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Numerical Solution of *Drinfel'd Sokolov Wilson* System Using Differential Quadrature and Finite Difference Methods

Drinfel'd Sokolov Wilson Sisteminin Diferansiyel Kuadratür ve Sonlu Farklar Metodu Kullanılarak Nümerik Çözümü

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Abstract

In this paper, the numerical solution of the initial value problem defined by the *Drinfel'd-Sokolov-Wilson* system is investigated. The equations in the system are discretized spatially by using the differential quadrature method (DQM) which is a domain discretization method and have the property of giving accurate solutions with a small number of discretization points. The resulting time-dependent system of ordinary differential equations is then solved by an explicit-implicit finite difference method (FDM). By using an explicit-implicit scheme for the time integration, the possible stability problems are eliminated. The proposed method is tested numerically and accurate solutions are obtained with a small number of discretization points with a low computational cost.

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Keywords: Differential quadrature method, *Drinfel'd-Sokolov-Wilson* equation, Finite difference method

Öz

Bu makalede, *Drinfel'd-Sokolov-Wilson* sistemi tarafından tanımlanan başlangıç değer probleminin nümerik çözümü incelenmiştir. Sistemdeki denklemler uzayda bir bölge ayrıştırma metodu olan ve az sayıda ayrıştırma noktası ile doğru çözümler verme özelliği olan diferensiyel kuadratür metodu kullanılarak ayrıştırılmıştır. Sonuçta oluşan zaman-bağımlı adi diferansiyel denklemler sistemi daha sonra bir açık-kapalı sonlu farklar metodu ile çözülmüştür. Açık-kapalı bir zaman metodu kullanılarak mümkün olan kararlılık problemleri bertaraf edilmiştir. Önerilen metot nümerik olarak test edilmiştir ve az sayıda ayrıştırma noktası ile yani düşük bir hesaplama maliyeti ile doğru çözümler elde edilmiştir.

Anahtar Kelimeler: Diferensiyel kuadratür metodu, *Drinfel'd-Sokolov-Wilson* denklemi, Sonlu farklar metodu

1. Introduction

When the real life problems are modelled, one often ends up with nonlinear equations or systems of nonlinear equations. When the initial condition is given for the model, the problem should be solved to predict the future behaviour. However, the models most often involve nonlinear systems which do not have exact analytical solutions or are not easy to solve. Thus, using accurate and efficient numerical techniques is very important. For this aim, there are many techniques for the numerical solutions offered in the literature. Among these techniques, there are domain discretization methods such as finite difference method (FDM), finite element method (FEM) which are computationally expensive. In order to overcome this disadvantage of FDM and FEM, boundary element method (BEM) can be used which discretizes only the boundary of the domain which results with a less number of discretization points. However, in order to apply the method, one should know the fundamental solution of the corresponding equation.

Although, the differential quadrature method also needs to discretize the problem domain, it gives accurate results by using less number of discretization points comparing to the other discretization methods such as FDM, FEM. It may even use less number of discretization points than BEM which only discretizes the boundary (Meral and Tezer 2011).

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Drinfel'd-Sokolov system, which is one of the nonlinear system playing an important role in mathematical physics, is introduced by Drinfel'd and Sokolov as an example of a system of nonlinear equations possesing Lax pairs of a special form (Wazwaz 2006). Then, after some calculations it is shown in (Wilson 1982) that the equation system given by Hirota and Satsuma is an example of the theory introduced by Drinfel'd and Sokolov. Then in 1985, *Drinfel'd-Sokolov-Wilson* equations are then introduced as the generalization of the Korteweg-de Vries and Sine-Gordon equations both also having an important role in mathematical physics.

In this study, the initial value problem (IVP) defined by *Drinfel'd-Sokolov-Wilson* system is solved numerically by using a combination of DQM and FDM. Unlike the other domain discretization methods, DQM has the advantage of giving accurate solutions by using smaller number of discretization points. Moreover, it is able to solve the corresponding initial value problem without any need of boundary condition. For the solution of the ordinary differential equations obtained after the DQM discretization of the space derivatives, FDM (an explicit-implicit scheme) is used. The method is tested on an IVP defined by *Drinfel'd-Sokolov-Wilson* system. It is seen that the proposed method in this study gives accurate results with a small number of discretization points and without stability problems.

2. Problem Definition

Drinfel'd-Sokolov-Wilson equations are given by

$$\frac{\partial u}{\partial t}+\rho v\frac{\partial v}{\partial x}=0 \tag{1}$$

$$\frac{\partial v}{\partial t}+q\frac{\partial^3 v}{\partial x^3}+ru\frac{\partial v}{\partial x}+sv\frac{\partial u}{\partial x}=0 \tag{2}$$

for $x\in\mathbb{R}$ where ρ,q,r,s are nonzero parameters and $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ stand for the time and space derivatives, respectively. System (1)-(2) is subjected to the initial conditions

$$u(x,0)=u_0(x) \tag{3}$$

$$v(x,0)=v_0(x) \tag{4}$$

where $x\in\mathbb{R}$.

3. Numerical Solution of the Problem

3.1. Differential Quadrature Method Discretization in Space

The differential quadrature method is used to discretize the space derivatives seen in System (1)-(2). The approach which will be used here has been given by Shu (Shu 2000) and it uses the Lagrange polynomials for approximating the solution and its derivatives at node x_i by using the *N*-th order Lagrange polynomial:

$$u_i=\sum_{j=1}^{N} w_{ij}u(x_j), \tag{5}$$

$$\left.\frac{\partial^m u}{\partial x^m}\right|_{x=x_i}=\sum_{j=1}^{N} w_{ij}^{(m)}u(x_j) \tag{6}$$

In Equations (5) and (6), w_{ij} denotes the value of the *j*-th degree Lagrange polynomial at x_i and $w_{ij}^{(m)}=w_j^{(m)}(x_i)$ $(m=1,2,3)$ are the weighting coefficients at grid points x_i $(i=1,2,...,N)$ with *N* being the number of grid points and *m* indicating the derivative order. The weighting coefficients are determined by a practical notation (Shu, 2000) with the recursive relations for $i,j=1,2,...,N$:

$$w_{ij}^{(1)}=\frac{M^{(1)}(x_i)}{(x_i-x_j)M^{(1)}(x_j)},\quad i\neq j$$
$$w_{ii}^{(1)}=-\sum_{j=1,\,j\neq i}^{N} w_{ij}^{(1)},$$
$$w_{ij}^{(m)}=m\;\; w_{ij}^{(1)}w_{ii}^{(m-1)}-\frac{w_{ij}^{(m-1)}}{x_i-x_j},\quad i\neq j$$
$$w_{ii}^{(m)}=-\sum_{j=1,\,j\neq i}^{N} w_{ij}^{(m)}$$

with

$$\begin{aligned}M(x)&=(x-x_1)(x-x_2)...(x-x_N)\\&=\Upsilon(x,x_k)(x-x_k)\\M^{(1)}(x_k)&=(x_k-x_1)(x_k-x_2)...\\&(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_N)\\&=\prod_{i=1,\,i\neq k}^{N}(x_k-x_i)\end{aligned} \tag{7}$$

where $\Upsilon(x_i,x_j)=M^{(1)}(x_i)\delta_{ij}$ and δ_{ij} is the Kronecker operator.

Using the DQM approach given by (5) and (6) to *Drinfel'd-Sokolov-Wilson* equations at nodes x_i gives the discretized equations of the form

$$\left.\frac{\partial u}{\partial t}\right|_{x=x_i}=-\rho v_i\sum_{j=1}^{N} w_{ij}^{(1)}v_j \tag{8}$$

$$\left.\frac{\partial v}{\partial t}\right|_{x=x_i}=-q\sum_{j=1}^{N} w_{ij}^{(3)}v_j-ru_i\sum_{j=1}^{N} w_{ij}^{(1)}v_j-sv_i\sum_{j=1}^{N} w_{ij}^{(1)}u_j \tag{9}$$

In many numerical methods, the uniformly distributed nodes are preferred to use since they are easy to implement. However, for the DQM discretized problems, it is seen (Shu, 2000) that using nonuniformly distributed nodes gives better results. In this study, the Chebyshev-Gauss-Lobatto

(CGL) points which are given on $[-1,1]$ by

$$x_n = \cos\frac{(n-1)\pi}{N-1} \quad n = 1,2,...,N \tag{10}$$

are used. These points are known as clustering through the end points -1 and 1 chosen as the roots of $|T_n(x)| = 1$ where $T_n(x)$ is the *n*-th order Chebyshev polynomial.

The spatially-discretized Equations (8) and (9) can be written in matrix-vector form as

$$\frac{\partial u}{\partial t} = [A_1]\{v\} \tag{11}$$

$$\frac{\partial v}{\partial t} = [C]\{v\} + [A_2]\{v\} + [A_3]\{u\} \tag{12}$$

where the vectors $\{u\},\{v\},\left(\frac{\partial u}{\partial t}\right)$ and $\left(\frac{\partial v}{\partial t}\right)$ each of which has size *N*, containing the unknowns and their time derivatives at the grid points, respectively. Moreover the entries of the matrices $[A_1],[A_2],[A_3]$ and $[C]$ are given by

$[A_1]_{ij} = -\rho w_{ij}^{(1)} v_i,\ [A_2]_{ij} = -r w_{ij}^{(1)} u_i,\ [A_3]_{ij} = -s w_{ij}^{(1)} v_i,$
$[C]_{ij} = -q w_{ij}^{(3)}, i,j = 1,2,...,N.$

3.2. Time Discretization

Equations (11)-(12) is a system of coupled ODEs at the grid points. In order to obtain the solution at the discretized space points at a desired time, a time discretization is needed. To this end, a combination of the forward and backward Euler methods are used by making use of the newly updated solution, i.e.,

$$\{u^{n+1}\} = \{u^n\} + \Delta t[A_1]\{v^n\} \tag{13}$$

$$[A_v]\{v^{n+1}\} = \{v^n\} + \Delta t[A_3]\{u^{n+1}\} \tag{14}$$

where $[A_v] = (I - \Delta t[C] + \Delta t[A_2])$ and I is the identity matrix of size *N*. In equation (14) the superscript *n* describes the time level with $t^n = n\Delta t$ and Δt being the time step.

The solution is obtained iteratively by using Equations (13) and (14) for the desired time level starting with the initial conditions (3)-(4).

4. Numerical Results

In this section, the numerical solution procedure developed for the *Drinfel'd-Sokolov-Wilson* equations (1)-(2) is applied to the test problem with $p = q = r = s = 1$. The initial conditions are taken appropriate with the exact solution (Zhang, 2011)

$$u(x,t) = 2\,\mathrm{sec}h^2(x-t) \tag{15}$$

$$v(x,t) = 2\,\mathrm{sec}h(x-t). \tag{16}$$

In order to measure the accuracy of the solution the absolute maximum errors

$$\tau_1^n = \max_{1\le i\le N} |u_{exact}(x_i,t^n) - u_{num}(x,t^n)| \tag{17}$$

$$\tau_2^n = \max_{1\le i\le N} |v_{exact}(x_i,t^n) - v_{num}(x,t^n)| \tag{18}$$

are made use of. In Equations (17) and (18), $u_{exact}(x_i,t^n), v_{exact}(x_i,t^n)$ and $u_{num}(x_i,t^n), v_{num}(x_i,t^n)$ denote the exact and numerical solutions, respectively; at time level t^n. Numerical tests show that, the fully implicit character of the time integration in (14) eliminates the stability problems and the choice of the time increment does not effect the accuracy of the results. Therefore, the time increment $\Delta t = 0.1$ is used throughout the simulations.

On the other hand, the expected character of the DQM discretization in terms of the number of grid points is also observed which can also be seen in Table 1. The table (Table 1) contains the maximum absolute errors τ_1 at time *t* = 5.0 with different number of grid points.

The maximum absolute errors for both solutions of the IVP defined by *Drinfel'd-Sokolov-Wilson* system (1)-(2) are given in Table 2 for several time levels with *N* = 5 CGL points. From the table one can see that the solutions agree well with the exact solution.

In Figure 1., the behaviours of the exact and DQM solutions for the IVP defined by *Drinfel'd-Sokolov-Wilson* system is analyzed. In order to see to the fact that DQM solution

Table 1. Maximum absolute error τ_1 with different number of grid points.

	N = 5	*N* = 8	*N* = 10	*N* = 15	*N* = 20
τ_1	1.8 x 10^{-21}	5.2 x 10^{-21}	6.2 x 10^{-21}	7.4 x 10^{-21}	7.4 x 10^{-21}

Table 2. Maximum absolute errors for different times.

	t = 0.1	*t* = 0.5	*t* = 1.0	*t* = 5.0	*Nt* = 10.0
τ_1	4.3 x 10^{-24}	3 x 10^{-22}	1.8 x 10^{-21}	7.9 x 10^{-18}	1.7 x 10^{-13}
τ_2	4.6 x 10^{-12}	2.6 x 10^{-11}	6.4 x 10^{-11}	4 x 10^{-9}	5.9 x 10^{-7}

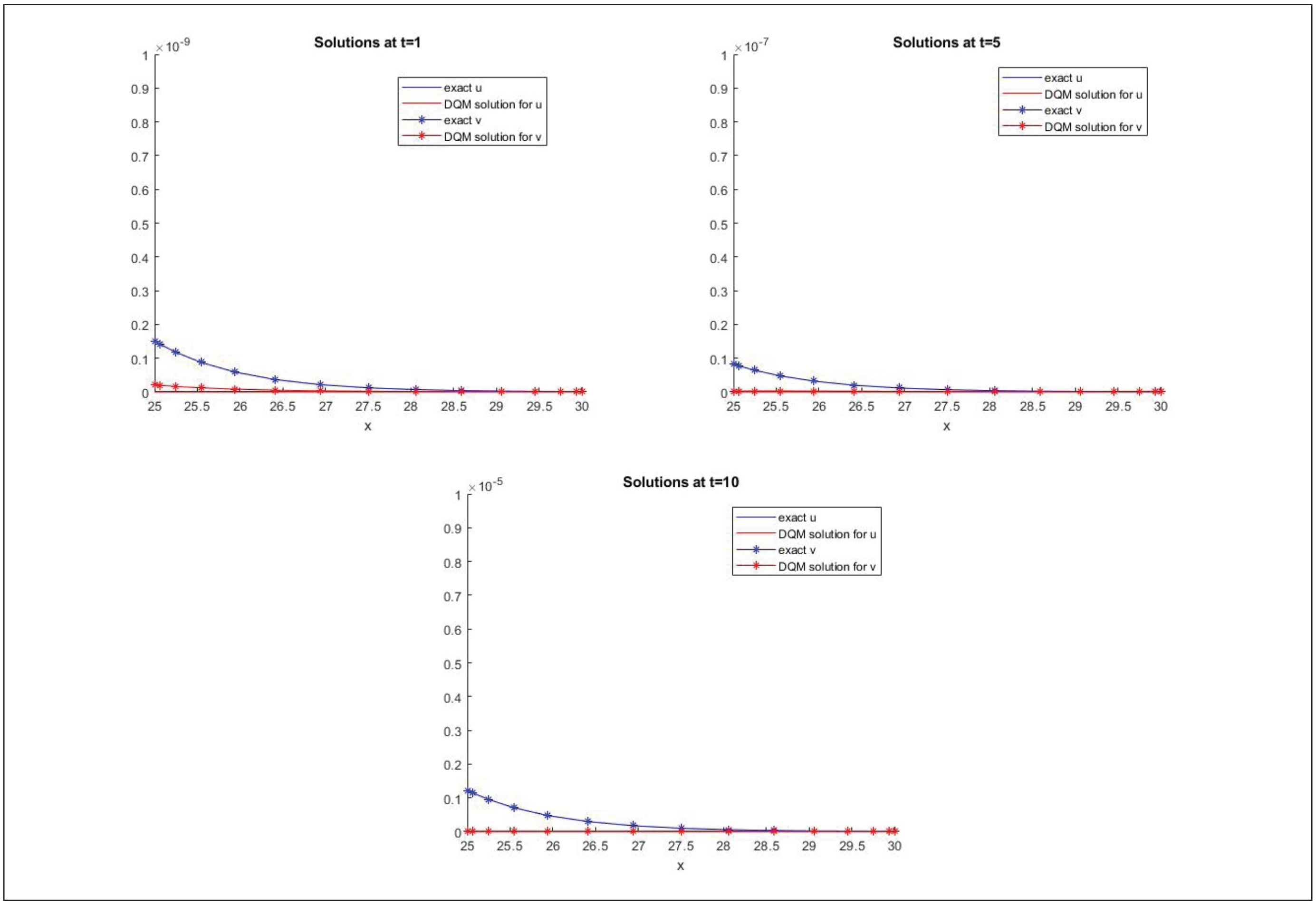

Figure 1. Solutions of the Drinfel'd-Sokolov-Wilson system at different times.

also have the behaviour of the exact solution, a bigger scale for the y-axis is used (otherwise all the solutions overlap and one cannot see the behaviour.) It can be easily observed also from the plots that the DQM and exact solutions are consistent with each other.

5. Conclusion

In this paper, a numerical procedure is developed for the solution of the *Drinfel'd-Sokolov-Wilson* equations. The solution method is a combination of the differential quadrature method in space and finite difference method in time. As a domain discretization method, differential quadrature method has the advantage of giving accurate solutions with a small number of discretization points and this is made use of in this study. For the solution of the obtained system of ODEs after the DQM discretization, a combination of the forward and backward Euler methods is used. The scheme is a combination of explicit and implicit methods which uses the updated solution values in the procedure. The numerical results agree well with the exact solution in terms of maximum absolute error. Moreover the consistency of exact and numerical solutions is observed with the plots at different times. For the future work, the proposed method can be applied to the other nonlinear evolution equations which take part in mathematical physics. Moreoever, other domain discretization methods in space direction and/or different time integration schemes can be applied and compared with the proposed method in terms of accuracy and computational cost.

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