &= -z\, sgn(x) \\ z' &= x\, sgn(y) - b \end{aligned} \tag{4}$$

where $b = \frac{\beta}{\sigma}$ (Li et al. 2015).

3. Variable Step Size Strategy for the Linear Systems (SSSLS)

Selection of the step size is one of the most important concepts in numerical integration of differential equation systems. It is not practical to use constant step size in numerical integration. If the used step size is large in numerical integration, the computed solution can diverge from the exact solution. And if the used step size is small, the calculation time, number of arithmetic operations, and the calculation errors start to increase. So, if the solution is changing rapidly, the step size should be chosen small. Inversely, if the solution is changing slowly, we should choose bigger step size.

Let us consider the linear system

$$X' = AX, X(t_0) = X_0 \tag{5}$$

on the region $D = \{(t,X): |t - t_0| \le T, |x_j - x_{j0}| \le b_j\}$, where $A = (a_{ij}) \in R^{N\times N}, X(t) = (x_j(t)), X_0 = (x_{j0})$, $x_{j0} = x_j(t_0), X(t), X_0$ and $b = (b_j) \in R^N$. The variable step size has been obtained as

$$h_i = \frac{1}{\alpha N^{\frac{4}{5}}}\left(\frac{2\delta_L}{\beta_{i-1}}\right)^{1/2}, \quad (6)$$

where

$\alpha = \max_{1\le i,j\le N}|a_{ij}|, \beta_{i-1} = \max_{1\le j\le N}(\sup_{t_{i-1}\le\tau_i<t_i}|z_j(\tau_i)|)$ and δ_L is the error level for the system (5) (Çelik Kızılkan and Aydın 2011). It means that: when the step size given by (6) is used, error occurs as much as maximum δ_L. We should specify that the effects which computation errors play in the choice of step size can be neglected in here.

4. Analysis of Linearized Lorenz System with Variable Step Size

In this section, we aimed to observe the trajectories of linearized Lorenz system and diffusionless Lorenz system. Therefore; we can rewrite the Linearized Lorenz system given by (3) as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -a & -sgn(x) \\ sgn(y) & 0 & -b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ c\,sgnx \\ 0 \end{pmatrix} \quad (7)$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -a & -sgn(x) \\ sgn(y) & 0 & -b \end{pmatrix}$ and $d = \begin{pmatrix} 0 \\ c\,sgnx \\ 0 \end{pmatrix}$. Let's get the numerical approach of system (7) by applying the variable step size strategy given in equation (6).

Firstly, for initial value $(1 \;\; 2 \;\; 6)^T$ let us take $\sigma = 4, r = 16$ and β =1. Because, it is stated that Lorenz system can be made marginally more elegant by using these values in (Sprott 2010). And we wonder how the linearized Lorenz system will be for these values. Figure 2 shows that these trajectories, which represent two different systems, resemble each other. Note that the transformation $(x,y,z) \rightarrow (\sigma x, \sigma y, \sigma z)$ is used for linearization.

Now, consider (7) for chaotic values, i.e. $a = \frac{1}{10}, b = \frac{8/3}{10}$ and $c = \frac{28}{10}$ and obtain strange attractor for initial values $(0 \;\; 1 \;\; 0)^T$. Appling SSSLS to (7), gives trajectories as shown in Figure 3 that resemble the familiar Lorenz attractor. In Figure 4, it is constructed a three-dimensional phase portrait of a chaotic solution of the linearized Lorenz system. The numerical computations, made to obtain the following figures, are on error level $\delta_L = 10^{-2}$.

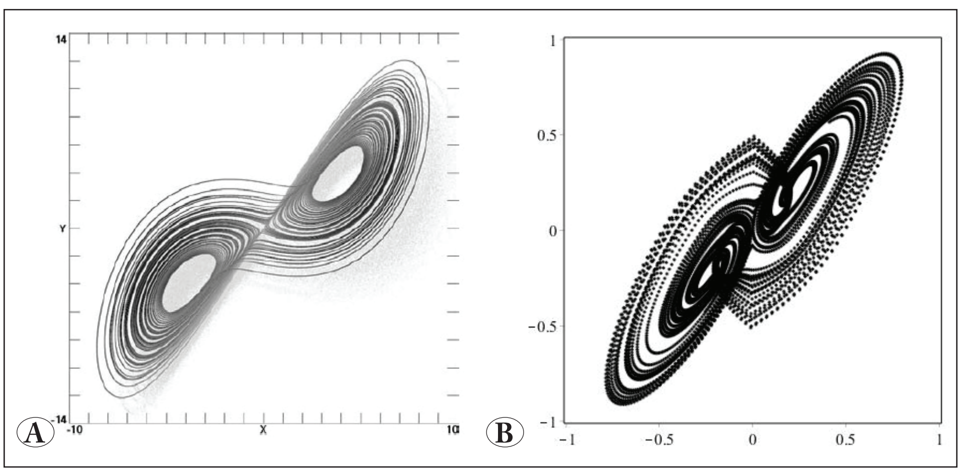

Figure 2. Projections of a trajectory in the x-y plane for $\sigma = 4, r = 16$ and β =1 with initial value value $(1 \;\; 2 \;\; 6)^T$ **(A)** Lorenz attractor in (Sprott 2010) **(B)** Linearized Lorenz system on error level $\delta_L = 10^{-1}$.

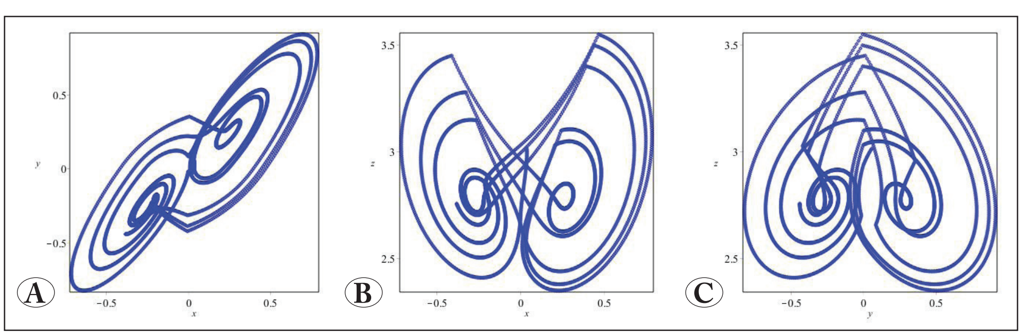

Figure 3. Phase portrait of system (7), $a = 0.1, b = 4/15, c = 2.8$ and initial value $(0 \;\; 1 \;\; 0)^T$ on error level $\delta_L = 10^{-2}$ **(A)** Projection in x-y plane, **(B)** Projection in x-z plane, **(C)** Projection in y-z plane.

For the parameters a = 0.1, b = 4/15, c = 2.8, it is stated that system (7) is no longer chaotic and a small change of b from b =4/15 to b =3/15 = 0.2 restores the chaos in (Li et al. 2015). Then, let apply SSSLS to system (7) with the parameters a = 0.1, b = 0.2, c = 2.8 and the initial value $(0 \;\; 1 \;\; 0)^T$. Phase portrait of linearized Lorenz system given in (Li et al. 2015) are shown in Figure 5 (A) and obtained with variable step size in Figure 5 (B).

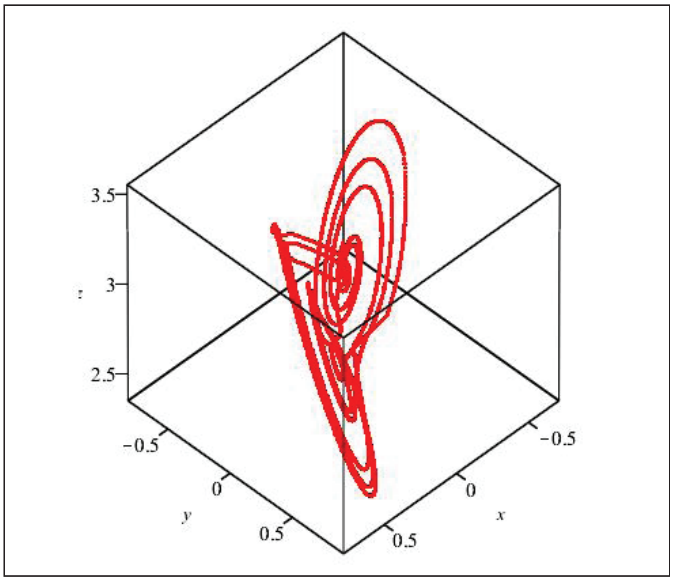

Figure 4. 3D view of system (7) in the xyz plane; $a = 0.1, b = 4/15, c = 2.8$ and initial value $(0 \;\; 1 \;\; 0)^T$ on error level $\delta_L = 10^{-2}$.

Diffusionless Lorenz system (4) can be written as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -sgn(x) \\ sgn(y) & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \qquad (8)$$

where $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -sgn(x) \\ sgn(y) & 0 & 0 \end{pmatrix}$ and $d = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$.

Let us apply variable step size strategy to system (8) for b=1 as in the (Li et al. 2015) In Figure 6 it can be seen that the trajectories obtained with variable step size are very close to the trajectories in (Li et al. 2015).

5. Conclusions

It is obvious that the behavior of the solutions of the Lorenz system changes more rapidly at some t values (see Figure 1). The step sizes used in these regions must be smaller. Because, if the selected step size is large in numerical integration, the computed solution can diverge from the exact solution. Inversely if the step size is small, the calculation time, number of arithmetic operations, and the calculation errors start to increase. Also, it is important to select the step size for convergence even if a constant step size is used.

In this paper, numerical solutions of the linearized Lorenz system and diffusionless Lorenz system have been obtained with variable step size. The obtained figures look like in the literature. The used strategy decided by performing error

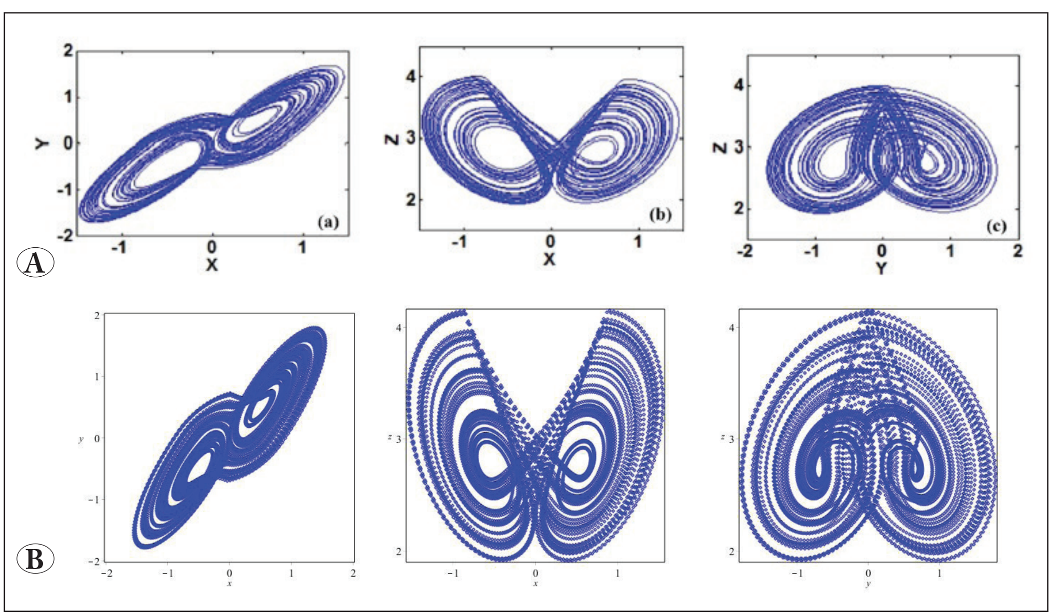

Figure 5. Phase portrait of system (7) $a = 0.1, b = 0.2, c = 2.8$ and initial value $(0 \;\; 1 \;\; 0)^T$ on error level $\delta_L = 10^{-1}$ **(A)** from [Li et all, 2015], **(B)** obtained with variable step size $\delta_L = 10^{-1}$.

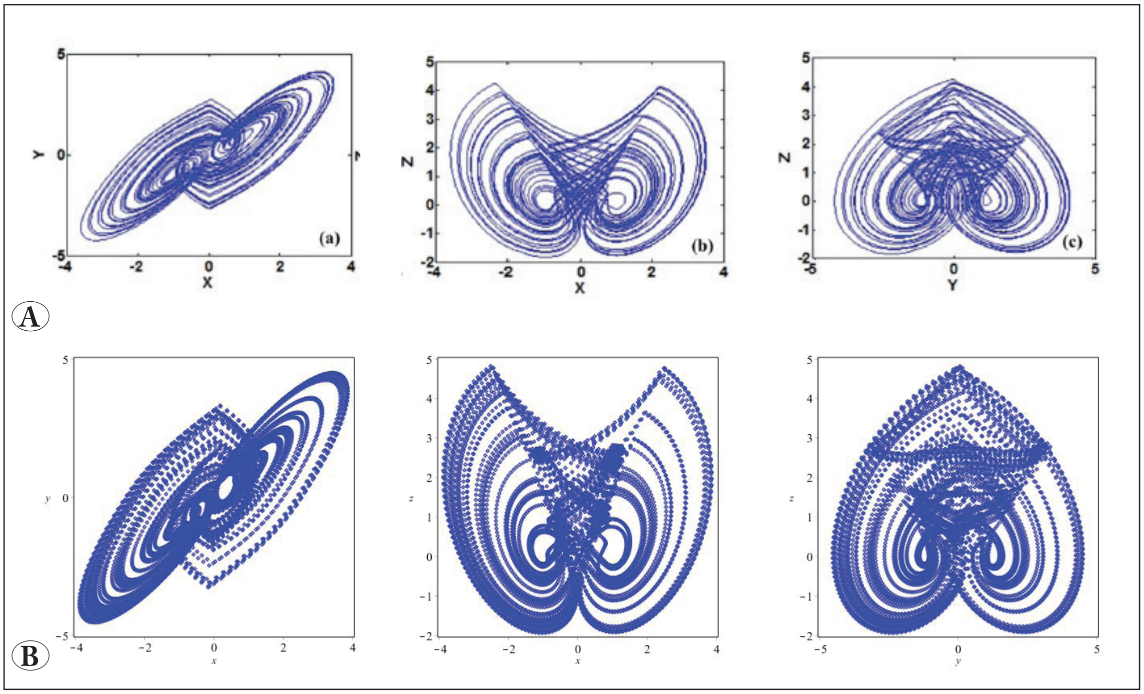

Figure 6. Phase portrait of system (8) for b = 1 initial value $(0 \ \ 1 \ \ 0)^T$ on error level $\delta_L = 10^{-1}$ **(A)** from [Li et all, 2015], **(B)** obtained with variable step size $\delta_L = 10^{-1}$.

checking at each step, where to use the small step size and where the large step size should be used. So, the variable step size allows to computation at the desired error level.

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