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On the Lie Symmetry Analysis and Traveling Wave Solutions of Time Fractional Fifth-Order Modified Sawada-Kotera Equation

Zaman Kesirli Beşinci Mertebeden Modifiye Edilmiş Sawada-Kotera Denkleminin Lie Simetri Analizi ve İlerleyen Dalga Çözümleri

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Abstract

In this paper, we study Lie symmetry analysis of the time fractional fifth-order modified Sawada-Kotera equation (FMSK) with Riemann-Liouville derivative. Applying the adapted the Lie group theory to the equation under study, two dimensional Lie algebra is deduced. Using the obtained nontrivial Lie point symmetry, it is shown that the equation can be converted into a nonlinear fifth order ordinary differential equation of fractional order in the meaning of the Erdelyi-Kober fractional derivative operator. In addition, we construct some exact traveling solutions for the FMSK using the sub-equation method.

Keywords: Lie symmetry, Riemann-Liouville derivative, Time fractional modified Sawada-Kotera equation, Traveling wave solutions

Öz

Bu makalede Riemann-Liouville türevine sahip beşinci mertebeden zaman kesirli modifiye edilmiş Sawada-Kotera denkleminin Lie grup analizi araştırılmıştır. Lie grup teorisinin denkleme uygulanmasıyla iki boyutlu Lie cebri elde edilmiştir. Aşikar olmayan Lie simetrisinin kullanılmasıyla denklemin Erdelyi-Kober kesirli türev operatörü cinsinden beşinci mertebeden kesirli adi diferensiyel denkleme dönüştürülebileceği gösterilmiştir. Bunun yanında alt-denklem metodu kullanılarak denklemin bazı tam ilerleyen dalga çözümlerine ulaşılmıştır.

Anahtar Kelimeler: Lie simetri, Riemann-Liouville türevi, Zaman kesirli modifiye edilmiş Sawada-Kotera denklemi, İlerleyen dalga çözümleri

1. Introduction

Fractional partial differential equations (FPDEs) which are extensions of integer order partial differential equations (PDEs) have been observed in many scientific disciplines such as physics, control theory, signal processing, systems identification, cosmology, and finance etc., since last two decades. There exist well written monographs on these type of equations by using the theory of derivatives and integrals of fractional order (Podlubny 1999, Kilbas et al. 2006, Yang et al. 2015).

When FPDEs are analyzed, one of the most important question is the construction of the exact solutions for the equation (Kudryashov and Loguinova 2008). Exact solutions help to analyze the stability of considered systems and to validate the results of numerical analysis of FPDEs (Sahadevan and Bakkyaraj 2014). Recently, we observe several efficient methods such as exp function method (Bekir et al. 2013a), fractional first integral method (Bekir et al. 2015), fractional sub-equation method (Zhang and Zhang 2011, Wang and Xu 2014), fractional (G'/G)-expansion method (Bekir and Güner 2013b), Jacobi elliptic-function method (Adem and Muatjetjeja 2015), simplest equation method (Adem and Lü 2016), multiple exp-function method (Adem 2016) and fractional Lie group method (Gazizov et al. 2009, Sahadevan and Bakkyaraj 2012, Wang et al. 2015, Rui and Zhang 2015). In addition, there exist important studies.

In this article, we will suggest first the fractional Lie group method and utilize this method to solve the following time fractional modified Sawada-Kotera (FMSK) equation:

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$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xxxxx} - (5u_x u_{xx} + 5uu_x^2 + 5u^2 u_{xx} - u^5)_x \quad (1)$$

where $0 < \alpha < 1$ is a parameter describing the order of the fractional time derivative. In addition, we intended to obtain the exact traveling wave solutions of the FMSK equation by sub-equation method. This equation (for the case of $\alpha = 1$) appeared first in work of Liang et al. (Liang and Guo 2012). Based on the modified Sawada-Kotera equation, they introduce a 3x 3 matrix spectral problem with two potentials and derive a hierarchy of new nonlinear evolution equations. We observe some further studies on Eq.(1). For example, in (Konno 1992, Naz et al. 2013) and (Liang and Guo 2012) conseration laws and integrabilty properties (Lax pair and explicit solutions) are investigated.

The remainder of this paper is organized as follows. In Sect. 2 we present Lie symmetry analysis of the fractional partial differential equations (FPDEs). Then, we apply Lie group classification on the time FMSK equation, and investigate the symmetry reductions of the time FMSK equation. Through the symmetry reductions, we transform the FPDEs into the fractional ordinary differential equations (FODEs) with a new independent variable. In section 3, some exact traveling wave type solutions are obtained by the sub-equation method. Concluding remarks are summarized in Section 4.

2. Lie Symmetry Analysis of FPDEs

We first present notation to be used and recall the definitions and theorems that appear in (Podlubny 1999, Wang and Xu 2014, Gazizov et al. 2009, Sahadevan and Bakkyaraj 2012, Wang et al. 2015, Kiryakova 1994).

The Riemann-Liouville derivative of order α is defined by the following expression

$$\partial_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \dfrac{f(\tau,x)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ n-1 < \alpha < n, n \in N \\ \dfrac{\partial^n f}{\partial t^n}, \alpha = n \in N. \end{cases} \quad (2)$$

Consider a scalar-time FPDE having the following form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx}). \quad (3)$$

Equation (3) is invariant under a one-parameter Lie group of point transformations

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ \bar{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x,t,u) + O(\epsilon^2), \\ \frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \epsilon\eta^x(x,t,u) + O(\epsilon^2), \\ \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2), \\ \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\eta^{xxx}(x,t,u) + O(\epsilon^2), \\ \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} &= \frac{\partial^4 u}{\partial x^4} + \epsilon\eta^{xxxx}(x,t,u) + O(\epsilon^2), \\ \frac{\partial^5 \bar{u}}{\partial \bar{x}^5} &= \frac{\partial^5 u}{\partial x^5} + \varepsilon\eta^{xxxxx}(x,t,u) + O(\varepsilon^2), \end{aligned} \quad (4)$$

where

$$\begin{aligned} \eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \\ \eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi), \\ \eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\tau) - u_{xxxxx} D_x(\xi), \end{aligned} \quad (5)$$

Here, D_x denotes the total derivative operator and is defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + ... \quad (6)$$

with the associated vector field of the form

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}, \quad (7)$$

where the coefficient functions $\xi(x,t,u), \tau(x,t,u)$, and $\eta(x,t,u)$ of the vector field are to be determined.

If the vector field (7) generates a symmetry of (3), then V must satisfy the Lie symmetry condition

$$pr^{(5)}V(\Delta_1)|_{\Delta_1=0} = 0, \quad (8)$$

where $\Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} - F(x,t,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx})$. Since the lower limit of the integral (2) is fixed and, therefore, it should be invariant regard to such transformations (4). The invariance condition yields

$$\tau(x,t,u)|_{t=0} = 0. \quad (9)$$

The α th extended infinitesimal has to do with the Riemann-Liouville fractional time derivative with (9), which reads (see, Gazizov et al. 2009):

$$\begin{aligned} \eta_\alpha^0 &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ &- \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x). \end{aligned} \quad (10)$$

where

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^{r}\frac{\partial^{m}}{\partial t^{m}}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^{k}}. \tag{11}$$

Definition $u=\theta(t,x)$ *is an invariant solution of Eq. corresponding to the infinitesimal generator if and only if*

1) $u=\theta(t,x)$ *satisfies Eq.(3),*

2) $u=\theta(t,x)$ *is an invariant surface of (3), namely, it fulfills the invariant surface condition:*

$$\tau(x,t,\theta)\theta_t+\xi(x,t,\theta)\theta_x=\eta(x,t,\theta)$$

3. Application of Lie Symmetry Analysis of Time FMSK equation

According to the Lie theory, applying the fifth prolongation $pr^5 V$ to Eq. (1), we obtain the following equation:

$$\begin{aligned}&\eta_{\alpha}^{0}-\eta^{xxxxx}+5(u_x+u^2)\eta^{xxx}+10(u_{xx}+2uu_x)\eta^{xx}\\&+5(u_{xxx}+3u_x^2+4uu_{xx}-u^4)\eta^{x}\\&+10(2u_xu_{xx}+uu_{xxx}-2u^3u_x)\eta=0.\end{aligned} \tag{12}$$

Substituting (5) and (10) into (12), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, one can find the determining equations for the symmetry group of Eq.(1). Solving these equations, we obtain the following forms of the coefficient functions:

$$\xi=\alpha xc_1+c_2,\ \tau=5tc_1,\ \eta=-\alpha uc_1, \tag{13}$$

where c_1 and c_2 are arbitrary constants. Therefore, we can obtain the corresponding vector fields

$$V=5tc_1\frac{\partial}{\partial t}+(\alpha xc_1+c_2)\frac{\partial}{\partial x}-\alpha uc_1\frac{\partial}{\partial u}. \tag{14}$$

Thus, the Lie algebra of infinitesimal symmetry of Eq.(1) is spanned by the two vector fields:

$$V_1=\frac{\partial}{\partial x},\ V_2=5t\frac{\partial}{\partial t}+\alpha x\frac{\partial}{\partial x}-\alpha u\frac{\partial}{\partial u}. \tag{15}$$

Case 1. $X_1=\frac{\partial}{\partial x}$.

Integration of the invariant surface condition

$$\frac{dt}{0}=\frac{dx}{1}=\frac{du}{0}$$

gives the similarity variables t,u. Thus we have the ansatz $u=f(t)$. Inserting it into Eq.(1) yields the reduced fractional ODE

$$\partial_t^{\alpha}f(t)=0.$$

Solving the above equation we obtain the following group invariant solution

$$u=a_1t^{\alpha-1}.$$

Case 2.

The similarity variable and similarity transformation corresponding to the infinitesimal generator X_2 can be obtained by solving the associated characteristic equation given by

$$\frac{dx}{\alpha x}=\frac{dt}{5t}=\frac{du}{-\alpha u}, \tag{16}$$

and the corresponding invariants are

$$\xi=xt^{\frac{-\alpha}{5}},\ \ u=t^{\frac{-\alpha}{5}}g(\xi). \tag{17}$$

The transformation (17) reduces (1) to the following nonlinear ordinary differential equation of fractional order:

$$\left(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}g\right)(\xi)=g_{\xi\xi\xi\xi\xi}-(5g_{\xi}g_{\xi\xi}+5gg_{\xi}^2+5g^2g_{\xi\xi}-g^5)_{\xi}, \tag{18}$$

with the Erdelyi-Kober fractional differential operator $P_{\beta}^{\tau,\alpha}$ of order (Kiryakova 2004), (see also Wang and Xu 2014, Sahadevan and Bakkyaraj 2012):

$$(P_{\beta}^{\tau,\alpha}g):=\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\xi\frac{d}{d\xi}\right)(K_{\beta}^{\tau+\alpha,n-\alpha}g)(\xi), \tag{19}$$

$$n=\begin{cases}[\alpha]+1, & \alpha\notin N,\\ \alpha, & \alpha\in N,\end{cases} \tag{20}$$

where

$$(K_{\beta}^{\tau,\alpha}g)(\xi):=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}g\left(\xi u^{\frac{1}{\beta}}\right)du, & \alpha>0,\\ g(\xi), & \alpha=0,\end{cases} \tag{21}$$

is the Erdelyi-Kober fractional integral operator.

Indeed, based on the Riemann-Liouville fractional derivative $(n-1<\alpha<n, n=1,2,3,....)$, one can have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{\frac{-\alpha}{5}}g\left(xs^{\frac{-\alpha}{5}}\right)ds\right]. \tag{22}$$

Letting $v=\frac{t}{s}$, one can get $ds=-\frac{t}{v^2}dv$, therefore (22) can be written as

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\frac{6\alpha}{5}}\frac{1}{\Gamma(n-\alpha)}\int_1^{\infty}(v-1)^{n-\alpha-1}v^{-(n+1-\frac{6\alpha}{5})}g\left(\xi v^{\frac{\alpha}{5}}\right)dv\right]. \tag{23}$$

In view of the Erdelyi-Kober fractional integral operator (19), one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big]. \quad (24)$$

Taking into consideration the relation $(\xi = xt^{\frac{-\alpha}{5}})$, we can obtain

$$t\frac{\partial}{\partial t}\varphi(\xi)=tx\Big(-\frac{\alpha}{5}\Big)t^{-\frac{\alpha}{5}-1}\varphi'(\xi)=-\frac{\alpha}{5}\xi\frac{\partial}{\partial \xi}\varphi(\xi). \quad (25)$$

Therefore, one can get

$$\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big]=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Big)(\xi)\Big)\Big]$$
$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{6\alpha}{5}-1}\Big(n-\frac{6\alpha}{5}-\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big]. \quad (26)$$

Repeating the similar procedure as above for $n-1$ times, one can obtain

$$\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big]=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big)\Big]$$
$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{6\alpha}{5}-1}\Big(n-\frac{6\alpha}{5}-\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big] \quad (27)$$
$$=\ldots=t^{-\frac{6\alpha}{5}}\prod_{j=0}^{n-1}\Big(1-\frac{6\alpha}{5}+j-\frac{\alpha}{5}\xi\frac{d}{d\xi}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi).$$

Now using (19), we get

$$\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}g\Big)(\xi)\Big]=t^{-\frac{6\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}g\Big)(\xi). \quad (28)$$

Substituting (28) into (24), one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\frac{6\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}g\Big)(\xi). \quad (29)$$

Thus, the time FMSK equation can be reduced into an FODE

$$\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}g\Big)(\xi)=g_{\xi\xi\xi\xi\xi}-(5g_\xi g_{\xi\xi}+5gg_\xi^2+5g^2g_{\xi\xi}-g^5)_\xi. \quad (30)$$

4. Traveling Wave Solutions of the Time FMSK Equation

Fractional sub-equation method

We consider the following general nonlinear FPDEs:

$$P(u,u_t,u_x,D_t^\alpha u,D_x^\alpha u,\ldots)=0,\ 0<\alpha\le 1, \quad (31)$$

where u is an unknown function, and P is a polynomial of u.

Very recently, (Zhang and Zhang 2011) proposed fractional sub-equation method for Eq. (31) (see also, Wang and Xu 2014, Bekir and Güner 2014).

By using the traveling wave variable

$$u(x,t)=u(\xi),\ \xi=x+ct+\xi_0, \quad (32)$$

where c is a nonzero constant we can rewrite Eq. (31) as the following nonlinear fractional ordinary differential equation (FODE):

$$P(u,cu',u',c^\alpha D_\xi^\alpha u,D_\xi^\alpha u,\ldots)=0,\ \ 0<\alpha\le 1. \quad (33)$$

where the prime denotes the derivation with respect to ξ.

Suppose that Eq. (33) can be expressed by a polynomial in ϕ as follows:

$$u(\xi)=\sum_{i=1}^{n}a_i(\varphi)^i+a_0, \quad (34)$$

where $a_i(i=1,\ldots,n)$ are constants while $\varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha\varphi(\xi)=\sigma+\varphi^2(\xi), \quad (35)$$

where σ is a constant. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(33). By substituting Eq.(34) into Eq.(33) and using Eq.(35), we collect all terms with the same order of φ. By equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i, σ and c. By solving the equation system and substituting a_i, σ and c and the general solutions of Eq. (35) into Eq. (34), we can obtain a variety of exact solutions of Eq.(31).

Application of the fractional sub-equation method to the time FMSK equation

For our purpose, we introduce the following transformations:

$$u(x,t)=u(\xi),\ \ \xi=x+ct, \quad (36)$$

where c is a constant. By substituting (36) into (1), then (1) is reduced into an nonlinear fractional ordinary differential equation (NFODE):

$$c^\alpha D_\xi^\alpha u=u_{\xi\xi\xi\xi\xi}-(5u_\xi u_{\xi\xi}+5uu_\xi^2+5u^2u_{\xi\xi}-u^5)_\xi. \quad (37)$$

Suppose that the solutions of Eq. (37) can be expressed by a polynomial in ϕ as follows:

$$u(\xi)=a_0+\sum_{i=1}^{n}a_i(\varphi)^i, a_1\neq 0 \quad (38)$$

Balancing the highest order derivative terms with nonlinear terms in Eq. (37), we get

$$u(\xi)=a_0+a_1\varphi. \quad (39)$$

Substituting (39) along with (35) into (37) and then letting the coefficients of $(\phi)^i$ be zero, one can get some algebraic equations about c, a_0, a_1. Solving the algebraic equatios by Maple, we obtain two cases:

Case 1:

$$a_0 = 0, a_1 = -2, \quad \sigma = \sigma,\ c = (16\sigma^2)^{\frac{1}{\alpha}}, \alpha = \alpha \tag{40}$$

Case 2:

$$a_0 = 0, a_1 = 1, \quad \sigma = \sigma,\ c = (\sigma)^{\frac{2}{\alpha}},\ \alpha = \alpha \tag{41}$$

By using Eq.(40), expression (38) can be written as

$$u(\xi) = a_0 + a_1\varphi \tag{42}$$

where $\xi = x + (16\sigma^2)^{\frac{1}{\alpha}}t$. By substituting the general solutions of Eq. (35) into Eq. (42), we have three types of travelling wave solutions of the time FMSK equation as follows.

When $\sigma < 0$,

$$u_1(\xi) = 2\sqrt{-\sigma}\tanh(-\sqrt{-\sigma}\,\xi), \tag{43}$$

$$u_2(\xi) = 2\sqrt{-\sigma}\coth(-\sqrt{-\sigma}\,\xi), \tag{44}$$

where, $\xi = x + (16\sigma^2)^{\frac{1}{\alpha}}t$,

When $\sigma > 0$,

$$u_3(\xi) = -2\sqrt{\sigma}\tan(\sqrt{\sigma}\,\xi), \tag{45}$$

$$u_4(\xi) = 2\sqrt{\sigma}\cot(\sqrt{\sigma}\,\xi), \tag{46}$$

$\xi = x + (16\sigma^2)^{\frac{1}{\alpha}}t$,

When $\sigma = 0$,

$$u_5(\xi) = \frac{2\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \tag{47}$$

$\xi = x + (16\sigma^2)^{\frac{1}{\alpha}}t$,

In view of (41), we can get new types of explicit solutions of Eq. (1) as follows:

$$u_1(\xi) = -\sqrt{-\sigma}\tanh(-\sqrt{-\sigma}\,\xi), \tag{48}$$

$$u_2(\xi) = -\sqrt{-\sigma}\coth(-\sqrt{-\sigma}\,\xi), \tag{49}$$

where $\sigma < 0, \xi = x + (\sigma)^{\frac{2}{\alpha}}t$;

$$u_3(\xi) = \sqrt{\sigma}\tan(\sqrt{\sigma}\,\xi), \tag{50}$$

$$u_4(\xi) = -\sqrt{\sigma}\cot(\sqrt{\sigma}\,\xi), \tag{51}$$

where $\sigma > 0, \xi = x + (\sigma)^{\frac{2}{\alpha}}t$;

$$u_5(\xi) = -\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \tag{52}$$

where $\sigma = 0, \xi = x + (\sigma)^{\frac{2}{\alpha}}t$.

5. Concluding Remarks

In this paper, we performed the fractional Lie group analysis to time FMSK equation based on the sense of Riemann--Liouville derivative. We deduced two dimensional Lie symmetry algebra. Using the nontrivial Lie point symmetry

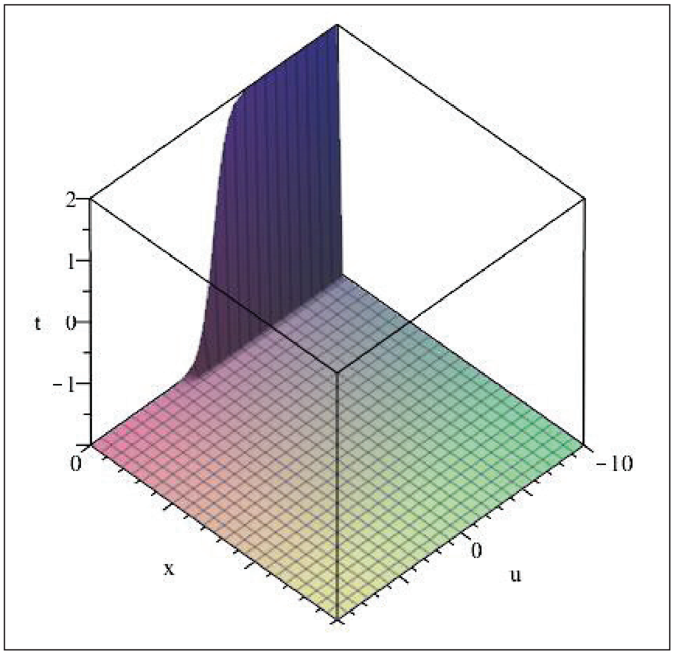

Figure 1. The profile of solution $u_1(x, t)$ in Case 1 where $\sigma = -1, c = 1, \alpha = 0,7$.

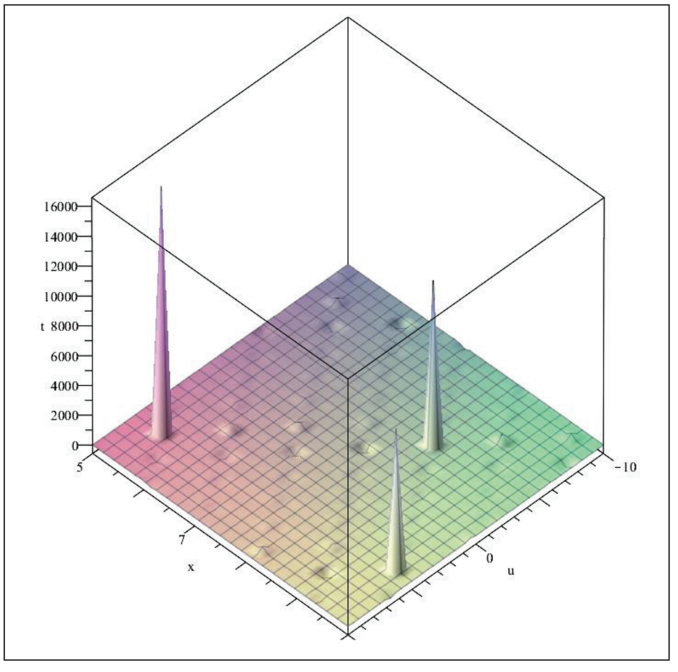

Figure 2. The profile of solution $u_4(x, t)$ in Case 1 where $\sigma = 1, c = 1, \alpha = 0,5$.

generator, we showed that time FMSK equation can be transformed into a NFODE. In addition, using the sub-equation method, we obtain hyperbolic, trigonometric and rational solutions.The obtained exact solutions can be used as benchmarks against the numerical simulations (Adem and Khalique 2012).

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