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Comparison Result of Some Gadjiev Ibragimov Type Operators

Bazı Gadjiev İbragimov Tipli Operatörlerin Karşılaştırma Sonuçları

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Abstract

The aim of this paper is to compare of approximation situation some of the Gadjiev and Ibragimov type operators. We give estimation the rate of convergence of these operators and we show the rate of convergence of these operators to a certain function by illustrative graphics using the Maple algorithms. Also, compare eror bounded of these operator for some values of n.

Keywords: Linear positive operator, Gadjiev Ibragimov type operators, Modulus of continuity

Öz

Bu çalışmada amacımız bazı Gadjiev- İbragimov tipli operatörlerin yaklaşım durumlarını karşılaştırmaktır. Bu operatörlerin yakınsaklık dereceleri verilecek ve belirlenen fonksiyona yakınsaklık hızları, Maple algoritması kullanılarak grafiksel olarak gösterilecektir. Aynı zamanda n nin bazı değerleri için bu operatörlerin hata sınırı karşılaştırılacaktır.

Anahtar Kelimeler: Doğrusal pozitif operatörler, Gadjiev Ibragimov tipi operatörler, Süreklilik modülü

1. Introduction

In recent years, the number of branches of mathematics related to approximation theory has been increasing steadly. Approximation theory belong to the common part of the theory of functional analysis. During the last 30 years, close relationship between approximation theory and numerical analysis, algebra, geometry and topology has been demonstrated. Gadjiev and Ibragimov defined a general sequence of positive operators in 1970. Several generalizations of well-known positive linear operators were introduced by several authors (Doğru 1997, Gadjiev and İspir 1999, Aral 2003, Ulusoy et.al. 2015).

In the present paper, we recall the construction of operators of Gadjiev and Ibragimov. It is known that Gadjiev Ibragimov operators include some well-known classical linear positive operators such as Bernstein, Bernstein-Chlodowsky, Szász and Baskakov operators.

The aim of this paper is to compare of approximation properties of some of the Gadjiev and Ibragimov operators.

We present some auxiliary results of the rate of convergence using modulus of continuity. Also, we show the rate of convergence of these operators to a certain function by illustrative graphics using the Maple algorithms. We compare rate of convergence and eror bound of these operators for some values of n.

Firstly we fix some notation. Let $C[a,b]$ denote the class of continuous real valued functions on the closed interval [*a*, *b*]. C_ρ denotes the subspace of all continuous functions satisfying and $|f(x)| \leq M_f(1+x^2)$ and $C_\rho^k[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ with $\lim_{|x|\to\infty} \dfrac{f(x)}{\rho(x)} = K_f < \infty$ where K_f is a constant depending only on *f*.

2. Definitions and Construction of Operators

2.1. Classical Gadjiev-Ibragimov Operators

Recall that the construction of Gadjiev-Ibragimov operators, given in (Gadjiev and Ibragimov 1970) are based on Taylor expansion of functions $K_n(x,t,u)$ of variables x,t,u which is an entire analytic function with respect to variable *u* for fixed $x,t \in [0,A]$, where $A > 0$.

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Let functions $K_n(x,t,u)$ of variables $x,t,u \in [0,A]$ satisfy the following conditions:

1) Each functions $K_n(x,t,u)$ is entire analytic function with respect to variable *u* for fixed $x,t \in [0,A]$;

2) For any natural number *n* and any $x \in [0,A], K_n(x,0,0) = 1$;

3) $\left\{(-1)^v \left[\frac{\partial^v}{\partial u^v} K_n(x,t,u)\right]_{\substack{u=u_1 \\ t=0}}\right\} \geq 0$, $v,n \in \mathbb{N}$, for any $x \in [0,A]$, fixed $u = u_1$;

4) $\frac{\partial^v}{\partial u^v} K_n(x,t,u)\Big|_{\substack{u=u_1 \\ t=0}} = -nx\left[\frac{\partial^{v-1}}{\partial u^{v-1}} K_{n+m}(x,t,u)\right]_{\substack{u=u_1 \\ t=0}}$ for any fixed $u = u_1$,

where $(n + m)$ ise natural number and *m* is a constant independent of v

Moreover, let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ be sequences of continuous functions on $[0,A]$ such that $\varphi_n(0) = 0, \psi_n(t) > 0$, for all $t \in [0,A]$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers having the properties

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = 1 \text{ and } \lim_{n\to\infty} \frac{1}{n^2 \psi_n(0)} = 0.$$

In (Gadjiev and Ibragimov 1970), the authors write the Taylor expansion of entire function $K_n(x,t,\varphi_n(t))$ in the powers of $(\varphi_n(t) - \alpha_n \psi_n(t))$ and taking $t = 0$, the expansion in the powers of $(-\alpha_n \psi_n(t))$ obtained since $\varphi_n(0) = 0$.

Under these conditions, classical Gadjiev-Ibrahimov operators for $n = 1,2,\ldots$, have the form

$$L_n(f,x) = \sum_{v=0}^{\infty} f\left(\frac{v}{n^2 \psi_n(0)}\right) K_n^{(v)}(x,0,\alpha_n \psi_n(0)) \frac{(-\alpha_n \psi_n(0))^v}{v!} \tag{1}$$

where the notation

$$K_n^{(v)}(x,0,\alpha_n \psi_n(0)) = \frac{\partial^v K_n(x,t,u)}{\partial u^v}\Bigg|_{\substack{u=\alpha_n\psi_n(t), \\ t=0}}$$

is used. In Gadjiev and Ibragimov 1970 it has been shown that for different functions $\psi_n(t)$, the operators in transformed to generalized Bernstein polynomials, Bernstein-Cholodowsky polynomials, Szasz operators, etc.

Remark 2.1.1 By choosing, $\{K_n(x,t,u)\}$ as follows, we obtain some known sequences of linear positive operators. Some of them follows:

By choosing

$$K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, m = -1 \;\; \psi_\lambda(0) = \frac{1}{n},$$

we have the operators defined by (1) are transformed into Bernstein polynomials.

For

$$\alpha_n = n, \psi_n(0) = \frac{1}{nb_n} \text{ and } \lim_{n\to\infty} b_n = \infty, \lim_{n\to\infty} \frac{b_n}{n} = 0$$

we obtain Bernstein-Chlodowsky polynomials.

By choosing

$K_n(x,t,u) = e^{-n}(t + ux), a_n = n, m = 0 \quad \psi_n(0) = \frac{1}{n}$, we get Szasz operators.

Theorem 2.1.2. Let $\{L_n\}$ be sequence of linear positive operators defined by (1) and *f* satisfy $|f(x)| \leq M_f(1 + x^2)$. Then for every function $f \in C[0,A]$,

$\lim_{n\to\infty} \| L_n(f,x) - f(x) \|_{C[0,A]} = 0$ (Gadjiev and Ibragimov 1970).

For interpretation of the next figures with colours, the reader is invited to the web version of this article.

Example 2.1.3

We give the graphics of approximation of functions $f(x) = \frac{\ln(x^3+1)}{e^{(x^2-4)}}$ by classical Gadjiev-Ibragimov operators by choosing next sequences. (n = 2(green), n = 4(red), n = 6(black), n = 8(cyan), n = 10(magenta) and *f*(blue)

For $K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, \quad \psi_n(0) = \frac{1}{n}$, (see Figure 1).

For $K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, \alpha_n = n, \psi_n(0) = \frac{1}{nb_n}$ and $b_n = \sqrt[5]{n}$ we get following graphics. (see Figure 2).

By choosing $K_n(x,t,u) = e^{-n(t+ux)}, a_n = n, \;\; \psi_n(0) = \frac{1}{n}$, we have following graphics. (see Figure 3).

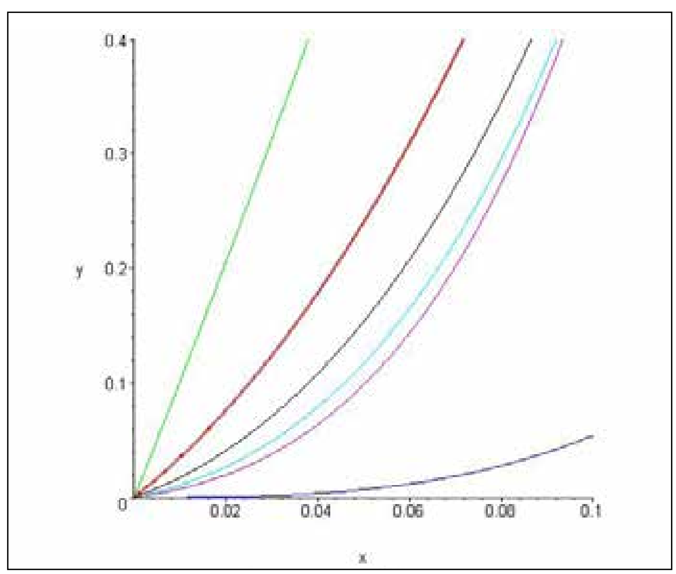

Figure 1. Approximation of $f(x)$ by $L_n(f, x)$.

We consider the following three modified form of these operators defined by (Doğru 1997, Gönül and Coşkun 2012, İspir et. al. 2008). Then we give the illustration of $L_n(f, x)$ for some sequences which aproximate to $f(x) = \frac{\ln(x^3+1)}{e^{(x^2-4)}}$ for every modification. So we compare these $L_n(f, x)$ for a certain function f.

2.2. Generalized Linear Positive Operators

Let λ and A be positive real numbers, $\{\varphi_\lambda(t)\}$ and $\{\psi_n(t)\}$ be the family of functions in $C[0, A]$ such that $\varphi_\lambda(0) = 0, \psi_\lambda(t) > 0$, for each $t \in [0,A]$. Let also $\{a_\lambda\}$ be a family of positive numbers such that

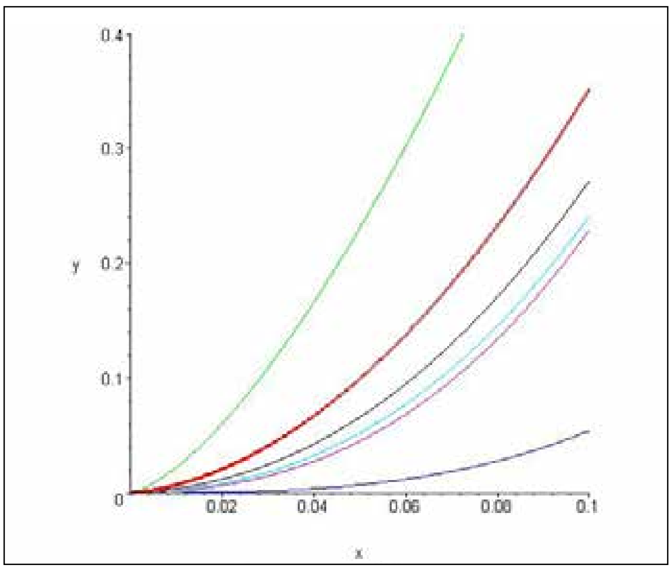

Figure 2. Approximation of $f(x)$ by $L_n(f, x)$.

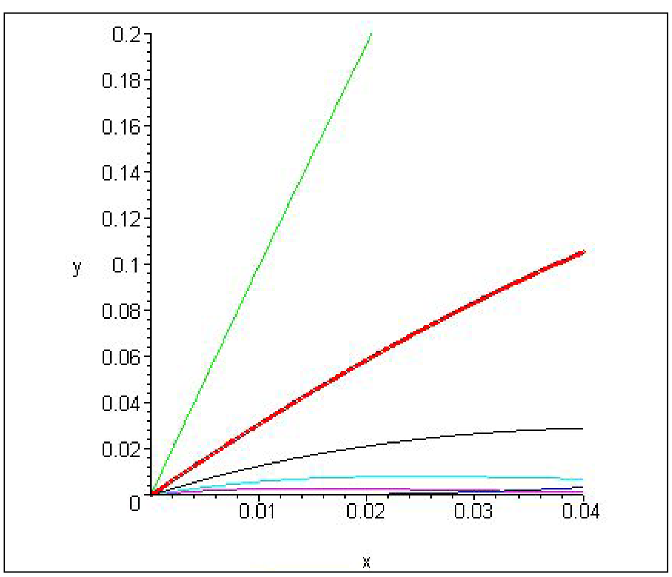

Figure 3. Approximation of $f(x)$ by $L_n(f, x)$.

$\lim_{\lambda\to\infty} \frac{\alpha\lambda}{\lambda} = 1$ and $\lim_{\lambda\to\infty} \frac{1}{\lambda^2\psi_\lambda(0)} = 0$.

Assume that a family of functions of three variables $\{K_\lambda(x,t,u)\}$; where $x,t \in [0,A], -\infty < u < \infty, \lambda \geq 0$ satisfies the following conditions:

1) Each function of this family is an entire analytic function with respect to u for fixed x and t of the interval $[0, A]$.
2) $K_\lambda(x,0,0) = 1$ for any $x \in [0,A]$ and for any $\lambda \geq 0$.
3) $\left\{(-1)^v \left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u)\right]_{\substack{u=u_1\\t=0}}\right\} \geq 0$, for any $x \in [0,A], \lambda \geq 0, v = 0,1,\ldots$.
4) $\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u)\Big|_{\substack{u=u_1\\t=0}} = -\lambda x\left[\frac{\partial^{v-1}}{\partial u^{v-1}} K_{h(\lambda)}(x,t,u)\right]_{\substack{u=u_1\\t=0}}$

for any $x \in [0,A], \lambda \in \mathbb{R}^+, v = 1,2,\ldots$ where $h(\lambda)$ is a nonnegative function satisfying the condition $\lim_{\lambda\to\infty} \frac{h(\lambda)}{\lambda} = 1$.

Consider the family of linear operators ;

$$L_\lambda(f;x) = \sum_{v=0}^{\infty} f\left(\frac{v}{\lambda^2\psi_\lambda(0)}\right)\left\{\left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u)\right]_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\t=0}}\right\}\frac{(-\alpha_\lambda\psi_\lambda(0))^v}{v!} \quad (2)$$

where $f \in C[0,A]$.

Note that for $\lambda = n$ and $h(\lambda) = m+n\,(m+n = 0,1,2,\ldots)$, the operators defined by (2) are reduced to the operators defined by (1).

Remark 2.2.1. By choosing, $\lambda = n\,(n = 1,2,\ldots)$ in $\{K_\lambda(x,t,u)\}$, Doğru 1997 obtained some known sequences of linear positive operators. Some of them as follows:

By choosing

$K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, \quad \psi_n(0) = \frac{1}{n},$

take $h(n) = n-1$ and the operators defined by (2) are transformed into Bernstein polynomials.

For $\alpha_n = n\psi_n(0) = \frac{1}{nb_n}, (\lim_{n\to\infty} b_n = \infty, \lim_{n\to\infty} \frac{b_n}{n} = 0)$, we obtain Bernstein-Chlodowsky polynomials.

By choosing $K_n(x,t,u) = e^{-n(t+ux)}, a_n = n, \psi_n(0) = \frac{1}{n}$, we have $h(n) = n$ and we get Szasz operators (Doğru 1997).

Let $K_n(x,t,u)$ is an entire analytic function and $K_n(x,t,u) = K_n(t+ux), \quad a_n = n, \psi_n(0) = \frac{1}{n}$ then we get (Baskakov 1957) operators, and for $a_n = n, \psi_n(0) = \frac{1}{a_n}$ and $\frac{n^2}{a_n} = \beta_n$, we get (Baskakov 1961) operators.

Theorem 2.2.2 Let $\{L_\lambda\}$ be sequence of linear positive operators defined by (2) and f satisfy $|f(x)| \leq M_f(1+x^2)$. Then for every function $f \in C[0,A]$,

$\lim_{\lambda\to\infty} \| L_\lambda(f,x) - f(x) \|_{C[0,A]} = 0$ (Doğru 1997)

Example 2.2.3

We give the graphics of approximation of the function $f(x) = \dfrac{\ln(x^3+1)}{e^{(x^2-4)}}$ by classical Gadjiev-Ibragimov operators by choosing next sequences. (n = 2(green), n = 4(red), n = 6(black), n = 8(cyan), n = 10(magenta), f (blue))

For $\lambda = n\,(n\in\mathbb{N}), K_n(x,t,u) = \left[1-\frac{ux}{1+t}\right]^n, a_n = n-1,$ $\psi_n(0) = \frac{1}{n},$ (see Figure 4).

For $\lambda = n\,(n\in\mathbb{N}), K_n(x,t,u) = \left[1-\frac{ux}{1+t}\right]^n, \alpha_n = n-1,$ $\psi_n(0) = \frac{1}{nb_n}$ and $b_n = \sqrt[3]{n}$ we get the following graphics. (see Figure 5).

By choosing $\lambda = n, K_n(x,t,u) = e^{-n(t+ux)}, a_n = n-1,$ $\psi_n(0) = \frac{1}{n},$ we have following graphics. (see Figure 6).

2.3. A Modified Gadjiev-Ibragimov Operators

We recall a sequence of linear positive operators defined in (Gönül and Coşkun 2012).

$$L_n(f,x) = \sum_{v=0}^{\infty} f\left(\frac{v}{\beta_n}\right) K_{n,v}(x) \frac{(-a_n)^v}{v!} \qquad (3)$$

Here $(a_n),(\beta_n); \lim_{n\to\infty}\beta_n = \infty,$ $\lim_{n\to\infty}\frac{a_n}{\beta_n} = 0$ and $\lim_{n\to\infty} n\frac{a_n}{\beta_n} = 1$ are two real sequence and $K_{n,v}(x)$ is the function depending on the parameters v and n to get the following conditions:

1) For any natural number n, any v = 0,1,2,... and for any $x\in[0,\infty)$ $(-1)^v K_{n,v}(x) \geq 0$

2) For any $x\in[0,\infty)$ $\sum_{v=0}^{\infty} K_{n,v}(x)\frac{(-a_n)^v}{v!} = 1;$

3) $K_{n,v}(x) = -nxK_{n+m,v-1}(x)$

for any $x\in[0,\infty)$ where $n + m$ is natural number and m is a constant independent of v.

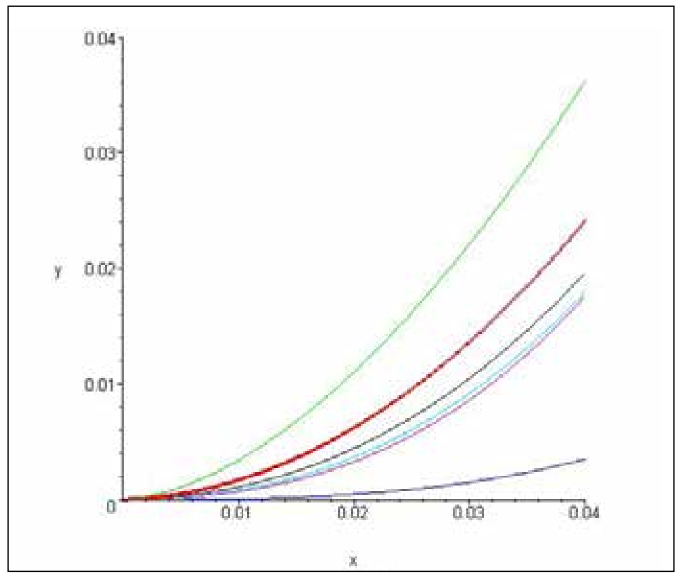

Figure 5. Approximation of $f(x)$ by $L_n(f, x)$.

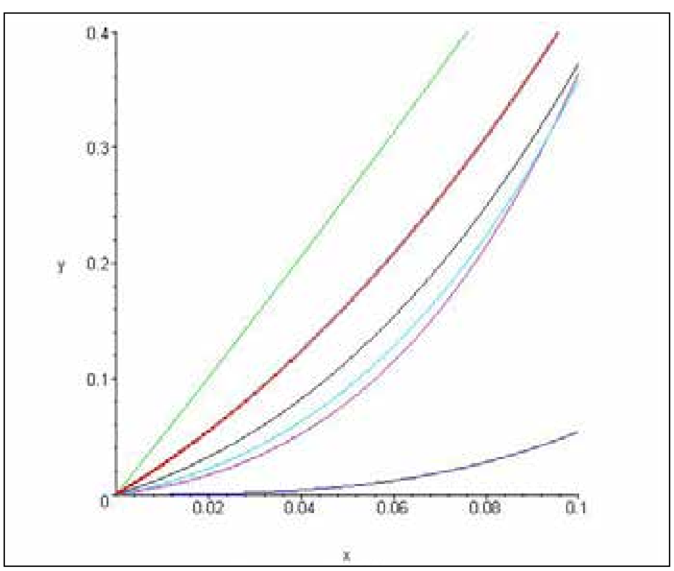

Figure 4. Approximation of $f(x)$ by $L_n(f, x)$.

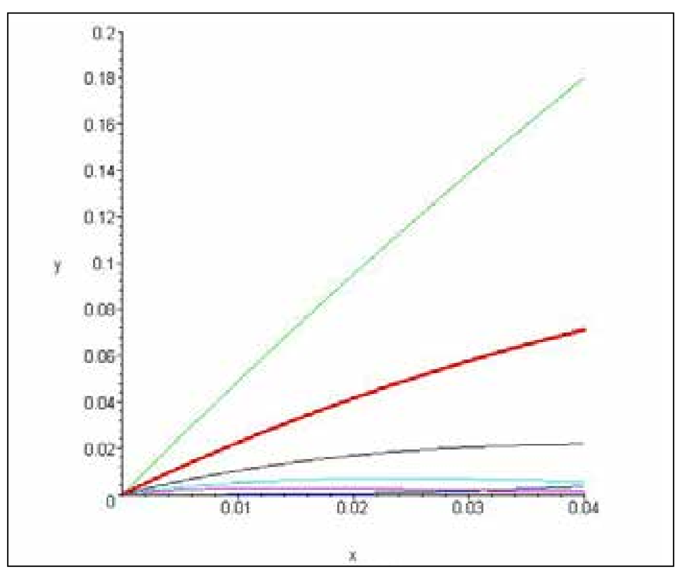

Figure 6. Approximation of $f(x)$ by $L_n(f, x)$.

Remark 2.3.1. Taking

$$K_{n,v}(x)=\begin{cases}(-1)^{v}\dfrac{n!}{(n-v)!}x^{v}(1-x)^{n-v}, x\in[0,1], v\le n\\ 0, \qquad\qquad\qquad\qquad otherwise\end{cases}$$

and choosing and $a_n=1$ and $\beta_n=n$; operators in (3) became the classical Bernstein polynomials.

If we take $\beta_n=\frac{n}{b_n}$ where β_n is the increasing sequences of positive number such that $\lim_{n\to\infty}b_n=\infty$ and $\lim_{n\to\infty}\frac{b_n}{n}=0$, then substituting x to be $\frac{x}{b_n}$ in $K_{n,v}(x)$ in (3) we obtain for $0\le x\le b_n$, the well-known Bernstein-Chlodowsky polynominals. In both cases m = -1. Choosing a_n = 1 and b_n = n and $K_{n,v}(x)=(-1)^{v}(nx)^{v}e^{-nx}$ we obtain Szasz operators m = 0 in (3).

Theorem 2.3.2 Let $\{L_n\}$ be sequence of linear positive operators defined by (3). Then for every function $f\in C_{\rho}^{k}[0,\infty)$,

$\lim_{n\to\infty}\| L_n(f,x)-f(x)\|_{\rho}=0$ (Gönül and Coşkun 2012).

Example 2.3.3

We give the graphics of approximation of functions $f(x)=\frac{\ln(x^3+1)}{e^{(x^2-4)}}$ by classical Gadjiev-Ibragimov operators by choosing next sequences. (n = 2(green), n = 4(red), n = 6(black), n = 8(cyan), n=10(magenta) and f (blue))

For $a_n=1, \beta_n=n$, (see Figure 7).

$$K_{n,v}(x)=\begin{cases}(-1)^{v}\dfrac{n!}{(n-v)!}x^{v}(1-x)^{n-v}, x\in[0,1], v\le n\\ 0, \qquad\qquad\qquad\qquad otherwise\end{cases}$$

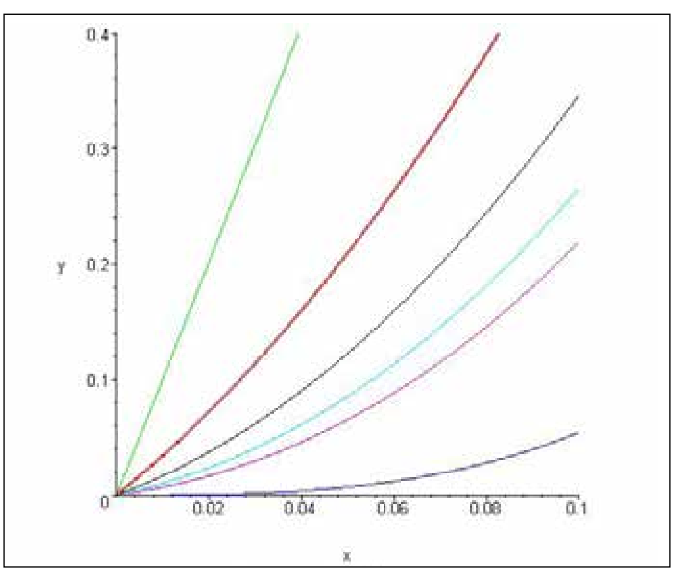

Figure 7. Approximation of $f(x)$ by $L_n(f, x)$.

Choosing $a_n=1$ and $\beta_n=\frac{n}{b_n}, a_n=\sqrt[5]{n}$, then we get next graphics. (see Figure 8).

We take $a_n=2$ and $b_n=2n-1$ and $K_{n,v}(x)=(-1)^{v}(nx)^{v}e^{-nx}$

In next section; approximation properties of operators given in (İspir et. al. 2008) are recalled for integrable functions.

2.4. Integral Form of Gadjiev Ibragimov Operators

Let $K_n(x,t,u)$ be sequence of entire analytic function with respect to variable $u\in\mathbb{R}$ for fixed $x,t\in[0,\infty)$ and $K_n(x,t,u)$ satisfy the following conditions:

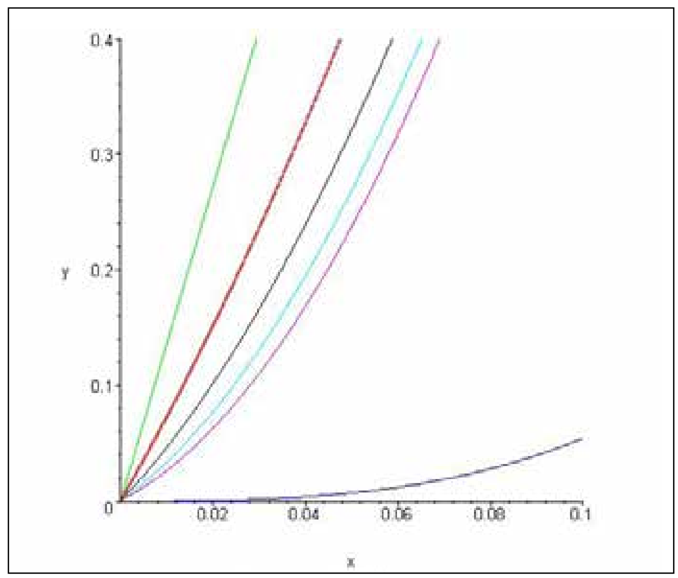

Figure 8. Approximation of $f(x)$ by $L_n(f, x)$.

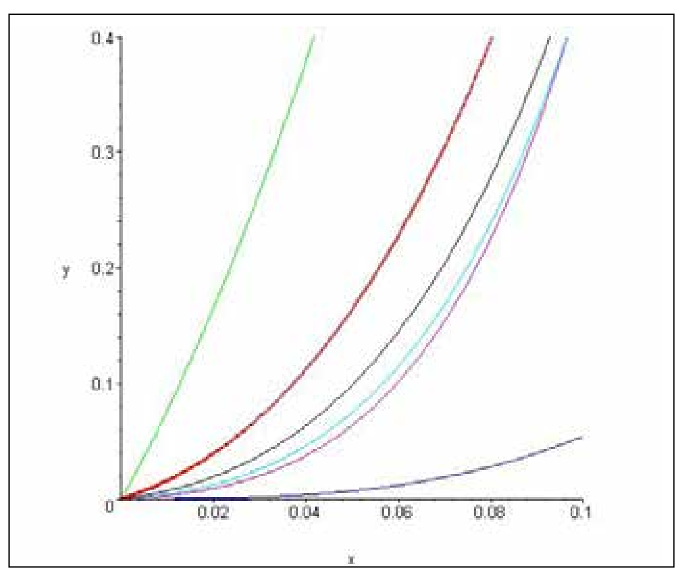

Figure 9. Approximation of $f(x)$ by $L_n(f, x)$.

1) For any natural number n and any $x \in [0,\infty), K_n(x,0,0) = 1$;

2) $\left\{(-1)^v \left[\frac{\partial^v}{\partial u^v} K_n(x,t,u)\right]_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right\} \geq 0, v \in \mathbb{N}_0,$

3) $\frac{\partial^v}{\partial u^v} K_n(x,t,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}} = -nx\left[\frac{\partial^{v-1}}{\partial u^{v-1}} K_{n+m}(x,t,u)\right]_{\substack{u=\alpha_n\psi_n(t)\\t=0}},$ $v \in \mathbb{N},$

where $(n + m)$ is a natural number or zero and m is a constant independent of v.

Moreover, let $(\varphi_n(t))_{n\in\mathbb{N}}$ and be sequences of continuous functions on $[0,\infty)$ such that

$\varphi_n(0) = 0, \psi_n(t) > 0$, for all $t \in [0,\infty)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers having the properties

$$\frac{\alpha_n}{n} = 1 + \sigma\left(\frac{1}{n^2\psi_n(0)}\right).$$

Under these conditions, integral form of operators in (1) is given as follows (İspir et. al. 2008).

$$L_n(f,x) = n^2\psi_n(0)\sum_{v=0}^{\infty} P_v(\alpha_n,\psi_n;K_n)\int_{I_{n,v}} f(t)\,dt \qquad (4)$$

where $I_{n,v} := \left[\frac{v}{n^2\psi_n(0)}, \frac{(v+1)}{n^2\psi_\lambda(0)}\right], n \in \mathbb{N}, v \in \mathbb{N}_0,$

$$P_v(\alpha_n,\psi_n;K_n) = \left(\frac{\partial^v}{\partial u^v} K_n(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_n(t)\\t=0}}\right)\frac{(-\alpha_\lambda\psi_n(0))^v}{v!}$$

and f is a member of the class of all measurable functions on $[0,\infty)$ and bounded on every compact subinterval of $[0,\infty)$.

Remark 2.4.1. By some special choosing of

$K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, \ \psi_n(0) = \frac{1}{n},$

the operators defined by (4) are transformed into Bernstein Kantorovich operators.

Taking

$K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, \ \psi_n(0) = \frac{1}{nb_n},$

the operators defined by (4) are transformed into Kantorovich type Bernstein Chlodowsky operators.

By choosing

$K_n(x,t,u) = e^{-n(t+ux)}, a_n = n, \psi_n(0) = \frac{1}{n},$

the operators defined by (4) are transformed into Szasz-Kantorovich operators.

Let ω be a positive continuous function on the real axis satisfying the condition

$$\int_{\mathbb{R}}^{\square} t^{2p}\omega(t)\,dt < \infty.$$

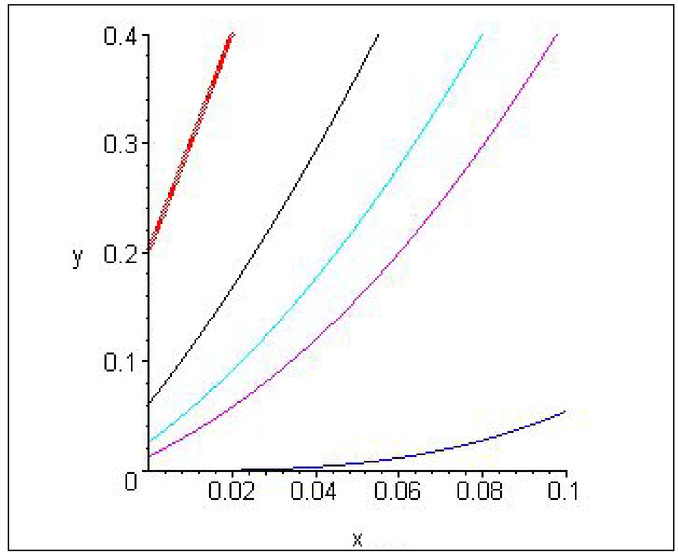

Figure 10. Approximation of $f(x)$ by $L_n(f, x)$.

We denote by $L_{p,\omega}(\mathbb{R})$, the linear space of measurable, p-absolutely integrable functions on $\mathbb{R}$ with respect to the weight function ω, i.e,

$$L_{p,\omega}(\mathbb{R}) = \left\{f:\mathbb{R}\to\mathbb{R}\,\middle|\, \|f\|_{p,\omega} := \left(\int_{\mathbb{R}}^{\square} |f(t)|^p\omega(t)\,dt\right)^{\frac{1}{p}} < \infty\right\}.$$

Theorem 2.4.2. Let $\{L_n\}$ be sequence of linear positive operators defined by (4). Then for every function $f \in L_{p,\omega}(\mathbb{R}_+)$

$\lim_{n\to\infty}\|L_n(f,x) - f(x)\|_{p,\omega} = 0$ (İspir et. al. 2008).

Example 2.4.3

We give the graphics of approximation of functions $f(x) = \frac{\ln(x^3+1)}{e^{(x^2-4)}}$ by Integral form of Gadjiev Ibragimov Operators by choosing next sequences. (n = 2(green), n = 4(red), n = 6(black), n = 8(cyan), n = 10(magenta) and f (blue))

$K_n(x,t,u) = \left[1 - \frac{ux}{1+t}\right]^n, a_n = n, \psi_n(0) = \frac{1}{n},$ (see Figure 10).

Choosing and $a_n = n$ and $\psi_n(0) = \frac{1}{nb_n}, b_n = \sqrt{n}$, then we get next graphics. (see Figure 11).

choosing $K_n(x,t,u) = e^{-n(t+ux)}, a_n = n$ and $\psi_n(0) = \frac{1}{n}$, then we get next graphics. (see Figure 11).

3. Rates of Convergence and Numerical Examples

In this section we want to find the rate of convergence of the sequence of operators $\{L_n\}$ defined by (Altın and Doğru 2004) and equality (1)-(3).

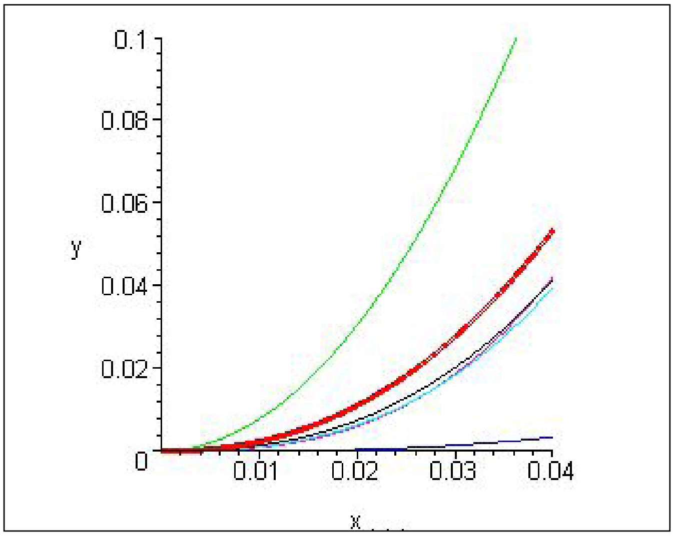

Figure 11. Approximation of $f(x)$ by $L_n(f,x)$.

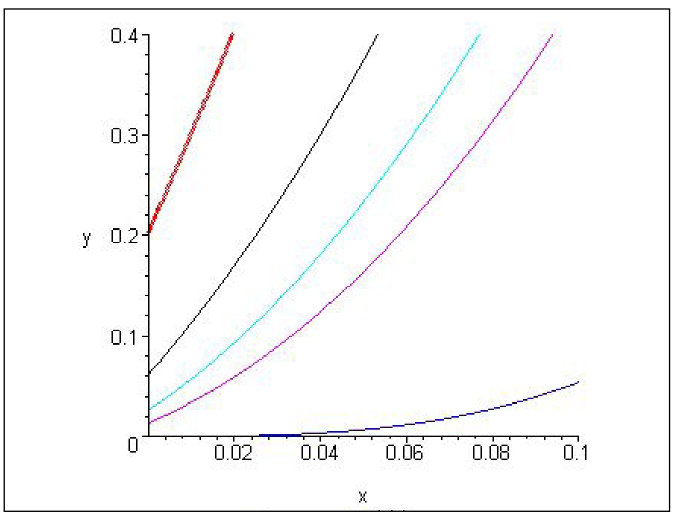

Figure 12. Approximation of $f(x)$ by $L_n(f,x)$.

Theorem 3.1 Let $f \in C[0,A]$, $\omega(f,\delta)$ be its weighted modulus of continuity. Then for a sufficiently large n, we get the inequality

$$|L_n(f,x) - f(x)| \leq 2\omega(f, \sqrt{\mu_{n,2}(x)})$$

where

$$\mu_{n,2}(x) = \left(\left(\frac{a_n}{n}\right)^2 \frac{n+m}{n} - 2\frac{a_n}{n} + 1 \right)x^2 + \frac{a_n}{n} \frac{1}{n^2 \psi_n(0)} x$$

(Altın and Doğru 2004).

Example 3.2 The error bound of the function

$$f(x) = \frac{x^2-3}{9e^4+3}, g(x) = \frac{1}{(x+2)^3 + (3x+2)^2 + 1},$$

for $x \in [0,2]$ and $(\alpha_n) = n-1, m = 1$ is given below.

Table 1. The error bound of function

$$f(x) = \frac{x^2-3}{9e^4+3}, g(x) = \frac{1}{(x+2)^3 + (3x+2)^2 + 1},$$

n	Error bound for continuity modulus of function f	Error bound for continuity modulus of function g
10	0.03427476695	0.3461882923
10^2	0.01060712489	0.1526418299
10^3	0.003291264094	0.05413815225
10^4	0.001033390884	0.01776621906
10^5	0.0003260143925	0.005684752530
10^6	0.0001030165412	0.001804397130
10^7	0.00003256883271	0.0005712747300
10^8	0.00001029838241	0.0001807201800
10^9	$0.3256555751\ 10^{-5}$	0.00005715540000
10^{10}	$0.1029805478\ 10^{-5}$	0.00001807461000

Theorem 3.3

Let f be continuous in $[0,\infty)$, satisfies $|f(x)| \leq M_f(1+x^2)$ and $\omega_{2A}(f,\delta)$ be its modulus of continuity on the finite interval $[0,2A]$ Then for the family of linear positive operators $\{L_n\}$ given by (2), the inequality

$$\| L_n(f,x) - f(x) \|_{C[0,A]} \leq \left[C_f M_f \left(5 + \frac{2}{A^2}\right) + 2 \right] \omega_{2A}(f, \delta_\lambda) \quad (5)$$

holds for all sufficiently large n where M_f is a positive constant which depends on f, C_f is as in the property of modulus of continuity and δ_λ is defined as follows (Doğru 1997).

$$\delta_\lambda = \left(\left\| \left(\frac{\alpha_\lambda^2}{\lambda^2} \frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda} + 1 \right) x^2 + \frac{\alpha_\lambda x}{\lambda} \frac{1}{\lambda^2 \psi(0)} \right\|_{C[0,A]} \right)^{\frac{1}{2}}$$

Example 3.4 The error bound of the function

$$f(x) = \frac{x^2-4}{4e^2+4}, \; g(x) = \frac{1}{(x+2)^3 + (3x+2)^2 + 1}, \text{for}$$

$x \in [0,2]$ and $(\alpha_n) = n-2, (h_n) = n+1$ is given below.

Table 2. The error bound of function

$f(x)=\dfrac{x^2-4}{4e^2+4}, g(x)=\dfrac{1}{(x+2)^3+(3x+2)^2+1}$,

n	Error bound for continuity modulus of function f	Error bound for continuity modulus of function g
10	0.5049696272	0.3479258270
10^2	0.1562746118	0.1526515849
10^3	0.04849014448	0.05413819068
10^4	0.01522493237	0.01776621960
10^5	0.004803165151	0.005684752530
10^6	0.001517741154	0.001804397130
10^7	0.0004798361234	0.0005712747300
10^8	0.0001517259135	0.0001807201800
10^9	0.00004797878700	0.00005715540000
10^{10}	0.00001517210865	0.00001807461000

Theorem 3.5 If $f\in C[0,A]$ then the inequality

$$\| L_n(f,x)-f(x)\|_{C[0,A]}\le K\omega\left(f,\sqrt{\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}}\right)$$

holds for sufficiently large n, where K is a constant independent of n (Gönül and Coşkun 2012).

Example 3.6 The error bound of the function $f(x)=\dfrac{x^2-3}{9e^4+3}$, $g(x)=\dfrac{1}{(x+2)^3+(3x+2)^2+1}$ $x\in[0,1]$, and $(\alpha_n)=1,(\beta_n)=n$ is given below.

Table 3. The error bound of function

$f(x)=\dfrac{x^2-3}{9e^2+3}, g(x)=\dfrac{1}{(x+2)^3+(3x+2)^2+1}$

n	Error bound for continuity modulus of function f	Error bound for continuity modulus of function g
10	0.04427443564	0.7928208060
10^2	0.01225133137	0.3406373368
10^3	0.003699263700	0.1203639828
10^4	0.001152315151	0.03948242720
10^5	0.0003626445714	0.01263284620
10^6	0.0001145033352	0.004009773800
10^7	0.00003619163916	0.001269499400
10^8	0.00001144305172	0.0004016004000
10^9	$0.3618435736\ 10^{-5}$	0.0001270120000
10^{10}	$0.1144232354\ 10^{-5}$	0.00004016580000

Theorem 3.7 Let $f\in C_\rho^0$ be given and let $\omega(f,\delta)$ be its weighted modulus of continuity. Then for a sufficiently large n, the inequality

$$\sup_{\substack{x\ge 0\\ |h|\le\delta}}\frac{|L_n(f,x)-f(x)|}{(1+x^2)(1+x^3)}\le K\omega\left(f,\frac{1}{\sqrt{n^2\psi_n(0)}}\right)$$

holds where K is a constant indepent on n,

$\omega(f,\delta)=\sup_{\substack{x\ge 0\\ |h|\le\delta}}\dfrac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}$ (Coşkun 2012).

Example 3.8 The error bound of the function $f(x)=\dfrac{x^2-3}{9e^4+3}, g(x)=\dfrac{(1+x^2)(2x+1)}{10}, x\in[0,1],$ and $(\alpha_n)=n$.

Table 4. The error bound of function

$f(x)=\dfrac{x^2-3}{9e^2+3}, g(x)=\dfrac{(1+x^2)(2x+1)}{10}$

n	Error bound for continuity modulus of function f	Error bound for continuity modulus of function g
10	0.003403286610	0.9863707760
10^2	0.001052665720	0.3100990099
10^3	0.0003245925914	0.09607857580
10^4	0.0001016328733	0.03012718728
10^5	0.00003203232816	0.009499744310
10^6	0.00001011865714	0.003001297199
10^7	$0.3198709445\ 10^{-5}$	0.0009488132095
10^8	$0.1011411479\ 10^{-5}$	0.0003000129972
10^9	$0.3198254608\ 10^{-6}$	0.00009486962970
10^{10}	$0.1011365977\ 10^{-6}$	0.00003000013000

4. Conclusion

We approached to the same function for each operator sequence (i.e. $L_n(f,x)\to f(x)$) we used. So readers can compare the rates of approximation of operators with the help of graphics. For a good approach, besides the used operators, the selection of the kernel function and the sequences is as important as the chosen function. This study will guide the comparison of other operators.

5. Appendix

We give algorithm of operators in (1) using Maple 13. First we choose f,α_n and β_n. Then we define $K_n(x,t,u)$ and $L_n(f,x)$. So we plot $f,L_n(f,x)$ for n = 2,4,6,8,10.

```
> restart;
> with(plots):
> f:=x->ln((x^3)+1)/exp((x^2)-4);
> printf(`Please enter the number of operator.\n`);m := scanf(`%d`)[1];
> 10
> printf(`Please enter the sum of the number of terms.\n`);r := scanf(`%d`)[1];
> 5
> for n from 1 to m do
> alpha(n):=n:
> psi(n):=1/n:
> K[n](x,t,u):=(1-((u*x)/(1+t)))^n:
> d(v):=diff(K[n](x,t,u),u$v):
> X[n](f,x):=sum(f(v/((n^2)*psi(n)))*d(v)*((alpha(n)*psi(n)))^v/v!,v=0..r):
>L[n](f,x):=subs(u=0,t=0,X[n](f,x)):
> end do:
> p1:=plot(f(x),x=0..0.1,y=0..0.4,color=blue):
>p2:=plot(L[2](f,x),x=0..0.1,y=0..0.4,color=green):
> p3:=plot(L[4](f,x),x=0..0.1,y=0..0.4,color=red):
> p4:=plot(L[6](f,x),x=0..0.1,y=0..0.4,color=black):
> p5:=plot(L[8](f,x),x=0..0.1,y=0..0.4,color=cyan):
> p6:=plot(L[10](f,x),x=0..0.1,y=0..0.4,color=magenta):
> display([p1,p2,p3,p4,p5,p6]);
```

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