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An Alternative Method to Undetermined Coefficients Method with aid of Fourier Transform

Fourier Dönüşümünün Yardımı ile Belirsiz Katsayılar Metoduna Alternatif bir Metod

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Abstract

In this paper, The Fourier transforms studied for special solution of ordinary differential equations. These equations of which right side is in the form $e^{\alpha x}P_m(x)$, where $P_m(x)$ is a polnomial from m.th degree, are nonhomogeneous constant coeffients. With this method we can solve all the equations that can be solved by the method of undeterminated coefficients.That is, this method is an alternative to the undetermined coefficients method.

Keywords: Dirac delta function, Fourier transform

Öz

Bu makalede adi diferansiyel denklemlerin özel çözümü için fourier dönüşümü çalışıldı $P_m(x)$. Dereceden polinom olmak üzere, bu denklemler sağ tarafı $e^{\alpha x}P_m(x)$ biçiminde olan sabit katsayılı homojen olmayan denklemlerdir. Bu metotla belirsiz katsayılar metodu ile çözebileceğimiz bütün denklemleri çözebiliriz. Yani bu metot belirsiz katsayılar metoduna bir alternatiftir.

Anahtar Kelimeler: Dirac delta fonksiyonu, Fourier dönüşümü

1. Introduction

Integral transform is a mapping from functions to functions that takes the following form

$$g(\alpha)=\int_a^b f(t).K(\alpha,t)\,dt \quad (1)$$

where is called the kernel of transform. In (1), is input fuction and is output function. Integral transforms have been used in solving many problems in applied mathematics, mathematical physics and engineering sciences. The best two known integral transforms are the Laplace transform and Fourier transform. The Fourier Transform, one of the gifts of Jean-Baptiste Joseph Fourier to the world of science, is an integral transform used in many areas of engineering It has been very useful for analyzing harmonic signals or signals for which there is known need for local information. Then the fourier transform analysis has also been very useful in many other areas such as quantum mechanics, wave motion, turbulence, etc (Bracewell 2000, Murray 1974, Lokenath 2015, Larry et al 1988 and Albert et al 2001). Furthermore it has been useful in methematics such as generalized integrals, integral equations, ordinary differential equations, partial differential equations can be solved by using the Fourier transform. Another example of its applications could be; voice of every human being can be expressed as the sum of sine and cosine. Since the electromagnetic spectrum of the frequency of each voice is different, the frequency of each sine and cosine sum will be different. In this way, a voice record can be found belongs to whom using the Fourier transform. In fact, our ear automatically runs this process instead of us.

There are some methods for finding of particular solutions nonhomogeneous differential equations with constant coefficients. For example, some of these methods are the undetermined coefficients method, the inverse operator methothe variation of parameters method, the reduction of order method, etc.

In this paper, we have studied a special solution of constant coefficients nonhomegeneous ordinary differential equations which right side is $e^{\alpha x}P_m(x)$, where $P_m(x)$ is a polynomial of order m and α is constant. We have obtained the solution by using fourier transform. These kind equations was studied in (Jia et al 2013, Ortigueira 2014). In (Jia et al 2013), Jia

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J.,Sogabe T. studied the equations by using upper triangular Toeplitz matrix. In (Ortigueira 2014), Ortigueira M.D., studied the equations by using eigenfuctions.

2. Basic Definition and Theorems

In this section well known basic definition and theorems with regard to fourier transform.

Definition 2.1. Fourier transform of $f(t)$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t).e^{-iwt}dt \tag{2}$$

is defined. Since integral (2) is function of w

$$\mathcal{F}[f(t)] = F(w)$$

is written.

Definition 2.2. If $\mathcal{F}[f(t)] = F(w)$ then $f(t)$ is called inverse Fourier transform of $F(w)$; where

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w).e^{iwt}dt \tag{3}$$

and it is showed by $f(t) = \mathcal{F}^{-1}[F(w)]$

Theorem 2.1. [Bracewell 2000, Murray 1974, Lokenath 2015, Larry et al 1988 and Albert et al 2001] The Fourier Transform is linear. Let $c_1, c_2 \in R$. Then

$$\mathcal{F}[c_1.f_1(t) + c_2.f_2(t)] = c_1.\mathcal{F}[f_1(t)] + c_2\mathcal{F}[f_2(t)]$$

Theorem 2.2. [Bracewell 2000, Murray 1974, Lokenath 2015, Larry et al 1988 and Albert et al 2001] Let $f(t)$ be continuous or partly continuous in the interval $(-\infty,\infty)$ and $f(t), f'(t), f''(t), ..., f^{(n-1)}(t) \to 0$ for $|t| \to \infty$. If $f(t), f'(t), f''(t), .., f^{(n-1)}(t)$ are absolutely integrable in the interval $(-\infty,\infty)$, then

$$\mathcal{F}[f^{(n)}(t)] = (iw)^n \mathcal{F}[f(t)] \tag{4}$$

Definition 2.3. The Dirac delta function can be rigorously thought of as a function on real line which is zero everywhere except at the orijin, where it is infinite,

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

The Dirac delta function has some properties, that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t).\delta(t-t_0)dt = f(t_0) \tag{5}$$

$$\int_{-\infty}^{\infty} f(t).\delta^{(n)}(t-t_0)dt = (-1)^n.f^{(n)}(t_0) \tag{6}$$

$$t.\delta'^{(t)} = -\delta(t) \tag{7}$$

Theorem 2.3. (Bracewell 2000, Larry 1988) The Fourier transform of the Dirac Delta function is 1. That is $\mathcal{F}[\delta(t)] = 1$.

Theorem 2.4. (Bracewell 2000, Larry 1988 and Albert 2001) Fourier transforms of some functions are following

i) $\mathcal{F}[1] = 2\pi.\delta(w)$

ii) $\mathcal{F}[t^n] = 2\pi.i^n.\delta^{(n)}(w)$

iii) $\mathcal{F}[t^n.f(t)] = i^n \frac{d^n\mathcal{F}[f(t)]}{dw^n}$

iv) $\mathcal{F}[e^{iw_0t}] = 2\pi\delta(w-w_0)$

v) If $\mathcal{F}[f(t)] = F(w)$, then $\mathcal{F}[e^{iw_0t}.f(t)] = F(w-w_0)$

vi) $\mathcal{F}[e^{at}] = 2\pi\delta(w+ia)$

vii) If $\mathcal{F}[f(t)] = F(w)$, then $\mathcal{F}[e^{a.t}.f(t)] = F(w+ia)$

Theorem 2.5 (Bracewell 2000, Larry 1988)

$$(w-w_0)^n.\delta^{(n)}(w-w_0) = (-1)^n.n!.\delta(w-w_0) \tag{8}$$

Theorem 2.6. (Bracewell 2000, Larry 1988)

$$\int_{-\infty}^{\infty} \frac{\delta(w-w_0)f(w)}{(w-w_0)^n}dw = \frac{1}{n!}\frac{d^nf(w)}{dw^n}(w=w_0)$$

Where $\delta(w-w_0)$ is defined as following

$$\delta(w-w_0) = \begin{cases} 0, & w \neq w_0 \\ \infty, & w = w_0 \end{cases}$$

3. Solutions of Differential Equations by using Fourier Transform

In this section, we have given some theorems finding a special solution of n.th order nonhomogeneous differential equations with constant coefficients by using fourier transforms.

Theorem 3.1. A special solution of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 . y' + a_0 y = e^{\alpha x} P_m(x)$$

is

$$y(x) = b_m (-i)^m \frac{d^m}{dx^m}\left(\frac{e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + a_{n-2}(iw)^{n-2} + \cdots + a_0}\right)(w=-i\alpha)$$
$$+ b_{m-1}(-i)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\left(\frac{e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + a_{n-2}(iw)^{n-2} + \cdots + a_0}\right)(w=-i\alpha)$$
$$+ \cdots + b_0 \frac{e^{ax}}{a_n (a)^n + a_{n-1}(a)^{n-1} + a_{n-2}(a)^{n-2} + \cdots + a_0}$$

Where $P_m(x)$ is polynomial m.th degree and α can be real and imaginer.

Proof: We let use fourier transform for

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 . y' + a_0 y = e^{\alpha x} P_m(x)$$
$$\mathcal{F}[a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 . y' + a_0 y] = \mathcal{F}[e^{\alpha x} P_m(x)]$$
$$\mathcal{F}[a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 . y' + a_0 y]$$
$$= \mathcal{F}[e^{\alpha x}(b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0)]$$
$$a_n \mathcal{F}[y^{(n)}] + a_{n-1}\mathcal{F}[y^{(n-1)}] + \cdots + a_0 \mathcal{F}[y]$$
$$= b_m \mathcal{F}[e^{\alpha x} x^m] + b_{m-1}\mathcal{F}[e^{\alpha x} x^{m-1}] + \cdots + b_0 \mathcal{F}[e^{\alpha x}]$$
$$[a_n (iw)^n + a_{n-1}(iw)^{n-1} + \cdots + a_0] Y(w)$$
$$= 2\pi[b_m i^m \delta^{(m)}(w+i\alpha) + b_{m-1} i^{m-1}\delta^{(m-1)}(w+i\alpha) + \cdots + b_0 . \delta(w+i\alpha)]$$
$$Y(w) = \frac{2\pi[b_m i^m \delta^{(m)}(w+i\alpha) + b_{m-1} i^{m-1}\delta^{(m-1)}(w+i\alpha) + \cdots + b_0 . \delta(w+i\alpha)]}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + \cdots + a_0}$$
$$y(x) = \mathcal{F}^{-1}[Y(w)]$$
$$= \int_{-\infty}^{\infty} \frac{b_m i^m \delta^{(m)}(w+i\alpha) + b_{m-1} i^{(m-1)}\delta^{(m-1)}(w+i\alpha) + \cdots + b_0 . \delta(w+i\alpha)}{a_n (iw)^n + a_{n-1)}(iw)^{n-1} + \cdots + a_0} . e^{iwx} dw$$
$$y(x) = b_m i^m \int_{-\infty}^{\infty} \frac{\delta^{(m)}(w+i\alpha) . e^{iwx}}{a_n (iw)^n + a_{n-1}/iw)^{n-1} + \cdots + a_0} dw$$
$$+ b_{m-1} i^{m-1} \int_{-\infty}^{\infty} \frac{\delta^{(m-1)}(w+i\alpha) . e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + \cdots + a_0} dw$$
$$+ \cdots + b_0 \int_{-\infty}^{\infty} \frac{\delta(w+i\alpha) . e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + \cdots + a_0} dw$$

From (6)

$$y(x) = b_m (-i)^m \frac{d^m}{dx^m}\left(\frac{e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + a_{n-2}(iw)^{n-2} + \cdots + a_0}\right)(w=-i\alpha)$$
$$+ b_{m-1}(-i)^{m-1}\frac{d^{m-1}}{dx^{m-1}}\left(\frac{e^{iwx}}{a_n (iw)^n + a_{n-1}(iw)^{n-1} + a_{n-2}(iw)^{n-2} + \cdots + a_0}\right)(w=-i\alpha)$$
$$+ \cdots + b_0 \frac{e^{ax}}{a_n (\alpha)^n + a_{n-1}(\alpha)^{n-1} + a_{n-2}(\alpha)^{n-2} + \cdots + a_0}$$

Example 3.1 We find a special solution following differential equation

$$y' + y = x + 1$$

Solution: From theorem 3.1. $e^{\alpha x} . P_m(x) = x + 1$. So,

$$\alpha = 0, n = 1, m = 1, b_0 = 1, b_1 = 1, a_0 = 1, a_1 = 1.$$
$$y(x) = -i\frac{d}{dw}\left(\frac{e^{iwx}}{iw+1}\right)(w=0) + \frac{1}{1}$$
$$= -i\left(\frac{ixe^{iwx}(iw+1) - ie^{iwx}}{(iw+1)^2}\right)(w=0) + 1$$
$$= -i(ix - i) + 1 = x$$

Example 3.2 We find a special solution following differential equation

$$y'' - 3y' + 2y = x.e^{3x}$$

Solution: From theorem 3.1. $n = 2, a_2 = 1, a_1 = -3, a_0 = 2,$
$m = 1, b_1 = 1, b_0 = 0, \alpha = 3$

$$\begin{aligned} y(x) &= -i\frac{d}{dw}\left(\frac{e^{iwx}}{-w^2 - 3iw + 2}\right)(w = -3i) \\ &= -i\frac{ixe^{iwx}(-w^2 - 3iw + 2) - (-2w - 3i)e^{iwx}}{(-w^2 - 3iw + 2)^2}(w = -3i) \\ &= -i\frac{2ixe^{3x} - 3ie^{3x}}{4} = \frac{2x - 3}{4}e^{3x} \end{aligned}$$

Example 3.3 We find a special solution following differential equation

$$y'' + 3y' + 4y = x.\cos x$$

Solution: If we consider that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Then from theorem we can write solution of equation as follow:

$$\begin{aligned} y(x) &= (-i)\frac{d}{dw}\left(\frac{e^{iwx}}{-w^2 + 3iw + 4}\right)(w = 1) + (-i)\frac{d}{dw}\left(\frac{e^{iwx}}{-w^2 + 3iw + 4}\right)(w = -1) \\ &= (-i)\left[\frac{ixe^{iwx}(-w^2 + 3iw + 4) - 4(-2w + 3i)e^{iwx}}{(-w^2 + 3iw + 4)^2}\right](w = 1) \\ &+ (-i)\left[\frac{ixe^{iwx}(-w^2 + 3iw + 4) - 4(-2w + 3i)e^{iwx}}{(-w^2 + 3iw + 4)^2}\right](w = -1) \\ &= (-i)\left[\frac{ix.e^{ix}(3 + 3i) - (3i - 2)e^{-ix}}{(3 + 3i)^2}\right] + (-i)\left[\frac{ix.e^{-ix}(3 - 3i) - (3i + 2)e^{-ix}}{(3 - 3i)^2}\right] \\ &= \frac{x}{3}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) + \frac{x}{3}\left(\frac{e^{ix} + e^{-ix}}{2}\right) - \frac{1}{3}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) - \frac{2}{9}\left(\frac{e^{ix} + e^{-ix}}{2}\right) \\ &= \frac{x}{3}(\cos x + \sin x) - \frac{1}{3}(\sin x + \frac{2}{3}\cos x) \end{aligned}$$

Example 3.4: We find a special solution following differential equation

$$y'' - 2y' + y = 5.e^{x}$$

Solution: From theorem 3.1

$$y(x) = 5\int_{-\infty}^{\infty}\frac{\delta(w + i)}{-(w + i)^2}e^{iwx}dw$$

From theorem 2.6

$$y(x) = -\frac{5}{2}\frac{d^2(e^{iwx})}{dw^2}(w = -i) = \frac{5x^2.e^{x}}{2}$$

Example 3.5: We find a special solution following differential equation

$$y''' + y'' - 4y' + 2y = 4x$$

Solution: From theorem 3.1 coeffients of equation are $n = 3, a_3 = 1, a_2 = 1, a_1 = -4, a_0 = 2, \alpha = 0, m = 1, b_1 = 4.$ So, a special solution of equation is that

$$\begin{aligned} y(x) &= -ib_1\frac{d}{dw}\left(\frac{e^{iwx}}{a_3(iw)^3 + a_2(iw)^2 + a_1(iw) + a_0}\right)(w = 0) \\ &= -4i\frac{d}{dw}\left(\frac{e^{iwx}}{(iw)^3 + (iw)^2 - 4iw + 2}\right)(w = 0) \\ &= -4i\left(\frac{ix.e^{iwx}((iw)^3 + (iw)^2 - 4iw + 2) - (3i(iw)^2 + 2i.iw - 4i)e^{iwx}}{((iw)^3 + (iw)^2 - 4iw + 2)^2}\right)(w = 0) \\ &= 2x + 4 \end{aligned}$$

4. Conclusion

In this study, we have found a special solution of n.th order constant coefficients ordinary differential equations of which right side are the form $e^{\alpha x}.P_m(x)$. It has been seen that the results were consistent with the studies in the literature. It may concluded that the proposed technique is efficient in finding the analytic solutions for a large class of constant coefficients ordinary differential equations.

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