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Solitary Wave and other Solutions of Nonlinear Space-Time Fractional Differential Equation Systems

Lineer Olmayan Uzay – Zaman Kesirli Diferensiyel Denklem Sistemlerinin Soliter Dalga ve Diğer Çözümleri

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Abstract

In this study, we have successfully found some travelling wave solutions of the variant Boussinesq system and fractional system of two-dimensional Burgers' equations of fractional order by using the $\left(\frac{G'}{G}\right)$-expansion method. These exact solutions contain hyperbolic, trigonometric and rational function solutions. The fractional complex transform is generally used to convert a partial fractional differential equation (FDEs) with modified Riemann-Liouville derivative into ordinary differential equation. We showed that the considered transform and method are very reliable, efficient and powerful in solving wide classes of other nonlinear fractional order equations and systems.

Keywords: The $\left(\frac{G'}{G}\right)$ -expansion method, exact solution, Fractional order system of variant Boussinesq equations, Fractional system of two-dimensional Burgers' equations.

Öz

Bu çalışmada, $\left(\frac{G'}{G}\right)$-açılım metodu kullanarak kesir mertebeli Boussinesq denklem sistemleri ve kesirli iki boyutlu Burgers' denklemlerinin bazı hareketli dalga çözümleri elde edilmiştir. Bu tam çözümler hiperbolik, trigonometrik ve rasyonel fonksiyon çözümlerini içermektedir. Kesirli karmaşık dönüşüm, genellikle, modifiye Riemann-Liouville türevi içeren kesirli kısmi diferansiyel denklemi adi diferensiyel denkleme dönüştürmek için kullanılır. Düşünülen dönüşüm ve metodun, diğer lineer olmayan kesir mertebeli denklemlerin ve sistemlerin çözümünde güvenilir, verimli ve etkili bir yol olduğu gösterilmiştir.

Anahtar Kelimeler: $\left(\frac{G'}{G}\right)$-açılım metodu, Tam çözüm, Kesir mertebeli variant Boussinesq denklem sistemi, İki boyutlu kesir mertebeli Burger denklem sistemi.

1. Introduction

Fractional calculus is used successfully in the field of mathematical physics, biology, engineering and the other field of applied sciences. Many important events are well described by differential equations of fractional order in control theory, acoustics, electromagnetics, electrochemistry, viscoelasticity, fluid flow, systems identification and signal and image processing. Fractional calculus has been used to described that are found to be best model by fractional differential equations (FDEs). Therefore, FDEs have been investigated by many researchers (1-3). For that reason, we need a reliable and efficient technique for the solution of fractional differential equations. So far, many powerful and efficient methods have been suggested to obtain numerical solutions and exact solutions of them. For example, numerical analytical methods for solving fractional problems are the adomian decomposition method (ADM) (4), the variational iteration method (VIM) (5), the homotopy perturbation method (HPM) (6), the homotopy analysis method (HAM) (7) and exact solution methods are the $\left(\frac{G'}{G}\right)$ -expansion method (8,9), the first integral method (10,11), the fractional sub-equation method (12,13), the modified trial equation method (14,15), the exp-function method (16,17) and the functional variable method (18).

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In this paper, we are going to use the $\left(\frac{G'}{G}\right)$-expansion method (19-22) for solving nonlinear fractional partial differential equations with modified Riemann-Liouville derivative. In Section 2, we describe the algorithm for using the $\left(\frac{G'}{G}\right)$-expansion method with fractional complex transform to solve FDEs. In Section 3 and 4, we will apply it to the fractional order system of variant Boussinesq and two-dimensional Burgers' equations. In the last section, conclusions are given.

2. The $\left(\frac{G'}{G}\right)$-Expansion Method and the Fractional Complex Transform

Let us consider the general nonlinear FDE as follow,

$$H(u,D_t^{\alpha}u,D_x^{\beta}u,D_y^{\psi},D_t^{2\alpha}u,D_t^{\alpha}D_x^{\beta}u,D_t^{\alpha}D_y^{\psi}u,D_x^{2\beta}u,D_x^{\beta}D_y^{\psi}u,\dots)=0,$$
$$0<\alpha,\beta,\psi<1 \tag{2.1}$$

where $u=u(x,y,t)$ is an unknown function. H is a polynomial of u and its partial fractional derivatives. By using the traveling wave transform (23)

$$u(x,y,t)=f(\theta), \tag{2.2}$$

$$\theta=\frac{ax^{\beta}}{\Gamma(1+\beta)}+\frac{by^{\psi}}{\Gamma(1+\psi)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{2.3}$$

where $a\neq 0, b\neq 0$ and $c\neq 0$ are constants. If we use chain rule, then we get

$$\begin{aligned}D_t^{\alpha}u&=\sigma'_t\frac{df}{d\theta}D_t^{\alpha}\theta\\ D_x^{\alpha}u&=\sigma'_x\frac{df}{d\theta}D_x^{\alpha}\theta\\ D_y^{\alpha}u&=\sigma'_y\frac{df}{d\theta}D_y^{\alpha}\theta\end{aligned} \tag{2.4}$$

where σ'_t,σ'_x and σ'_y are named the sigma indexes (24). We can take them $\sigma'_t=\sigma'_x=\sigma'_y=\phi$ where ϕ is a constant.

Setting (2.2) and (2.4) into (2.1), we get the following nonlinear ODE;

$$\Psi(U,U',U'',\dots)=0. \tag{2.5}$$

where the primes denote the derivations with respect to θ.

Suppose that the solution of the equation (2.5) is expressed by a polynomial of $\left(\frac{G'}{G}\right)$ as follows:

$$U(\theta)=\sum_{i=0}^{z}a_i\left(\frac{G'}{G}\right)^i,\ a_z\neq 0 \tag{2.6}$$

where $a_i(i=0,1,2,\dots,z)$ are constants, while $G(\theta)$ satisfies the following second order LODE

$$G''(\theta)+\lambda G'(\theta)+\mu G(\theta)=0, \tag{2.7}$$

where λ and μ are constants. The positive integer z can be determined by taking the homogeneous balance in the Eq.(2.5). We have collected all terms with the same order of $\left(\frac{G'}{G}\right)$ by putting the Eq.(2.6) into the Eq.(2.5) and using the equation (2.7). By equalizing each coefficient of the polynomial to zero, we have obtained a set of algebraic equations system for $a_i(i=0,1,2,\dots,z),a,b,c$. Thus we can get a variety of exact solutions of equation (2.1), by solving the equations system.

3. $\left(\frac{G'}{G}\right)$-Expansion Method for Fractional Order System of Variant Boussinesq Equations

In this study, the fractional order system of variant Boussinesq equations is proposed on $\left(\frac{G'}{G}\right)$-expansion method

$$\begin{aligned}&D_t^{\alpha}v+D_x^{\alpha}(uv)+D_x^{3\alpha}u=0\\ &D_t^{\alpha}u+D_x^{\alpha}v+D_x^{\alpha}u=0\end{aligned} \tag{3.1}$$

where is the parameter of the order of the fractional derivative, and $0<\alpha\leq 1$. For a model of the water waves, $u(x,t)$ is the velocity, $v(x,t)$ is the total depth and the subscripts denote partial derivatives (25). Gepreel and Mohamed, have found numerical solutions of Eqs. (3.1) by using the HAM. In (26), Yan studied the travelling wave solutions of these equations and obtained three new types of travelling wave solutions by using fractional sub-equation method. When $\alpha=1$, Eqs.(3.1) is called the variant Boussinesq equations. Some researchers investigated exact and analytical solutions of these equations. Wang (27) has obtained exact solutions of them by homogeneous balance method. In (28), several types of explicit and exact solutions contain Wang's results. The system of the Eqs. (3.1) has been obtained by using an improved Sine-cosine method and Wu elimination method. Yomba (29) has found travelling wave solutions of classic version by using the extended Fan's sub-equation method. In (30), Lu has obtained abundant Jacobi elliptic function solutions of the variant Boussinesq equations. Soliman and Abdo, by using modified extended direct algebraic (MEDA) method obtained multiple exact complex solutions of system of variant Boussinesq equations (31). Wu and He (32) have obtained a solitary wave solutions by exp-function method. Recently, Zhao and his colleagues, used the $\left(\frac{G'}{G}\right)$-expansion method and obtained soliton solutions of these equations (33).

If we use the following transformations

$$u(x,t)=U(\xi),\ \xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)},$$
$$v(x,t)=V(\xi),\ \xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \quad (3.2)$$

where $c\neq 0$ and $k\neq 0$ are constants.

By using of Eq. (3.2) and (2.4), Eq. (3.1) can be turned into an ODEs

$$cV'+k(UV)'+\phi^2k^3U'''=0, \quad (3.3)$$

$$-cU'+kV'+kUU'=0, \quad (3.4)$$

where "U"$=\frac{dU}{d\xi}$ and "V"$=\frac{dV}{d\xi}$. By once integrating, we obtain

$$-cV+kUV+\phi^2k^3U''+\xi_0=0, \quad (3.5)$$

$$-cU+kV+\frac{k}{2}U^2+\xi_1=0, \quad (3.6)$$

where ξ_0 and ξ_1 are integration constants.

If we consider the homogeneous balance between the terms UV and U'' in (3.5), then we get

$$z_1+z_2=z_1+2, \quad (3.7)$$

$$z_2=2. \quad (3.8)$$

Analogously, if we take the terms V and U^2 in (3.6) we get,

$$2z_1=z_2. \quad (3.9)$$

$$z_2=2. \quad (3.10)$$

According to the solutions of (3.8) and (3.10) the polynomial of $\left(\frac{G'}{G}\right)$ can be expressed as follows:

$$U(\xi)=a_0+a_1\frac{G'}{G},a_1\neq 0 \quad (3.11)$$

$$V(\xi)=b_0+b_1\frac{G'}{G}+b_2\ \frac{G'^2}{G},\ b_2\neq 0 \quad (3.12)$$

where a_0,a_1,b_0,b_1 and b_2 are constants. By using Eqs. (3.11), (3.12) and (2.7) we obtain that

$$U''(\xi)=2a_1\frac{G'^3}{G}+3a_1\lambda\frac{G'^2}{G}+(2a_1\mu+a_1\lambda^2)\frac{G'}{G}+a_1\lambda\mu, \quad (3.13)$$

$$U^2(\xi)=a_1^2\frac{G'^2}{G}+2a_0a_1\frac{G'}{G}+a_0^2. \quad (3.14)$$

If we substitute Eqs.(3.11)-(3.14) into Eqs.(3.5) and (3.6), then we can get the coefficients of $\left(\frac{G'}{G}\right)$ and equalizing them to zero we obtain the system as follow:

3: $kb_2a_1+2\phi^2k^3a_1=0,$
2: $-cb_2+kb_2a_0+kb_1a_1+3\phi^2k^3a_1\lambda=0,$
1: $kb_0a_1-cb_1+\phi^2k^3a_1\lambda^2+2\phi^2k^3a_1\mu+kb_1a_0=0,$
0: $kb_0a_0-cb_0+\phi^2k^3a_1\lambda\mu+\xi_0=0,$
2: $kb_2+\frac{1}{2}ka_1^2=0,$
1: $ka_0a_1+kb_1-ca_1=0,$ (3.15)
0: $-ca_0+kb_0+\frac{1}{2}ka_0^2+\xi_1=0,$

If we solve this system by Maple, then we get

(a)

$$a_0=\frac{c-\phi k^2\lambda}{k},\quad a_1=-2\phi k,\quad b_0=-2\phi^2k^2\mu,$$
$$b_1=-2\phi^2k^2\lambda,\quad b_2=-2\phi^2k^2,\quad \xi_0=0,$$
$$\xi_1=\frac{c^2+4\phi^4k^4\mu-\phi^2k^4\lambda^2}{2k}\quad k=k\quad c=c \quad (3.16)$$

and

(b)

$$a_0=\frac{c+\phi k^2\lambda}{k},\quad a_1=2\phi k,\quad b_0=-2\phi^2k^2\mu,$$
$$b_1=-2\phi^2k^2\lambda,\quad b_2=-2\phi^2k^2,\quad \xi_0=0,$$
$$\xi_1=\frac{c^2+4\phi^4k^4\mu-\phi^2k^4\lambda^2}{2k}\quad k=k\quad c=c \quad (3.17)$$

By using Eq. (3.16), Eqs. (3.11) and (3.12) can be written as

$$U(\xi)=\frac{c-\phi k^2\lambda}{k}-2\phi k\frac{G'}{G}, \quad (3.18)$$

$$V(\xi)=-2\phi^2k^2\mu-2\phi^2k^2\lambda\frac{G'}{G}-2\phi^2k^2\frac{G'^2}{G}. \quad (3.19)$$

(a): By substituting general solutions of Eq. (2.6) into Eqs. (3.18) and (3.19) we have three types of exact solutions of the nonlinear fractional order variant Boussinesq equation as follows:

If $\lambda^2-4\mu>0$,

$$U_1(\xi)=\frac{c}{k}$$
$$-\phi k\sqrt{\lambda^2-4\mu}\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}} \quad (3.20)$$

$$V_1(\xi)=\frac{\phi^2k^2(\lambda^2-4\mu)}{2}1$$
$$-\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}\right)^2 \quad (3.21)$$

where $\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}$. In particular, if $C_1=0,C_2\neq 0,\lambda>0,\mu=0$, then U_1 and V_1 become

$$u_1(x,t)=\frac{c}{k}-\phi k\lambda\tanh\left\{\frac{\lambda}{2}\left(\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right\}, \quad (3.22)$$

$$v_1(x,t)=\frac{\phi^2k^2\lambda^2}{2}\sec h^2\left\{\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right\},\tag{3.23}$$

In particular, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0,$ then U_1 and V_1 become

$$u_2(x,t)=\frac{c}{k}-\phi k\lambda\coth\left\{\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right\},\tag{3.24}$$

$$v_2(x,t)=-\frac{\phi^2k^2\lambda^2}{2}-\operatorname{csc}h^2\left\{\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right\}.\tag{3.25}$$

If $\lambda^2-4\mu<0$, ,

$$U_2(\xi)=\frac{c}{k}-\phi k\sqrt{\lambda^2-4\mu}\,\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi},\tag{3.26}$$

$$V_2(\xi)=\frac{\phi^2k^2(\lambda^2-4\mu)}{2}1+\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2\tag{3.27}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(\mathrm{A}+\alpha)}$. In particular, if $C_1=0, C_2\neq 0, \lambda>0, \mu=0$, then U_2 becomes $u_2(x,t)$ and V_2 becomes $v_2(x,t)$.

On the other hand, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0$, then U_2 becomes $u_1(x,t)$ and V_2 becomes $v_1(x,t)$.

If $\lambda^2-4\mu=0$,

$$u_3(x,t)=\frac{c}{k}-\frac{2\phi kC_2}{C_1+C_2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)},\tag{3.28}$$

$$v_3(x,t)=-2\phi^2k^2\frac{C_2}{C_1+C_2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}^2.\tag{3.29}$$

(b): Analogously, by using Eq. (3.16) with Eqs. (3.11) and (3.12) can be expressed as

$$U(\xi)=\frac{c+\phi k^2\lambda}{k}+2\phi k\frac{G'}{G},\tag{3.30}$$

$$V(\xi)=-2\phi^2k^2\mu-2\phi^2k^2\lambda\frac{G'}{G}-2\phi^2k^2\frac{G'}{G}^2.\tag{3.31}$$

By writing general solutions of Eq. (2.7) into Eqs. (3.30) and (3.31) we have three kinds of exact solutions as follows:

If $\lambda^2-4\mu>0$,

$$U_3(\xi)=\frac{c}{k}+\phi k\sqrt{\lambda^2-4\mu}\,\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}},\tag{3.32}$$

$$V_3(\xi)=\frac{\phi^2k^2(\lambda^2-4\mu)}{2}1-\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}\right)^2\tag{3.33}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}$. In particular, if $C_1=0, C_2\neq 0, \lambda>0, \mu=0$ then U_3 becomes

$$u_4(x,t)=\frac{c}{k}+\phi k\lambda\tanh\left\{\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right\},\tag{3.34}$$

and V_3 becomes $v_1(x,t)$.

In particular, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0$ then U_3 becomes

$$u_5(x,t)=\frac{c}{k}+\phi k\lambda\coth\left\{\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right\},\tag{3.35}$$

and V_3 becomes $v_2(x,t)$.

If $\lambda^2-4\mu<0$,

$$U_4(\xi)=\frac{c}{k}+\phi k\sqrt{4\mu-\lambda^2}\,\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi},\tag{3.36}$$

$$V_4(\xi)=\frac{\phi^2k^2(\lambda^2-4\mu)}{2}1+\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2\tag{3.37}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}$. In particular, if $C_1=0, C_2\neq 0, \lambda>0, \mu=0$, then U_4 becomes $u_5(x,t)$ and V_4 becomes $v_2(x,t)$.

On the other hand, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0$, then U_4 becomes $u_4(x,t)$ and V_4 becomes $v_1(x,t)$.

If $\lambda^2 - 4\mu = 0$,

$$u_6(x,t) = \frac{c}{k} + \frac{2\phi k C_2}{C_1 + C_2\left(kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}, \tag{3.38}$$

$$v_4(x,t) = -2\phi^2 k^2 \left(\frac{C_2}{C_1 + C_2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}\right)^2. \tag{3.39}$$

which are the exact solutions of the nonlinear fractional order variant Boussinesq equations. We note that the exact solutions established in (3.22) - (3.23), (3.24) - (3.25), (3.28) - (3.29), (3.34), (3.35) and (3.38) - (3.39) are new exact solutions to these equations.

4. $\left(\frac{G'}{G}\right)$-Expansion Method for Fractional Space-time System of 2D Burger' Equations

In this case, we'll deal with the fractional system of two-dimensional Burgers' equations (34)

$$\begin{aligned} D_t^\alpha u + uD_x^\alpha u + vD_y^\alpha u &= \frac{1}{K}[D_x^{2\alpha} u + D_y^{2\alpha} u] \\ D_t^\alpha u + uD_x^\alpha v + vD_y^\alpha v &= \frac{1}{K}[D_x^{2\alpha} v + D_y^{2\alpha} v] \end{aligned} \tag{4.1}$$

Where K is Reynolds number. α is a parameter which is the order of the fractional space-time derivative and $0 < \alpha \le 1$. Elhanbaly and Abdou, have found exact solutions of fractional order system of two dimensional Burgers' equations by using the sub-equation method. When $\alpha = 1$ Eqs. (4.1) is the classical system of Burgers' equations. Burgers' equation has been obtained to describe various type of phenomena such as a mathematical model of turbulence (35) and the approximate theory of flow through a shock wave traveling in a viscous fluid (36). The numerical solutions of this equation system have been investigated by several authors. For example, the analytic solution of Eqs. (4.1) was given by Fletcher using the Hopf-Cole transformation and he has discussed the comparison of a number of different numerical approaches (37,38). In (39), a fully implicit finite-difference method is used to solve two dimensional Burgers' equations. Wubs and Goede (40) have applied an explicit-implicit method and Goyon (41) used several multi-level schemes. El-Sayed and Kaya, obtained the numerical and analytical solutions of these system by using Adomian decomposition method (ADM) (42). Recently; Biazar and Aminikhah (43), used the variational iteration method (VIM) and obtained numerical solutions of these equations then Zhu et al. (44), used the discrete Adomian decomposition method (ADM) and obtained numerical solutions.

If we use the following wave transformations

$$\begin{aligned} u(x,y,t) = U(\xi),\ \ \xi = \frac{ax^\alpha}{\Gamma(1+\alpha)} + \frac{by^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \\ v(x,y,t) = V(\xi),\ \ \xi = \frac{ax^\alpha}{\Gamma(1+\alpha)} + \frac{by^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{4.2}$$

where $a \neq 0, b \neq 0$ and $c \neq 0$ are constants.

With similar procedure in section 3, Eq. (4.1) can be turned into an ODEs

$$-cU + aUU' + bVU' = \frac{1}{K}(a^2\phi U'' + b^2\phi U''), \tag{4.3}$$

$$-cV' + aUV' + bVV' = \frac{1}{K}(a^2\phi V'' + b^2\phi V''), \tag{4.4}$$

where $"U'" = \frac{dU}{d\xi}$ and $"V'" = \frac{dV}{d\xi}$.

By homogeneous balance between the terms UU' and U'' in (43a), then we get,

$$\begin{aligned} z_1 + z_1 + 1 &= z_1 + 2, \\ z_1 &= 1. \end{aligned} \tag{4.5}$$

Similarly, between VV' and V'' in Eq.(4.4) we get,

$$\begin{aligned} z_2 + z_2 + 1 &= z_2 + 2, \\ z_2 &= 1. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we can express the polynomial of $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = a_0 + a_1\frac{G'}{G}, a_1 \neq 0, \tag{4.7}$$

$$V(\xi) = b_0 + b_1\frac{G'}{G}, b_1 \neq 0. \tag{4.8}$$

We can drive the equations by using Eqs. (4.7), (4.8) and (2.7) as follows:

$$U'(\xi) = -a_1\frac{G'^2}{G} - a_1\lambda\frac{G'}{G} - a_1\mu, \tag{4.9}$$

$$\begin{aligned} U''(\xi) &= 2a_1\frac{G'^3}{G} + 3a_1\lambda\frac{G'^2}{G} \\ &+ (2a_1\mu + a_1\lambda^2)\frac{G'}{G} + a_1\lambda\mu, \end{aligned} \tag{4.10}$$

$$V'(\xi) = -b_1\frac{G'^2}{G} - b_1\lambda\frac{G'}{G} - b_1\mu, \tag{4.11}$$

$$\begin{aligned} V''(\xi) &= 2b_1\frac{G'^3}{G} + 3b_1\lambda\frac{G'^2}{G} \\ &+ (2b_1\mu + b_1\lambda^2)\frac{G'}{G} + b_1\lambda\mu, \end{aligned} \tag{4.12}$$

By setting Eqs.(4.7)-(4.12) into Eqs.(4.3) and (4.4), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i = 0, \ldots, 3)$ and equalizing them to zero we get the system as follows

$$\begin{aligned}
3:&\ \frac{2\phi a^2 a_1}{K}+\frac{2\phi b^2 a_1}{K}+ba_1b_1+aa_1^2=0,\\
2:&\ -aa_1a_0+ca_1-\frac{3\phi a^2 a_1\lambda}{K}-bb_0a_1-ba_1b_1\lambda\\
&-\frac{3\phi b^2 a_1\lambda}{K}-aa_1^2\lambda=0,\\
1:&\ -\frac{2\phi a^2 a_1\mu}{K}-bb_1a_1\mu+ca_1\lambda-\frac{\phi a^2 a_1\lambda^2}{K}\\
&-\frac{2\phi b^2 a_1\mu}{K}-aa_1^2\mu-\frac{\phi b^2 a_1\lambda^2}{K}-aa_0a_1\lambda-bb_0a_1\lambda=0,\\
0:&\ -\frac{\phi a^2 a_1\lambda\mu}{K}-aa_0a_1\mu-\frac{\phi b^2 a_1\lambda\mu}{K}+ca_1\mu-bb_0a_1\mu=0,\\
3:&\ \frac{2\phi b^2 b_1}{K}+\frac{2\phi a^2 b_1}{K}+ab_1a_1+bb_1^2=0,\\
2:&\ -bb_1^2\lambda-bb_1b_0-aa_0b_1+cb_1-\frac{3\phi b^2 b_1\lambda}{K}\\
&-ab_1a_1\lambda-\frac{3\varphi a^2 b_1\lambda}{K}=0,\\
1:&\ -\frac{2\phi b^2 b_1\mu}{K}-bb_1^2\mu+cb_1\lambda-\frac{\phi a^2 b_1\lambda^2}{K}-bb_0b_1\lambda-ab_1a_0\lambda\\
&-ab_1a_1\mu-\frac{2\phi a^2 b_1\mu}{K}-\frac{\phi b^2 b_1\lambda^2}{K}=0,\\
0:&\ -\frac{\varphi b^2 b_1\lambda\mu}{K}-bb_0b_1\mu-\frac{\phi a^2 b_1\lambda\mu}{K}+cb_1\mu-aa_0b_1\mu=0.
\end{aligned} \tag{4.13}$$

By solving the algebraic equations above given, we get

$$\begin{aligned}
&a_0=a_0,\quad a_1=-\frac{2\phi a^2+2\phi b^2+bb_1K}{aK},\\
&b_0=b_0,\quad b_1=b_1,\\
&a=a,\quad b=b,\\
&\qquad c=\frac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{K}.
\end{aligned} \tag{4.14}$$

where λ and μ are arbitrary constants.

If we put Eq. (4.14) into Eq. (4.7) and (4.8), it yields

$$U(\xi)=a_0-\frac{2\phi a^2+2\phi b^2+bb_1K}{aK}\frac{G'}{G}, \tag{4.15}$$

$$V(\xi)=b_0+b_1\frac{G'}{G}. \tag{4.16}$$

By writing general solutions of Eq. (2.7) into (4.15) and (4.16), we get three kinds of solutions as follows:

When $\lambda^2-4\mu>0$,

$$\begin{aligned}
U_1(\xi)&=a_0+\frac{2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K}{2aK}\\
&-\frac{(2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K)\sqrt{\lambda^2-4\mu}}{2aK}\\
&\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}},
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
V_1(\xi)&=b_0-\frac{b_1\lambda}{2}+\frac{b_1\sqrt[3]{\lambda^2-4\mu}}{2}\\
&\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu\xi}}
\end{aligned} \tag{4.18}$$

where $\xi=\frac{ax^\alpha}{\Gamma(1+\alpha)}+\frac{by^\alpha}{\Gamma(1+\alpha)}$.
$-\frac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{K\Gamma(1+\alpha)}t^\alpha.$

In particular, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0$ then $U_1(\xi)$ and $V_1(\xi)$ become

$$\begin{aligned}
u_1(x,y,t)&=a_0+\frac{2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K}{2aK}\\
&-\frac{(2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K)\lambda}{2aK}\coth\left\{\frac{\lambda}{2}\left(\frac{ax^\alpha}{\Gamma(1+\alpha)}\right.\right.\\
&\left.\left.+\frac{by^\alpha}{\Gamma(1+\alpha)}-\frac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{KT(1+\alpha)}t^\alpha\right)\right\},\\
v_1(x,y,t)&=b_0-\frac{b_1\lambda}{2}+\frac{b_1\lambda}{2}\coth\left\{\frac{\lambda}{2}\left(\frac{ax^\alpha}{\Gamma(1+\alpha)}\right.\right.\\
&\left.\left.+\frac{by^\alpha}{\Gamma(1+\alpha)}-\frac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{KT(1+\alpha)}t^\alpha\right)\right\}
\end{aligned} \tag{4.19}$$

When $\lambda^2-4\mu<0$,

$$\begin{aligned}
U_2(\xi)&=a_0+\frac{2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K}{2aK}\\
&-\frac{(2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K)\sqrt{4\mu-\lambda^2}}{2aK}\\
&\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}-C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}},
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
V_2(\xi)&=b_0-\frac{b_1\lambda}{2}+\frac{b_1\sqrt{4\mu-\lambda^2}}{2}\\
&\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}-C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2\xi}}
\end{aligned} \tag{4.21}$$

where $\xi=\frac{ax^\alpha}{\Gamma(1+\alpha)}+\frac{by^\alpha}{\Gamma(1+\alpha)}$.
$-\frac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{KT(1+\alpha)}t^\alpha.$

In particular, if $C_1\neq 0, C_2=0, \lambda>0, \mu=0$ then $U_2(\xi)$ becomes $u_1(x,y,t)$ and $V_2(\xi)$ becomes $v_1(x,y,t)$.

When $\lambda^2-4\mu=0$,

$$\begin{aligned}
u_2(x,y,t)&=a_0+\frac{2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K}{2aK}\\
&-\frac{(2\lambda\phi a^2+2\lambda\phi b^2+\lambda bb_1K)C_2}{C_1+C_2\dfrac{ax^\alpha}{\Gamma(1+\alpha)}+\dfrac{by^\alpha}{\Gamma(1+\alpha)}-\dfrac{aa_0K+bb_0K+\phi b^2\lambda+\phi a^2\lambda}{KT(1+\alpha)}t^\alpha}
\end{aligned} \tag{4.22}$$

$$v_2(x,y,t) = b_0 - \frac{b_1\lambda}{2} + \frac{b_1 C_2}{C_1 + C_2 \frac{ax^\alpha}{\Gamma(1+\alpha)} + \frac{by^\alpha}{\Gamma(1+\alpha)} - \frac{aa_0K + bb_0K + \phi b^2\lambda + \phi a^2\lambda}{KT(1+\alpha)} t^\alpha}, \quad (4.23)$$

which are the exact solutions of the system of two-dimensional Burger's equations. We see that the exact solutions established in (4.19), (4.22) and (4.23) are new exact solutions to these equations.

5. Conclusion

In this paper, the $\left(\frac{G'}{G}\right)$-expansion method is used for finding exact solutions of the fractional order system of Variant Boussinesq equations and fractional system of two-dimensional Burgers' equations with modified Riemann-Liouville derivative. The obtained results show that the $\left(\frac{G'}{G}\right)$-expansion method and fractional complex transform are reliable, efficient and powerful method for solving nonlinear FDEs and systems. As we know that these new solutions have not been studied in literature, they can be important for some special physical phenomena.

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7. References

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