



Distribution of the Lower Boundary Functional of the Step Process of Semi-Markov Random Walk with Delaying Barrier

Tutan Bariyerli Yarı-Markov Rastgele Yürüyüş Sıçrama Sürecinin Alt Sınır Fonksiyonelinin Dağılımı

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Abstract

In this paper, a step process of semi-Markovian random walk with delaying barrier on the zero-level is constructed and the Laplace transformation of the distribution of first crossing time of this process into the delaying barrier is obtained. Also, the expectation and standard diversion of a boundary functional of the process are given.

Keywords: Delaying screen, Laplace transformation, Semi-Markovian random walk

Öz

Bu çalışmada sıfır seviyesinde tutan bariyerli bir yarı-Markov rastgele yürüyüş basamak süreci kurulmuş ve bu sürecin tutan bariyere ilk kez düşme anının dağılımının Laplace dönüşümü elde edilmiştir. Ayrıca sürecin bir sınır fonksiyonelinin beklenen değer ve standart sapması verilmiştir.

Anahtar Kelimeler: Tutan bariyer, Laplace dönüşümü, Yarı-markov rastgele yürüyüş

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1. Introduction

In recent years, random walks with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. Many good monographs in this field exist in literature (see references (1-3) and etc.).

In particular, a number of very interesting problems of stock control, queues and reliability theories can be expressed by means of random walks with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite volumes or queueing systems with finite waiting time or sojourn time by means of random walks with two delaying barriers. Furthermore, the functioning of stochastic systems with spare equipment can be given by random walks with two barriers, one of them is delaying and the other one is any type barrier.

It is known that the most of the problems of stock control theory is often given by means of random walks or random walks with delaying barriers (see References 1-2 etc.). Numerous studies have been done about step processes of semi-Markovian random walk with two barriers of their practical and theoretical importance. But in the most of these studies the distribution of the process has free distribution. Therefore the obtained results in this case are cumbersome and they will not be useful for applications (1,2,3,4 and etc.).

In this study, we considered a semi-Markov random walk with delaying barrier on the zero-level, and this process represents the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. The processes of this type have been widely studied in literature (see, for

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example References (1-4). But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the quantity of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with delaying barrier at zero-level. This type problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc.

2. Construction of the Process

Suppose that $\{(\xi_i, \eta_i)\}, i = 1, 2, 3, \dots$, is a sequence of identically and independently distributed pairs of random variables, defined on any probability space (Ω, \mathcal{F}, P) such that ξ_i 's are positive valued, i.e.,

$$P\{\xi_i > 0\} = 1, i = 1, 2, 3, \dots$$

In addition, the random variables ξ_i and η_i are mutually independent as well. Also let us denote the distribution function of ξ_1 and η_1

$$\Phi(t) = P\{\xi_1 < t\}, t \in \mathbb{R}^+ \text{ and } F(x) = P\{\eta_1 < x\}, x \in \mathbb{R},$$

respectively.

By using the random pairs (ξ_i, η_i) we can construct a step process of a semi-Markov random walk

$$X_1(t) = \sum_{i=1}^n \eta_i, \text{ if } \sum_{i=0}^{n-1} \xi_i \leq t < \sum_{i=0}^n \xi_i, n \geq 1$$

where $\xi_0 = 0$ and $\eta_0 = z \geq 0$. Let this process is delayed by a barrier at zero:

$$X(t) = X_1(t) - \inf_{0 \leq s \leq t} \{0, X_1(s)\}.$$

This process forms a step process of semi-Markovian random walk with delaying barrier on the zero-level.

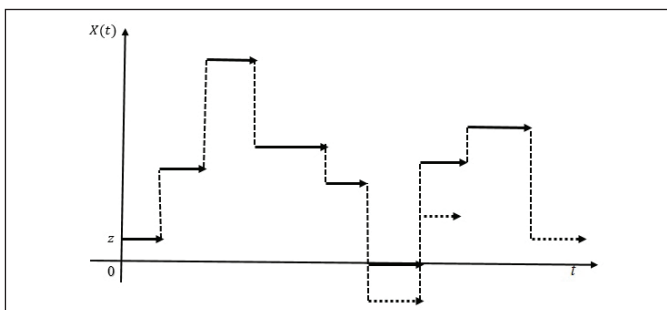


Figure 1. A view of a step process of Semi-Markov Random Walk with delaying barrier on the zero-level.

Now, we introduce the random variable $\tau_0 = \tau_0(w) = \inf\{t: X(t) = 0\}$. We set $\tau_0 = \infty$ if $X(t) > 0$ for all t . It is obvious that the random variable τ_0 is the time of the first crossing of the the process $X(t)$ into the delaying barrier at zero level. In this case τ_0 is rewritten as $\tau_0 = \min\{k \geq 1: X(t) = 0\}$.

The aim of the present work is to determine the Laplace transform of the distribution of the random variable τ_0 . Denote the Laplace transform of the distribution of the random variable τ_0 by

$$L(\theta) = E[e^{-\theta\tau_0}]$$

and the Laplace transform of the conditional distribution of the random variable τ_0 . by

$$L(\theta | z) = E[e^{-\theta\tau_0} | X(0) = z], \theta > 0, z \geq 0.$$

According to the total probability formula, we can write

$$L(\theta) = \int_{z=0}^{\infty} L(\theta | z) dP(X(0) < z).$$

Let us denote the conditional distribution of random variable of τ_0 and the Laplace transformation of the conditional distribution with

$$N(t | z) = P[\tau_0 > t | X(0) = z],$$

and

$$\bar{N}(\theta | z) = \int_{t=0}^{\infty} e^{-\theta t} N(t | z) dt,$$

respectively. Then, the Laplace transformation of the absolute distribution has the following form:

$$\bar{N}(\theta) = \int_{z=0}^{\infty} \bar{N}(\theta | z) dP(X(0) < z).$$

Finally, let us denote the Laplace transformation of the random variable ξ_1 by

$$\varphi(\theta) = E[e^{-\theta\xi_1}].$$

Thus, we can easily obtain that

$$\bar{N}(\theta | z) = \frac{1 - L(\theta | z)}{\theta}$$

or, equivalently,

$$L(\theta | z) = 1 - \theta \bar{N}(\theta | z).$$

3. Derivation of The Integral Equation for $\bar{N}(\theta | z)$

In this section we give an integral equation for $\bar{N}(\theta | z)$. For this, we can state the following theorem:

Theorem 1. Under above representations, we get

$$\bar{N}(\theta | z) = \frac{1 - \varphi(\theta)}{\theta} + \varphi(\theta) \int_{y=0}^{\infty} \bar{N}(\theta | y) d_y P\{\eta_1 < y - z\}.$$

Proof: According to the total probability formula, it is obvious that

$$\begin{aligned}
 N(t|z) &= P\{\tau_0 > t; \xi_1 > t | X(0) = z\} + \\
 &P\{\tau_0 > t; \xi_1 < t | X(0) = z\} \\
 &= P\{\xi_1 > t\} + \int_{s=0}^t \int_{y=0}^{\infty} P\{\xi_1 \in ds; z + \eta_1 \in dy\} \\
 &P\{\tau_0 > t - s | X(0) = z\}.
 \end{aligned}$$

Thus, we have an integral equation for the distribution of the random variable τ_0 as follows:

$$\begin{aligned}
 N(t|z) &= P\{\xi_1 > t\} \\
 &+ \int_{s=0}^t \int_{y=0}^{\infty} P\{\xi_1 \in ds\} N(t-s|y) d_y P\{\eta_1 < y-z\}.
 \end{aligned}
 \tag{3.1}$$

By applying the Laplace transform to this equation, i.e., multiplying both sides of (3.1) by $e^{-\theta t}$, integrating with respect to t from 0 to ∞ , and taking into account the definition of $\bar{N}(\theta|z)$ we have

$$\begin{aligned}
 \bar{N}(\theta|z) &= \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1 > t\} dt \\
 &+ \int_{y=0}^{\beta} \int_{t=0}^{\infty} e^{-\theta t} \int_{s=0}^t P\{\xi_1 \in ds\} N(t-s|y) d_y P\{\eta_1 < y-z\} \\
 &= \frac{1 - \varphi(\theta)}{\theta} \\
 &+ \varphi(\theta) \int_{y=0}^{\infty} \bar{N}(\theta|y) d_y P\{\eta_1 < y-z\}.
 \end{aligned}
 \tag{3.2}$$

Therefore, the Theorem is proved.

This integral equation can be solved by method of successive approximations for arbitrarily distributed random variables ξ_i and $\eta_i, i \geq 1$, but it is unsuitable for applications. On the other hand, this equation has a solution in explicit form in the class of compound Laplace distributions. For example, let the random walk follow the compound Laplace distribution. We introduce random variable $\eta_1 = \eta_1^+ + \eta_2^+ - \eta_1^-$ in which random variables $\eta_i^+, i = 1, 2$ and η_i^- are distributed as follows:

$$F_{\eta_i^+}(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\lambda t}, & t > 0, \lambda > 0, i = 1, 2, \end{cases}$$

and

$$F_{\eta_i^-}(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\mu t}, & t > 0, \mu > 0, \end{cases}$$

respectively. Then it is easy to see that the distribution function of random variable η_1 has the form

$$\begin{aligned}
 F_{\eta_1}(t) &= P\{\eta_1 < t\} \\
 &= \begin{cases} \frac{\lambda^2}{(\lambda + \mu)^2} e^{\mu t}, & t < 0, \\ 1 - \frac{\mu}{\lambda + \mu} \left[1 + \frac{\lambda}{\lambda + \mu} + \lambda t \right] e^{-\lambda t}, & t > 0, \end{cases}
 \end{aligned}
 \tag{3.3}$$

and the probability density function of its has the form

$$\begin{aligned}
 F_{\eta_1}(t) &= [F_{\eta_1}(t)]' \\
 &= \begin{cases} \frac{\lambda^2 \mu}{(\lambda + \mu)^2} e^{\mu t}, & t < 0, \\ \left[\frac{\lambda^2 \mu}{\lambda + \mu} \left[\frac{\lambda}{\lambda + \mu} + t \right] \right] e^{-\lambda t}, & t > 0, \end{cases}
 \end{aligned}
 \tag{3.4}$$

(see, Nasirova and Omarova 2007). The distribution function given by (3.3) is called the compound Laplace distribution function of order (2,1) and denote by $L(2^+, 1^-)$. In this case, the integral equation (3.2) can be rewritten as

$$\begin{aligned}
 \bar{N}(\theta|z) &= \frac{1 - \varphi(\theta)}{\theta} \\
 &+ \frac{\lambda^2 \mu}{(\lambda + \mu)^2} \varphi(\theta) \int_{y=0}^z e^{\mu(y-z)} \bar{N}(\theta|y) dy \\
 &+ \frac{\lambda^2 \mu}{\lambda + \mu} \varphi(\theta) \int_{y=z}^{\infty} \left[\frac{\lambda}{\lambda + \mu} + y - z \right] e^{-\lambda(y-z)} \bar{N}(\theta|y) dy.
 \end{aligned}
 \tag{3.5}$$

By using the integral equation (3.5), we obtain the ordinary differential equation with constant coefficients

$$\begin{aligned}
 \bar{N}'''(\theta|z) - (2\lambda - \mu) \bar{N}''(\theta|z) \\
 + \lambda(2\mu - \lambda) \bar{N}'(\theta|z) + \lambda^2 \mu [1 - \varphi(\theta)] \bar{N}(\theta|z) \\
 = \frac{\lambda^2 \mu}{\theta} [1 - \varphi(\theta)].
 \end{aligned}
 \tag{3.6}$$

On the other hand, since $\bar{N}(\theta|z) = \frac{1 - L(\theta|z)}{\theta}$, by substituting this expression in (3.5), we can write the integral equation in $\bar{M}(\theta|z)$

$$\begin{aligned}
 \bar{M}(\theta|z) &= 1 - \varphi(\theta) + \\
 &+ \frac{\lambda^2 \mu}{(\lambda + \mu)^2} \varphi(\theta) \int_{y=0}^z e^{\mu(y-z)} \bar{M}(\theta|y) dy \\
 &+ \frac{\lambda^2 \mu}{\lambda + \mu} \varphi(\theta) \int_{y=z}^{\infty} \left[\frac{\lambda}{\lambda + \mu} + y - z \right] e^{-\lambda(y-z)} \bar{M}(\theta|y) dy,
 \end{aligned}
 \tag{3.7}$$

where $\bar{M}(\theta|z) = \theta \bar{N}(\theta|z)$. Therefore, from (3.6), we get a homogen differential equation:

$$\begin{aligned}
 \bar{M}'''(\theta|z) - (2\lambda - \mu) \bar{M}''(\theta|z) \\
 + \lambda(2\mu - \lambda) \bar{M}'(\theta|z) + \lambda^2 \mu [1 - \varphi(\theta)] \bar{M}(\theta|z) = 0.
 \end{aligned}
 \tag{3.8}$$

The characteristic equation of this differential equation is

$$k^3(\theta) - (2\lambda - \mu)k^2(\theta) + \lambda(2\mu - \lambda)k(\theta) + \lambda^2\mu[1 - \varphi(\theta)] = 0. \tag{3.9}$$

and so, it has a general solution in the form

$$\overline{M}(\theta | z) = C_1(\theta)e^{k_1(\theta)z} + C_2(\theta)e^{k_2(\theta)z} + C_3(\theta)e^{k_3(\theta)z} + 1, \tag{3.10}$$

where $k_1(\theta), k_2(\theta)$ and $k_3(\theta)$ are roots of the characteristic equation (3.9).

Now, we should find $C_i(\theta), i = 1, 2, 3$. The following system is obtained from the initially boundary conditions for differential equation (3.8) from integral equation (3.7), by taking zero instead of z :

$$\begin{cases} \overline{M}(\theta | 0) = 1 - \varphi(\theta) + \frac{\lambda^2\mu}{\lambda + \mu}\varphi(\theta)\int_{y=0}^{\infty}\left[\frac{\lambda}{\lambda + \mu} + y\right]e^{-\lambda y}\overline{M}(\theta | y)dy, \\ \overline{M}'(\theta | 0) = \frac{\lambda^2\mu}{\lambda + \mu}\varphi(\theta)\int_{y=0}^{\infty}\left[-\frac{\mu}{\lambda + \mu} + \lambda y\right]e^{-\lambda y}\overline{M}(\theta | y)dy, \\ \overline{M}''(\theta | 0) = \frac{\lambda^2\mu}{\lambda + \mu}\varphi(\theta)\int_{y=0}^{\infty}\left[-\frac{\lambda + 2\mu}{\lambda + \mu} + \lambda y\right]e^{-\lambda y}\overline{M}(\theta | y)dy. \end{cases} \tag{3.11}$$

By substituting these expressions on (3.10), we have

$$\begin{cases} \overline{M}(\theta | 0) = C_1(\theta) + C_2(\theta)e^{k_2(\theta)} + C_3(\theta) + 1, \\ \overline{M}'(\theta | 0) = k_1(\theta)C_1(\theta) + k_2(\theta)C_2(\theta) + k_3(\theta)C_3(\theta), \\ \overline{M}''(\theta | 0) = k_1^2(\theta)C_1(\theta) + k_2^2(\theta)C_2(\theta) + k_3^2(\theta)C_3(\theta). \end{cases} \tag{3.12}$$

Now, by multiplying the first equality by λ^2 and second equality by -2λ and then adding these equalities side by side, we have

$$\overline{M}''(\theta | 0) - 2\lambda\overline{M}'(\theta | 0) + \lambda^2\overline{M}(\theta | 0) = -\lambda^2\varphi(\theta).$$

Thus we can write the following linear equation system of $C_i(\theta), i = 1, 2, 3$:

$$\begin{cases} \sum_{i=1}^3 (k_i(\theta) - \lambda)^2 C_i(\theta) = -\lambda^2\varphi(\theta), \\ \sum_{i=1}^3 \left[k_i(\theta) - \lambda - \frac{\lambda^2\mu\varphi(\theta)}{(k_i(\theta) - \lambda)(\lambda + \mu)} \right] C_i(\theta) = \frac{\lambda^2\varphi(\theta)}{\lambda + \mu}, \\ \sum_{i=1}^3 \left[k_i^2(\theta) - \lambda k_i(\theta) - \frac{\lambda^2\mu\varphi(\theta)}{(k_i(\theta) - \lambda)(\lambda + \mu)} \right] C_i(\theta) = \frac{\lambda^2\mu\varphi(\theta)}{\lambda + \mu}. \end{cases} \tag{3.13}$$

Taking into account the expressions in (3.12), we conclude that all equations of system (3.13) can be reduced to the single equation

$$(k_1(\theta) - \lambda)^2 C_1(\theta) + (k_2(\theta) - \lambda)^2 C_2(\theta) + (k_3(\theta) - \lambda)^2 C_3(\theta) = -\lambda^2\varphi(\theta).$$

In this case, the only single solution is

$$C_1(\theta) = \frac{\lambda^2\varphi(\theta)}{(k_1(\theta) - \lambda)^2}, C_2(\theta) = C_3(\theta) = 0,$$

and therefore, we have

$$\overline{M}(\theta | z) = \frac{\lambda^2\varphi(\theta)}{(k_1(\theta) - \lambda)^2}e^{k_1(\theta)z} + 1.$$

From this, we can write

$$L(\theta | z) = 1 - \overline{M}(\theta | z) = \frac{\lambda^2\varphi(\theta)}{(k_1(\theta) - \lambda)^2}e^{k_1(\theta)z}. \tag{3.14}$$

By using (3.14) and the definition of $L(\theta)$, we get

$$\begin{aligned} L(\theta) &= \int_{z=0}^{\infty} L(\theta | z) dP(\eta_1^+ < z) \\ &= \lambda^2 \int_{z=0}^{\infty} ze^{-\lambda z} L(\theta | z) dz \\ &= \frac{\lambda^2\varphi(\theta)}{(k_1(\theta) - \lambda)^2} \int_{z=0}^{\infty} ze^{-\lambda z} e^{k_1(\theta)z} dz \\ &= \frac{\lambda^2\varphi(\theta)}{(k_1(\theta) - \lambda)^2} \int_{z=0}^{\infty} ze^{(k_1(\theta) - \lambda)z} dz \\ &= \frac{\lambda^4\varphi(\theta)}{(k_1(\theta) - \lambda)^4}. \end{aligned} \tag{3.15}$$

We now determine $E[\tau_0]$ and $D[\tau_0]$, expectation and standart diversion of the random variable τ_0 . Since $L(\theta)$ is a Laplace transformation, it is written $E[\tau_0] = -L'(0)$ and $D[\tau_0] = -L''(0) - [L'(0)]^2$. Thus, by considering (3.15), we can write

$$L^{(\theta)} = \frac{\lambda^4[\varphi'(\theta)(\lambda - k_1(\theta)) + 4k_1'(\theta)\varphi(\theta)]}{(\lambda - k_1(\theta))^5}$$

and

$$\begin{aligned} L''(\theta) &= \frac{\lambda^4}{(k_1(\theta) - \lambda)^5} \{ [\varphi''(\theta)(\lambda - k_1(\theta)) \\ &+ 8k_1'(\theta)\varphi'(\theta) + 4k_1''(\theta)\varphi(\theta)] \\ &+ (\lambda - k_1(\theta)) + 20[k_1'(\theta)]^2\varphi(\theta) \}. \end{aligned}$$

If we put $\theta = 0$, then we get

$$L'(0) = \frac{\lambda\varphi'(0) + 4k_1'(0)}{\lambda}$$

and

$$L''(0) = \frac{\lambda^2\varphi''(0) + 8\lambda k_1'(0)\varphi'(0) + 4\lambda k_1''(0) + 20[k_1'(0)]^2}{\lambda^2}.$$

From (3.9) we have

$$\begin{aligned} k_1'(0) &= \frac{\lambda\mu}{\lambda - 2\mu}\varphi'(0), k_1''(0) \\ &= \frac{1}{\lambda(\lambda - 2\mu)}[\lambda^2\mu\varphi''(0) + 2(2\lambda - \mu)[k_1'(0)]^2] \end{aligned}$$

and therefore,

$$L'(0) = \frac{\lambda + 2\mu}{\lambda - 2\mu} E[\xi_1], L''(0) = \frac{\lambda + 2\mu}{\lambda - 2\mu} E[\xi_1] + \frac{8\lambda^2 + 4\lambda\mu^2 - 16\mu^3}{(\lambda - 2\mu)^2} (E[\xi_1])^2.$$

Thus, it is obtained that

$$E[\tau_0] = -L'(0) = -\frac{\lambda + 2\mu}{\lambda - 2\mu} E[\xi_1]$$

and

$$D[\tau_0] = -L''(0) - [L'(0)]^2 = \frac{\lambda + 2\mu}{\lambda - 2\mu} D[\xi_1] + \frac{4\lambda\mu(\lambda + \mu) - 16\mu^3}{(\lambda - 2\mu)^3} (E[\xi_1])^2.$$

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