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Kinetik Tipten Bir Denklem İçin Bir Ters Problem

An Inverse Problem for a Kinetic Type Equation

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Öz

Bu çalışmada, kinetik tipten bir denklem için bir ters problemin çözümünün tekliği araştırılmıştır. Burada kullanılan temel metot Amirov (2001) tarafından geliştirilmiş olup uygun diverjans terim ve pozitif karesel formların elde edilmesine dayanmaktadır.

Anahtar Kelimeler: Kinetik tipten denklem, Ters problem, Teklik teoremi

Abstract

In this work, uniqueness of solution of an inverse problem for a kinetic type equation is investigated. Here, the main method used was developed by Amirov (2001) and based on the obtaining suitable divergence term and positive quadratic forms.

Keywords: Kinetic type equation, Inverse problem, Uniqueness theorem

1. Giriş

Kinetik denklemler; gaz dinamiği, plazma fiziği, biyoloji ve sosyo-ekonomi gibi birçok alanda çok parçacıklı sistemlerin istatistiksel modeli olarak karşımıza çıkmaktadır, (Liboff 2003). Burada sunulan çalışmanın literatürdeki mevcut sonuçlardan farkı, farklı sınır şartları ile bölgenin geometrisine ihtiyaç duyulmadan problemin çözümünün tekliğinin ispatlanmasıdır. Daha açık olarak, problemin ele alındığı bölgenin bazı kümelerin kartezyen çarpımı olarak verilmesi gerekmemektedir. Kinetik denklemler için ters problemlerin çözülebilirliği ve yaklaşık çözümleri Amirov (2001), Anikonov (2001), Amirov vd. (2009), Amirov ve Gölgeleyen (2010), Yildiz vd. (2010) tarafından incelenmiştir. Bu çalışmalarda sınır koşulu olarak, problemin çözümü bölgenin tüm sınırında verilmiş ve çözülebilirlik bölgenin geometrisi kullanılarak ispatlanmıştır.

Bu makalede bir

$$\Omega = \{(x,y): x \in D \subset \mathbb{R}^n, y \in G \subset \mathbb{R}^n, n \geq 1\}$$

bölgesinde

$$\{H,u\} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial v_i}\frac{\partial H}{\partial x_i}\right),$$

$$I_1(u) = \int_G \Phi(x,v,v')u(x,v')dv'$$

olmak üzere

$$\{H,u\} + I_1(u) = \lambda(x,v) \tag{1}$$

denklemi

$$\nabla u|_{\partial\Omega} = u_0, \nabla_v \lambda|_{\Gamma_1} = \lambda_0, \tag{2}$$

$$u(x_0,v_0) = u_1, (x_0,v_0) \in \Omega \tag{3}$$

koşulları ile birlikte ele alınacaktır. Burada Ω bölgesinin sınırı $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ olup $\Gamma_1 = \partial D \times G, \Gamma_2 = D \times \partial G$ şeklinde tanımlıdır. Ayrıca ∂D ve ∂G sınırları C^2 sınıfındandır.

Problem 1. (1) denklemini (2), (3) koşulları ile sağlayan $u(x,v)$ ve $\lambda(x,v)$ fonksiyonlarının bulunması problemini ele alalım.

Teorem 1. Kabul edelim ki Hamiltonian fonksiyonu $H(x,v) \in C^2(\overline{\Omega})$ ve saçılım çekirdeği $\Phi(x,v,v') \in C^1(\overline{D} \times \overline{G} \times \overline{G})$ verilsin. α_1, α_2 pozitif tam sayılar olmak üzere $(x,v) \in \overline{\Omega}$ ve her $\xi \in \mathbb{R}^n$ için

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$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial v_i \partial v_j}\xi^i\xi^j \ge \alpha_1 |\xi|^2, \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial_{x_i}\partial_{x_j}}\xi^i\xi^j \leqslant -\alpha_2|\xi|^2, \tag{4}$$

$$\alpha_1 - 1 > 0, \alpha_2 - \alpha_3 > 0,$$
$$\alpha_3 = C\max_{1\le j\le n}\left\{\max_{x\in D}\int_G\int_G \left(\frac{\partial\Phi}{\partial v_j}\right)^2 dv'dv\right\} \tag{5}$$

şartları sağlansın. Burada C, Steklov eşitsizliğinden gelen bir sabittir. Ayrıca (1)'de, $\lambda(x,v)$ fonksiyonu

$$\widehat{L}\lambda \equiv \sum_{j=1}^{n} \frac{\partial^2 \lambda}{\partial v_j \partial x_j} = 0 \tag{6}$$

diferensiyel denklemini sağlasın. Bu durumda Problem 1, $u\in C^2(\overline{\Omega})$ ve $\lambda\in C^2(\overline{\Omega})$ olacak şekilde en çok bir (u,λ) çözümüne sahiptir.

İspat. Kabul edelim ki $u\in C^2(\Omega), \lambda\in C^2(\Omega)$ olmak üzere, (u,λ) fonksiyonlar çifti Problem 1'in $u_0=\lambda_0=u_1=0$ şartını sağlayan bir çözümü olsun. (6) koşulundan

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(u\frac{\partial\lambda}{\partial v_j}\right) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial\lambda}{\partial v_j} + \sum_{j=1}^{n} u\left(\frac{\partial^2\lambda}{\partial v_j\partial x_j}\right)$$

eşitliğinin sağ tarafındaki ikinci terim sıfır olacağından,

$$\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial\lambda}{\partial v_j} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(u\frac{\partial\lambda}{\partial v_j}\right) \tag{7}$$

yazılır. Diğer yandan

$$2\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\left[\frac{\partial}{\partial v_j}(\{H,u\}+I_1(u))\right]$$
$$= 2\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\left[\frac{\partial}{\partial v_j}\left(\sum_{i=1}^{n}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right.\right.$$
$$\left.\left. + \int_G \Phi(x,v,v')u(x,v')dv'\right)\right]$$

eşitliğinde parantezler açılırsa

$$= 2\sum_{i,j=1}^{n}\left[\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right]$$
$$+2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\int_G \Phi(x,v,v')u(x,v')dv'\right)$$
$$= 2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\left(\frac{\partial^2 H}{\partial v_i\partial v_j}\frac{\partial u}{\partial x_i} + \frac{\partial H}{\partial v_i}\frac{\partial^2 u}{\partial x_i\partial v_j}\right.$$
$$\left. - \frac{\partial^2 H}{\partial x_i\partial v_j}\frac{\partial u}{\partial v_i} - \frac{\partial H}{\partial x_i}\frac{\partial^2 u}{\partial v_i\partial v_j}\right)$$
$$+2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\int_G \Phi(x,v,v')u(x,v')dv'\right)$$
$$= 2\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial v_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} + 2\sum_{i,j=1}^{n}\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial v_j}$$
$$-2\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial v_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j} - 2\sum_{i,j=1}^{n}\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial v_i\partial v_j}$$
$$+2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\int_G \Phi(x,v,v')u(x,v')dv'\right) \tag{8}$$

elde edilir. Bulunan son eşitlikteki ikinci ve dördüncü terimler

$$2\sum_{i,j=1}^{n}\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial v_j} = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial v_j}\right)$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right)$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial v_j}$$
$$-\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j} - \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}, \tag{9}$$

$$-2\sum_{i,j=1}^{n}\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial v_i\partial v_j} = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)$$
$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right)$$
$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_i}\right) - \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}$$
$$+\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j} + \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial v_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_i} \tag{10}$$

şeklinde yazılabilir. (9), (10) bağıntıları (8) eşitliğinde yerine yazılırsa

$$2\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial v_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial v_j}\right)$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right)$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial v_j}$$
$$-\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j} - \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i\partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)$$
$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right) - \sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_i}\right)$$
$$-\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j} + \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}$$
$$+\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i\partial v_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_i}$$
$$+2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\int_G \Phi(x,v,v')u(x,v')dv'\right) = 2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial\lambda}{\partial v_j}$$

elde edilir. Burada gerekli düzenlemeler yapıldığında

$$\sum_{i,j=1}^{n}\left(\frac{\partial^2 H}{\partial v_i\partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\partial^2 H}{\partial x_i\partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left[\frac{\partial u}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\frac{\partial H}{\partial v_i} - \frac{\partial u}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right]$$
$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left[\frac{\partial u}{\partial v_j}\left(\frac{\partial u}{\partial x_i}\frac{\partial H}{\partial v_i} - \frac{\partial u}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right]$$
$$+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right) - \sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{\partial H}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial v_j}\right) \tag{11}$$
$$+2\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial v_j}\left(\int_G \Phi(x,v,v')u(x,v')dv'\right)$$
$$= 2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(u\frac{\partial\lambda}{\partial v_j}\right)$$

bulunur. (2) koşulu dikkate alınarak, (7) ve (11) eşitliklerinden

$$\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-\frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)d\Omega$$
$$+2\sum_{j=1}^{n}\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \frac{\partial}{\partial v_j}\Phi(x,v,v')u(x,v')d\acute{v}\right)d\Omega \quad (12)$$
$$=2\sum_{j=1}^{n}\int_{\Omega}\frac{\partial}{\partial x_j}\left(u\frac{\partial \lambda}{\partial v_j}\right)d\Omega=0$$

elde edilir. (12) eşitliğinin sol tarafındaki ikinci terimde $2ab \geq -(a^2+b^2)$ eşitsizliği dikkate alınırsa

$$2\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \frac{\partial \Phi}{\partial v_j}u\,dv'\right)d\Omega \geq -\int_{\Omega}\left(\left(\frac{\partial u}{\partial x_j}\right)^2\right.$$
$$\left.+\left(\int_G \frac{\partial \Phi}{\partial v_j}u\,dv'\right)^2\right)d\Omega$$

yazılır. Son eşitsizlikte sağ tarafın ikinci terimine Cauchy-Schwarz eşitsizliği uygulanırsa

$$2\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \frac{\partial \Phi}{\partial v_j}u\,dv'\right)d\Omega \geq -\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\right)^2 d\Omega$$
$$-\int_D\int_G\left(\int_G\left(\frac{\partial \Phi}{\partial v_j}\right)^2 dv'\right)\left(\int_G (u)^2 dv'\right)dv dx \quad (13)$$

elde edilir. Fubini Teoreminden

$$2\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \frac{\partial \Phi}{\partial v_j}u\,dv'\right)d\Omega \geq -\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\right)^2 d\Omega$$
$$-\int_D\left(\int_G (u(x,v'))^2 dv'\right)\beta_j(x)\,dx \quad (14)$$

yazılır. Burada

$$\beta_j(x)=\int_G\int_G\left(\frac{\partial \Phi}{\partial v_j}\right)^2 dv' dv$$

olarak tanımlıdır. Steklov eşitsizliği göz önünde bulundurulursa

$$-\int_D\left(\int_G (u)^2 dv'\right)\beta_j(x)\,dx \geq -C(\max_{x\in D}\beta_j(x))\int_{\Omega}\left(\frac{\partial u}{\partial v_j}\right)^2 d\Omega$$

eşitsizliğinden, $\alpha_3 = C\max_{1\leq j\leq n}\{\max_{x\in D}\beta_j(x)\}$ olmak üzere

$$2\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \frac{\partial \Phi}{\partial v_j}u\,dv'\right)d\Omega \geq$$
$$-\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\right)^2 d\Omega-\alpha_3\int_{\Omega}\left(\frac{\partial u}{\partial v_j}\right)^2 d\Omega \quad (15)$$

biçiminde yazılır. (12) denklemi (4) koşulu ve (15) eşitsizliği ile birlikte tekrar değerlendirilirse

$$\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-\frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)d\Omega$$
$$+2\sum_{j=1}^{n}\int_{\Omega}\left(\frac{\partial u}{\partial x_j}\int_G \Phi(x,v,v')u(x,\acute{v})dv'\right)d\Omega$$
$$\geq \alpha_1\int_{\Omega}(\nabla_v u)^2 d\Omega+\alpha_2\int_{\Omega}(\nabla_x u)^2 d\Omega-\int_{\Omega}(\nabla_x u)^2 d\Omega$$
$$-\alpha_3\int_{\Omega}(\nabla_v u)^2 d\Omega$$

bulunur ve gerekli düzenlemeler ile

$$(\alpha_1-1)\int_{\Omega}|\nabla_x u|^2 d\Omega+(\alpha_2-\alpha_3)\int_{\Omega}|\nabla_v u|^2 d\Omega \leq 0 \quad (16)$$

yazılır. Dolayısıyla (5) koşulundan dolayı $|\nabla_x u|^2=|\nabla_v u|^2=0$ dır. Son olarak (3) koşulundan u = 0 bulunur. Denklem (1)'den, Ω da λ = 0 olup verilen teoremin ispatı tamamlanır.

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