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Global Dynamics of a Third-Order Rational Difference Equation

Üçüncü Mertebeden Rasyonel Bir Fark Denkleminin Global Dinamikleri

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Abstract

In this paper, we will investigate the global dynamics of the following non-linear difference equation

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_n^{p_1}x_{n-1}^{p_2}},\ \ n=0,1,...$$

where the parameters A, B, C, p_1, p_2 are non-negative numbers and the initial values x_{-2},x_{-1},x_0 are non-negative numbers.

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Keywords: Difference equation, Equilibrium point, Global stability, Recursive sequence

Öz

Bu makalede negatif olmayan A, B, C, p_1, p_2 parametreleri ve negatif olmayan x_{-2},x_{-1},x_0 başlangıç koşulları için

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_n^{p_1}x_{n-1}^{p_2}},\ \ n=0,1,...$$

lineer olmayan fark denkleminin global dinamiklerini araştıracağız.

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Anahtar Kelimeler: Fark denklem, Denge noktası, Global kararlılık, Tekrarlı dizi

1. Introduction

The theory of difference equations is an interesting and fruitful topic as it supports the analysis and modelling of various daily life phenomena. Hence, especially in the last twenty years, there has been a great interest in the study of qualitative analysis of difference equations and systems of difference equations (see [1-16] the references cited therein). The applications of difference equations have been the keystone of various applied sciences. For example, physics, computer sciences, population biology, economics, probability theory, genetics and so on.

The aim of this paper is to study the local asymptotic stability of equilibria, the periodic nature and the global behavior of solutions of the following rational recursive sequence

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_n^{p_1}x_{n-1}^{p_2}},\ n=0,1,... \qquad (1.1)$$

where the parameters A,B,C,p_1,p_2 are non-negative numbers and the initial values x_{-2},x_{-1},x_0 are positive numbers such that the denominator is always positive.

In [5] El-Owaidy *et al.* investigated the global character of the following rational recursive sequence

$$x_{n+1}=\frac{Ax_{n-1}}{B+Cx_{n-2}^{k}},\ \ n=0,1,...$$

with non-negative parameters and non-negative initial values.

By generalizing the results due to El-Owaidy *et al.* [5], in [3], Chen *et al.* studied the dynamical behavior of the following rational difference equation

$$x_{n+1}=\frac{Ax_{n-r}}{B+Cx_{n-s}},\ n=0,1,...$$

where $r,s\in Z$, the parameters are positive real numbers and the initial values $B+Cx_{n-s}>0, \forall n\geq 0$.

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Also, in [2] Chen and Li studied the dynamical behavior of the following higher order difference equation

$$x_{n+1}=\frac{Ax_{n-k}^2}{B+Cx_{n-s}},\ n=0,1,...$$

where $k,s\in Z$, the parameters are non-negative real numbers and $A>0$, the initial values are nonnegative numbers such that the denaminator is always positive.

In [1] Ahmed investigated the global asymptotic behavior and the periodic character of the difference equation

$$x_{n+1}=\frac{Ax_{n-1}}{B+C\prod_{i=l}^{k}x_{n-2i}^{p_i}},\ n=0,1,...$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [7], Erdogan *et al.* investigated the dynamical behavior of positive solutions of the following higher order difference equation

$$x_{n+1}=\frac{Ax_{n-1}}{B+C\sum_{k=1}^{t}x_{n-2k}^{p}\prod_{i=1}^{k}x_{n-2i}^{q}},\ n=0,1,...$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

As far as we examine, there is exactly no paper dealing with Eq.(1.1).

Therefore, in this paper, we focus on Eq.(1.1) in order to fill in the gap.

2. Notations and Terminology

For the sake of completeness and the readers convenience, we are including some basic results (one can see [11-13]).

Let *I* be an interval of real numbers and let f be a continuously differentiable function. Then for any condition

$x_{-2},x_{-1},x_0\in I$,

the difference equation

$$x_{n+1}=f(x_n,x_{n-1},x_{n-2})\ n=0,1,... \tag{2.1}$$

has a unique positive solution $\{x_n\}_{n=-2}^{\alpha}$.

Definition 2.1. *An equilibrium point of Eq.*(2.1) *is a point $\bar{x}$ that satisfies*

$f(\bar{x},\bar{x},\bar{x})=\bar{x}$

The point $\bar{x}$ is also said to a fixed point of the function f.

Definition 2.2. *Let $\bar{x}$ be a positive equilibrium of* (2.1).

(a) $\bar{x}$ is stable if for every, $\varepsilon>0$ there is $\delta>0$ such that for every positive solution $\{x_n\}_{n=-2}^{\alpha}$ of (2.1) *with, $\sum_{i=-2}^{0}|x_i-\bar{x}|<\delta, |x_n-\bar{x}|<\varepsilon$ holds for $n\in N$.*

(b) $\bar{x}$ is locally asymptotically stable if $\bar{x}$ is stable and there is $\gamma>0$ such that $\lim x_n=\bar{x}$ holds for every positive solution $\{x_n\}_{n=-2}^{\alpha}$ of (2.1) with

$\sum_{i=-2}^{0}|x_i-\bar{x}|<\gamma$.

(c) $\bar{x}$ is a global attractor if

$\lim x_n=\bar{x}$

holds for every positive solution $\{x_n\}_{n=-2}^{\alpha}$ of (2.1).

(d) $\bar{x}$ is globally asymptotically stable if $\bar{x}$ is both stable and global attractor.

Definition 2.3. *The linearized equation of (2.1) about the equilibrium point $\bar{x}$ is*

$$y_{n+1}=t_0y_n+t_1y_{n-1}+t_2y_{n-2},\ n=0,1,2,... \tag{2.2}$$

where

$$t_0=\frac{\partial f}{\partial x_n}(\bar{x},\bar{x},\bar{x}), t_1=\frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},\bar{x}), t_2=\frac{\partial f}{\partial x_{n-2}}(\bar{x},\bar{x},\bar{x}).$$

The characteristic equation of (2.2) is

$$\mu^3-t_0\mu^2-t_1\mu-t_2=0. \tag{2.3}$$

The following result, known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of equation (2.1).

Theorem 2.1. *(The Linearized Stability Theorem)*

Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then, the following statements are true:

(i) If all roots of (2.3) have absolute value less than one, then the equilibrium point $\bar{x}$ of (2.1) is locally asymptotically stable.

(ii) If at least one of the roots of (2.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ of (2.1) is unstable. Also, the equilibrium point $\bar{x}$ of (2.1) is called a saddle point if (2.3) has roots both inside and outside the unit disk.

Theorem 2.2. *Assume that $\alpha_2,\alpha_1,\alpha_0$ are real numbers. Then a necessary and sufficient condition for all roots of the equation to lie inside the unit disk is*

$$|\alpha_2+\alpha_0|<1+\alpha_1, |\alpha_2-3\alpha_0|<3-\alpha_1 \text{ and } \alpha_0^2+\alpha_1-\alpha_0\alpha_2<1. \tag{2.4}$$

3. Special Cases for Eq.(1.1)

In this part of the paper, we examine that the structure of positive solutions of Eq.(1.1) when one or more of the parameters of Eq.(1.1) are zero. The following special equations emerge;

If $A=0$, then

$$x_{n+1}=0,\ n=0,1,... \quad (3.1)$$

if $B=0$, then

$$x_{n+1}=\frac{Ax_{n-2}}{Cx_n^{p_1}x_{n-1}^{p_2}},\ n=0,1,... \quad (3.2)$$

if $C=0$, then

$$x_{n+1}=\frac{Ax_{n-2}}{B},\ n=0,1,... \quad (3.3)$$

if $p_1=0$, then

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_{n-1}^{p_2}},\ n=0,1,... \quad (3.4)$$

if $p_2=0$, then

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_n^{p_1}},\ n=0,1,... \quad (3.5)$$

In each of the above five equations, we suppose that all parameters in the equations are positive. Eq.(3.1) is trivial case. Eq.(3.3) is linear. Also, Eq.(3.2) can be reduced to a linear difference equation by the change of variables $x_n=e^{y_n}$. Eq.(3.4) was investigated in [2,3] and Eq.(3.5) was investigated in [2.3].

4. Dynamics of Eq. (4.1)

In this section, we will prove our main results, namely, we investigate some dynamics of Eq.(4.1)

Note that Eq.(1.1) can be reduced to the following non-linear difference equation

$$y_{n+1}=\frac{ry_{n-2}}{1+y_n^{p_1}y_{n-1}^{p_2}},\ n=0,1,... \quad (4.1)$$

by the change of variables $x_n=\left(\frac{B}{C}\right)^{\frac{1}{p_1+p_2}}x_n$ with $r=\frac{A}{B}$. So, we shall investigate Eq.(4.1).

Before giving the main results of this paper, we will give the non-hyperbolic point definition. If the characteristic equation of the linearized equation associated with Eq.(4.1) about the equilibrium point has a root which is equal to one, then the equilibrium point is a called a non-hyperbolic point.

Lemma 4.1. *We have the following cases for the equilibrium points of Eq (4.1);*

(i) $\overline{y_0}$ *is always the equilibrium point of Eq (4.1).*

(ii) If $r>1$, *then Eq.(4.1) has the positive equilibrium* $\overline{y_1}=(r-1)^{\frac{1}{p_1+p_2}}$.

(iii) If $r<1$ *and* $\frac{1}{p_1+p_2}$ *is an even positive integer, then Eq.(4.1) has the positive equilibrium* $\overline{y_2}=(r-1)^{\frac{1}{p_1+p_2}}$.

which is always in the interval (0.1).

Proof. The proof is easily obtained from the definition of equilibrium point.

In the following Theorems, we investigate the local asymptotic behavior of the equilibria and the global behavior of solutions of Eq.(4.1) with $r,p_1,p_2>0$ and non-negative initial conditions.

Theorem 4.2. *For Eq (4.1), we have the following results.*

(i) Assume that $r<1$, *then the zero equilibrium point is locally asymptotically stable.*

(ii) Assume that $r>1$, *then the zero equilibrium point is unstable.*

(iii) Assume that $r=1$, *then the zero equilibrium point is non-hyperbolic point.*

(iv) The positive equilibrium point $\overline{y_1}=(r-1)^{\frac{1}{p_1+p_2}}$. *is unstable.*

(v) Assume that $r\in(0,1)$ *and* $\frac{1}{p_1+p_2}$ *is an even positive integer, then the positive equilibrium point is unstable.*

Proof. The linearized equation associated with Eq.(4.1) about zero equilibrium has the form

$$z_{n+1}-rz_{n-2}=0\ \ n=0,1,2,... \quad (4.2)$$

The characteristic equation of Eq.(4.1) about zero equilibrium, is

$$\mu^3-r=0 \quad (4.3)$$

then the proof of (i),(ii),(iii) follows immediately from Theorem 2.1.

Now, we shall prove the case (iv). The linearized equation associated with Eq.(4.1) about

$\overline{y_1}=(r-1)^{\frac{1}{p_1+p_2}}$ is

$$z_{n+1}+p_1\left(1-\frac{1}{r}\right)z_n+p_2\left(1-\frac{1}{r}\right)z_{n-1}-z_{n-2}=0\ \ n=0,1,2,... \quad (4.4)$$

The characteristic equation of Eq.(4.1) about is $\overline{y_1}=(r-1)^{\frac{1}{p_1+p_2}}$

$$\mu^3+p_1\left(1-\frac{1}{r}\right)\mu^2+p_2\left(1-\frac{1}{r}\right)\mu-1=0 \quad (4.5)$$

If we consider Theorem 2.2, then a necessary and sufficient condition for all roots of the equation (4.5) to lie inside the unit disk is

$$\frac{(p_1+p_2)(r-1)}{r}<0 \quad (4.6)$$

which is impossible since $p_1,p_2>0$ and $r>1$. Thus, the proof of this case is complete.

It remains the proof of (v). The linearized equation associated with Eq.(4.1) about $\overline{y_2}=(r-1)^{\frac{1}{p_1+p_2}}$ is as (4.4) and the characteristic equation of it about $\overline{y_2}=(r-1)^{\frac{1}{p_1+p_2}}$ is as (4.5).

Set

$$f(\mu)=\mu^3+p_1\left(1-\frac{1}{r}\right)\mu^2+p_2\left(1-\frac{1}{r}\right)\mu-1, \quad (4.7)$$

Then

$$f(1)=\frac{(p_1+p_2)(r-1)}{r}<0$$

and

$$\lim_{\mu\to\infty}f(\mu)=\infty,$$

so f has at least a root in $(1,\infty)$. Hence, the proof of (v) follows immediately from Theorem 2.1. Consequently, the proof is completed.

Theorem 4.3. *If* $r<1$, *then the solutions of Eq.(4.1) are bounded.*

Proof. Assume that $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq.(4.1). Then, we have

$$y_{n+1}=\frac{ry_{n-2}}{1+y_n^{p_1}y_{n-1}^{p_2}} \le ry_{n-2}.$$

Thus, we obtain

$$y_{n+1}\le y_{n-2}.$$

We can divided the sequence $\{y_n\}_{n=-2}^{\infty}$ to three subsequence bounded above by the initial conditions as follows:

$$y_{-2}\ge y_1\ge y_4\ge y_7\ge ...$$
$$y_{-1}\ge y_2\ge y_5\ge y_8\ge ...$$
$$y_0\ge y_3\ge y_6\ge y_9\ge ...$$

Hence we chose

$$M=\max\{y_{-2},y_{-1},y_0\}$$

which leads to

$$0\le y_n\le M, \forall n\ge -2.$$

The proof is completed.

Theorem 4.4. *Assume that* $r<1$, *then the zero equilibrium point of Eq.(4.1) is globally asymptotically stable.*

Proof. We know by Theorem 4.2 that the zero equilibrium point of Eq.(4.1) is locally asymptotically stable, and so it suffices to show that it is a global attractor.

From Eq.(4.1), we have

$$0\le y_{n+1}=\frac{ry_{n-2}}{1+y_n^{p_1}y_{n-1}^{p_2}}\le ry_{n-2} \text{ for all } n\ge 0.$$

By induction

$$y_{3n+1}\le r^{n+1}y_{-2},$$
$$y_{3n+2}\le r^{n+1}y_{-1},$$
$$y_{3n+3}\le r^{n+1}y_0.$$

So, we obtain that the subsequences

$$\{y_{3n+1}\}\to 0, \{y_{3n+2}\}\to 0, \{y_{3n+3}\}\to 0$$

for $r<1$. Thus,

$$\lim_{n\to\infty} y_n=0$$

This completes the proof.

Theorem 4.5. *Assume that at least one of the initial conditions is different from zero. Then, Eq.(4.1) has three prime periodic solution if*

$$r=1+W$$

where $W=x_{-1}^{p_1}x_{-2}^{p_2}=x_0^{p_1}x_{-1}^{p_2}=x_{-2}^{p_1}x_0^{p_1}$.

Proof. Assume that there exists a distinct prime period 3 solutions of Eq.(4.1). Thus, we have the following algebraic system of 3 equations

$$x_1=x_{-2}, x_2=x_{-1}, x_3=x_0. \quad (4.8)$$

Solving the algebraic system (4.8), we get

$$x_1=\frac{rx_{-2}}{1+x_0^{p_1}x_{-1}^{p_2}}=x_{-2},$$

then,

$$\frac{rx_{-2}}{1+W}=x_{-2}.$$

So

$$(r-(1+W))x_{-2}=0.$$

Similarly, by the solving the further algebraic equations we get

$$(r-(1+W))x_{-1}=0,$$
$$(r-(1+W))x_0=0.$$

Hence, from assumption we get $r = 1 + W$. Thus, the proof is completed.

Theorem 4.6. *Assume that* $p_2+2\ge p_1$, *then Eq.(4.1) has no two prime period solutions. If* $p_2+2<p_1$ *and* $r(p_1-p_2-2)>p_1-p_2$, *then Eq.(4.1) has two prime period solutions.*

Proof. Assume that a prime two periodic solution exists in the following form

$$\{...,x_1,x_2,x_1,x_2,...\}$$

of Eq.(4.1). From Eq.(4.1), we get the following equalities:

$$x_1=\frac{rx_2}{1+x_2^{p_1}x_1^{p_2}} \text{ and } x_2=\frac{rx_1}{1+x_1^{p_1}x_2^{p_2}}$$

That is,

$$rx_2-x_1=x_1^{p_2+1}x_2^{p_1} \text{ and } rx_1-x_2=x_1^{p_1}x_2^{p_2+1}.$$

This implies that

$$\left(\frac{x_1}{x_2}\right)^{p_2+1-p_1} = \frac{rx_2 - x_1}{rx_1 - x_2}$$

Now if we set $\theta = \frac{x_1}{x_2}$, then we get

$$r - \theta = \theta^{p_2+1-p_1}(r\theta - 1) \tag{4.9}$$

As $\theta^{p_1+1-p_2} > 0$ always, we obtain the relation $\frac{1}{r} < \theta < r$ We consider the following cases:

Case 1. $p_2 + 1 \geq p_1$. We shall show that Eq.(4.9) has no positive real roots except for $\theta = 1$.

If $p_2 + 2 - p_1 = 0$, then from Eq.(4.9) we get $\theta = 1$. Now suppose that $p_2 + 2 - p_1 > 0$ Clearly $\theta = 1$ is a root of Eq.(4.9). Consider the function

$$h(\theta) = \theta^{p_2+2-p_1} - \theta^{p_2+1-p_1} + \theta - r.$$

The derivative of the function *h* is

$$h'(\theta) = (p_2 + 2 - p_1)r\theta^{p_2-1-p_1} - (p_2 - 1 - p_1)\theta^{p_2-p_1} + 1.$$

For all values of $\theta \geq 0$, we have

$$h'(\theta) = (p_2 + 1 - p_1)\theta^{p_2-p_1}(r\theta - 1) + r\theta^{p_2+1-p_1} + 1 > 0.$$

That is, h is an increasing function. Therefore, $\theta = 1$ is the unique zero of the function h.

Case 2. $p_2 + 2 - p_1 < 0$. From Eq.(4.9) we get

$$\theta^{p_1-p_2} - r\theta^{p_1-p_2-1} + r\theta - 1 = 0.$$

Let

$$g(\theta) = \theta^{p_1-p_2} - r\theta^{p_1-p_2-1} + r\theta - 1$$

Using simple analysis, if $r > \frac{p_1 - p_2}{p_1 - p_2 - 2}$, then the function g has a zero θ_0 other than $\theta = 1$.

Now, by a simple calculation, and satisfy the relation

$$x^2 - y^2 = x^{p_1}y^{p_2+2} - y^{p_1}x^{p_2+2}.$$

If we set $x_{-2} = x_0 = x$ and $x_{-1} = y$. Then

$$x_1 = \frac{ry}{1 + y^{p_1}x^{p_2}} = \frac{ry}{1 + \left(\frac{x^2}{y^2} - 1 + x^{p_1+2}y^{p_2-2}\right)}$$

and $\frac{ry^2}{x(1 + x^{p_1}y^{p_2})} = y$

$$x_2 = \frac{rx}{1 + x^{p_1}y^{p_2}} = \frac{rx}{1 + \left(\frac{y^2}{x^2} - 1 + y^{p_1+2}x^{p_2-2}\right)}$$

$$= \frac{rx^2}{y(1 + y^{p_1}x^{p_2})} = x$$

This completes the proof.

5. References

1. **Ahmed, AM. 2011.** On the dynamics of a higher-order rational difference equation, *Discrete Dynamics in Nature and Society*, Volume 2011, Article ID 419789, 8 pages.
2. **Chen, D., Li, X. 2012.** Some dynamical properties for a class of nonlinear difference equation, *ADMS*, 1(2), 1-12.
3. **Chen, D., Li, X., Wang, Y. 2009.** Dynamics for non-linear difference equation $x_{n+1} = \frac{Ax_{n-r}}{B + Cx_{n-s}}$, $n = 0, 1, ...,$ *Advances in Difference Equations*, Volume 2009, Article ID 235691, 13 pages.
4. **Camouzis, E., Ladas, G. 2007.** *Dynamics of third-order rational difference equations with open problems and conjectures*, Vol. 5. CRC Press.
5. **El-Owaidy, HM., Ahmed, AM., Youssef, AM. 2005.** The dynamics of the recursive sequence $x_{n+1} = Ax_{n-1}/B + Cx_{n-2}^k$, *App. Math. Let.*, 18(9): 1013-1018.
6. **Elsayed, EM. 2014.** New method to obtain periodic solutions of period two and three of a rational difference equation. *Nonlinear Dyn.*, 79(1): 241-250.
7. **Erdogan, ME., Cinar, C. 2013.** On the dynamics of the recursive sequence $x_{n+1} = Ax_{n-1}/\left(B + C\sum_{k=1}^{t} x_{n-2k}^{p} \prod_{i=1}^{k} x_{n-2i}^{q}\right)$. *Fasciculi Math.*, 50: 59-66.
8. **Gocen, M., Cebeci, A. 2018.** On the periodic solutions of some systems of higher order difference equations, *Rocky Mountain Journal of Mathematics, 48*(3): 845-858.
9. **Gocen, M., Guneysu, M. 2018.** The global attractivity of some rational difference equations. *Journal of Comput. Anal. and App., 25*(7): 1233-1243.
10. **Karatas, R. 2010.** Global behavior of a higher order difference equation, *Comput. Math. App.*, 60(3): 830-839.
11. **Kocić, V., Ladas, G. 1993.** *Global behavior of nonlinear difference equations of higher order with applications,* Kluwer Academic Publishers, Dordrecht.
12. **Kulenović, MRS., Ladas, G. 2001.** *Dynamics of second order rational difference equations*, Chapman & Hall/CRC.
13. **Sedaghat, H. 2003.** *Nonlinear difference equations theory with applications to social science models*, Vol. 15. Springer Science & Business Media.
14. **Tasdemir, E., Soykan, Y. 2016.** On The Periodicies of The Difference Equation $x_{(n+1)}=x_n x_{(n-1)}+\alpha$. *Karaelmas Fen ve Mühendislik Dergisi, 6*(2): 329-333.
15. **Tasdemir, E., Soykan, Y. 2019.** Dynamical analysis of a nonlinear difference equation. *Journal of Comput. Anal. & App., 26*(2): 288-301.
16. **Tasdemir, E., Soykan, Y. 2017.** Long-Term Behavior of Solutions of the Non-Linear Difference Equation x_{n+1}=xn−1xn−3−1. *Gen. Math. Notes, 38*(1): 13-31.