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Research Article

Integrability Conditions and Tachibana Operators According to ${}^{CC}F-\frac{1}{2}\gamma(NF)$ on Semi Cotangent Bundle $t^{*}(M_n)$

Semi Cotangent Demette ${}^{CC}F-\frac{1}{2}\gamma(NF)$ Yapısına Göre Tachibana Operatörleri ve İntegrallenebilme Şartları

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Abstract

The main aim of this paper is to find integrability conditions by calculating Nijenhuis Tensors $N({}^{cc}X,{}^{cc}Y),N({}^{cc}X,{}^{vv}\omega),N({}^{vv}\omega,{}^{vv}\theta)$of almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ and to show the results of Tachibana operators applied ${}^{cc}X$ and ${}^{vv}\omega$ according to structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ in semi cotangent bundle $t^{*}(M_n)$.

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Öz

Bu çalışmanın temel amacı ${}^{cc}F-\frac{1}{2}\gamma(NF)$ almost kompleks yapısının $N({}^{cc}X,{}^{cc}Y),N({}^{cc}X,{}^{vv}\omega)$ *ve* $N({}^{vv}\omega,{}^{vv}\theta)$ ve ${}^{cc}F-\frac{1}{2}\gamma(NF)$ Nijenhuis tensörlerini hesaplayarak integrallenebilme şartlarını bulmak ve $t^{*}(M_n)$ semi cotangent demeti içerisinde ${}^{cc}F-\frac{1}{2}\gamma(NF)$ yapısına göre ${}^{cc}X$ ve ${}^{vv}\omega$ ye uygulanan Tachibana operatörlerinin sonuçlarını göstermektir.

Anahtar Kelimeler: Almost kompleks yapı, Komple lift, Integrallenebilme şartları, Semi cotangent demet, Tachibana operatörü, Vertikal lift

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class $C^{\infty}$and let $T(M_n)$ the tangent bundle determined by a natural projection (submersion) . $\pi_1\colon T(M_n)\to M_n$. We use the notation $(x^{i})=(x^{\bar{\alpha}},x^{\alpha})$, where the indices i,j,…run from 1 to 2n, the indices $\alpha,\beta,\ldots$ from 1 to n and the indices $\bar{\alpha},\bar{\beta},\ldots$ from n+1 to 2_n, x^{α} are coordinates in $M_n, x^{\bar{\alpha}}=y^{\alpha}$are fibre coordinates of the tangent bundle $T(M_n)$. If $(x^{i'})=(x^{\bar{\alpha}'},x^{\alpha'})$ is another system of local adapted coordinates in the tangent bundle $T(M_n)$ then we have

$$\begin{cases} x^{\bar{\alpha}'}=\dfrac{\partial x^{\alpha'}}{\partial x^{\beta}}y^{\beta}, \\ x^{\alpha'}=x^{\alpha'}(x^{\beta}). \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$(A_{j}^{i'})=\left(\frac{\partial x^{i'}}{\partial x^{j}}\right)=\begin{pmatrix} A_{\beta}^{\alpha'} & A_{\beta\varepsilon}^{\alpha'}y^{\varepsilon} \\ 0 & A_{\beta}^{\alpha'} \end{pmatrix},$$

where $A_{\beta}^{\alpha'}=\dfrac{\partial x^{\alpha'}}{\partial x^{\beta}},A_{\beta\varepsilon}^{\alpha'}=\dfrac{\partial^{2}x^{\alpha'}}{\partial x^{\beta}\partial x^{\varepsilon}}$.

Let $T_{x}^{*}(M_n)\,(x=\pi_1(x),x=(x^{\bar{\alpha}},x^{\alpha})\in T(M_n))$ be the cotangent space at a point x of M_n. If p_{α} are components of $p\in T_{x}^{*}(M_n)$with respect to the natural coframe $\{dx^{\alpha}\}$, i.e. $p=p_i dx^i$, then by definition the set $t^{*}(M_n)$ of all points $(x^{I})=(x^{\bar{\alpha}},x^{\alpha},x^{\bar{\bar{\alpha}}}),x^{\bar{\bar{\alpha}}}=p_{\alpha};I,J,\ldots=1,\ldots 3n$ with projection $\pi_2\colon t^{*}(M_n)\to T(M_n)\big(i.e.\pi_2\colon(x^{\bar{\alpha}},x^{\alpha},x^{\bar{\bar{\alpha}}})\to(x^{\bar{\alpha}},x^{\alpha})\big)$ is a semi-cotangent (pull-back (Yıldırım and Salimov 2014)) bundle of the cotangent bundle by submersion $\pi_1\colon T(M_n)\to M_n$ (For definition of the pull-back bundle, see for example (Husemöller 1994, Lawson and Michelsohn 1989, Pontryagin 1962, Steenrod 1951, Yıldırım 2015, Yıldırım and Salimov 2014)). It is remarkable fact that the semi-cotangent (pull-back) bundle has a degenerate symplectic structure (Yıldırım and Salimov 2014)

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$$\omega=(\omega_{AB})=dp=\begin{pmatrix}0&0&0\\0&0&-\delta_\beta^\alpha\\0&\delta_\alpha^\beta&0\end{pmatrix}$$

It is clear that the pull-back bundle $t^{*}(M_n)$ of the cotangent bundle $T^{*}(M_n)$ also has the natural bundle structure over M_n, its bundle projection $\pi:t^{*}(M_n)\to M_n$ being defined by $\pi:(x^{\bar\alpha},x^{\alpha},x^{\bar{\bar\alpha}})\to(x^{\alpha})$ and hence $\pi=\pi_1\circ\pi_2$ (Yıldırım 2015). Thus $(t^{*}(M_n),\pi_1\circ\pi_2)$ is the composite bundle (Pontryagin 1962, p.9) or step-like bundle (Lawson and Michelsohn 1989).

1.1. Complete Lift of Vector Fields

We denote by $\Im_q^p(T(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p,q) on $T(M_n)$ and M_n respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞-functions on $T(M_n)$ and M_n, respectively.

To a transformation (1.1) of local coordinates of $T(M_n)$ there corresponds on $t^{*}(M_n)$ the coordinate transformation (Yıldırım 2015)

$$\begin{cases}x^{\bar\alpha'}=\dfrac{\partial x^{\alpha'}}{\partial x^{\beta}}y^{\beta},\\ x^{\alpha'}=x^{\alpha'}(x^{\beta}),\\ x^{\bar{\bar\alpha}'}=\dfrac{\partial x^{\beta}}{\partial x^{\alpha'}}p_{\beta.}\end{cases}\qquad(1.2)$$

The Jacobian of (1.2) is given by

$$\bar A=(A_J^{I'})=\begin{pmatrix}A_\beta^{\alpha'}&A_{\beta\varepsilon}^{\alpha'}y^{\varepsilon}&0\\0&A_\beta^{\alpha'}&0\\0&p_\alpha A_\beta^{\beta'}A_{\beta'\alpha'}^{\alpha}&A_{\alpha'}^{\beta}\end{pmatrix},\qquad(1.3)$$

where

$$A_\beta^{\alpha'}=\frac{\partial x^{\alpha'}}{\partial x^{\beta}},A_{\alpha'}^{\beta}=\frac{\partial x^{\beta}}{\partial x^{\alpha'}},A_{\beta\varepsilon}^{\alpha'}=\frac{\partial^2x^{\alpha'}}{\partial x^{\beta}\partial x\varepsilon},A_{\beta'\alpha'}^{\alpha}=\frac{\partial^2x^{\alpha'}}{\partial x^{\beta'}\partial x^{\alpha'}}$$

It is easily verified that the condition Det $\overline{A}\neq 0$ is equivalent to the condition:

$Det(A_\beta^{\alpha'})\neq 0.$

Let $X\in\Im_0^1(T(M_n))$, $i.e.X=X^{\alpha}\partial_\alpha$. The complete lift ${}^{c}X$ of X to tangent bundle is defined by ${}^{c}X=X^{\alpha}\partial_\alpha+(y^{\beta}\partial_\beta X^{\alpha})\partial_{\bar\alpha}$ (Yano and Ishihara 1973, p.15). On putting

$${}^{cc}X=({}^{cc}X^{\alpha})=\begin{pmatrix}y^{\varepsilon}\partial_\varepsilon X^{\alpha}\\X^{\alpha}\\-p_\varepsilon(\partial_\alpha X^{\varepsilon})\end{pmatrix},\qquad(1.4)$$

from (1.3), we easily see that ${}^{cc}X'=\overline{A}({}^{cc}X)$. The vector field ${}^{cc}X$ is called the complete lift of ${}^{c}X\in\Im_0^1(T(M_n))$ to $t^{*}(M_n)$ (Yıldırım 2015).

Now, consider $\omega\in\Im_1^0(T(M_n))$ and $F\in\Im_1^1(T(M_n))$ then ${}^{vv}\omega$ (vertical lift) and $\gamma F\in\Im_0^1(t^{*}(M_n))$ have respectively, componenets on the semi-cotangent bundle $t^{*}(M_n)$ (Yıldırım and Salimov 2014)

$${}^{vv}\omega=\begin{pmatrix}0\\0\\\omega_\alpha\end{pmatrix},\gamma F=(\gamma F^{I})=\begin{pmatrix}0\\0\\p_\beta F_\alpha^{\beta}\end{pmatrix}\qquad(1.5)$$

with respect to the coordinates $(x^{\bar\alpha},x^{\alpha}x^{\bar{\bar\alpha}})$ where ω_α and F_α^{β} are local components of ω and F.

For $T\in\Im_2^1(M_n)$, we can define an affinor field $\gamma T\in\Im_1^1(t^{*}(M_n))$ (Yıldırım and Salimov 2014):

$$\gamma T=((\gamma T)_J^I)=\begin{pmatrix}0&0&0\\0&0&0\\0&p_\varepsilon T_{\beta\alpha}^{\varepsilon}&0\end{pmatrix},\qquad(1.6)$$

where $T_{\beta^\alpha}^{\varepsilon}$ are local components of T in M_n.

On the other hand, ${}^{vv}f$ the vertical lift of function f on $t^{*}(M_n)$ is defined by (Yıldırım and Salimov 2014):

$${}^{vv}f={}^{v}f\circ\pi_2=f\circ\pi_1\circ\pi_2=f\circ\pi.\qquad(1.7)$$

Theorem 1.1.1. For any vector fields X,Y on $T(M_n)$ and $f\in\Im_0^0(M_n)$, we have (Yıldırım 2015)

i) ${}^{cc}(X+Y)={}^{cc}X+{}^{cc}Y,$

ii) ${}^{cc}X{}^{vv}f={}^{vv}(Xf).$

Theorem 1.1.2. Let $X,Y\in\Im_0^1(T(M_n))$. For the Lie product, we have

i) $[{}^{cc}X,{}^{cc}Y]={}^{cc}[X,Y](i.e.L_{{}^{cc}X}({}^{cc}Y)={}^{cc}(L_XY)),$

ii) $[{}^{cc}X,{}^{vv}\omega]={}^{vv}(L_X\omega),$

iii) $[{}^{cc}X,\gamma F]=\gamma(L_XF)$

for any $\omega\in\Im_1^0(M_n)$ and $F\in\Im_1^1(T(M_n))$, where L_X the operator of Lie derivation with respect to X (Yıldırım 2015).

1.2. Complete Lift of Tensor Fields of Type (1,1)

Suppose now that $F\in\Im_1^1(T(M_n))$ and F has local components F_β^α in a neighborhood U of $M_n, F=F_\beta^\alpha\partial_\alpha\otimes dx^{\beta}$. If we take account of (1.3), we can prove that ${}^{cc}F_{J'}^{I'}=A_I^{I'}{}^{cc}F_J^I$, where ${}^{cc}F$ is an affinor field defined by

$${}^{cc}F=({}^{cc}F_J^I)=\begin{pmatrix}F_\beta^\alpha&y^{\varepsilon}\partial_\varepsilon F_\beta^\alpha&0\\0&F_\beta^\alpha&0\\0&p_\sigma(\partial_\beta F_\alpha^\sigma-\partial_\alpha F_\beta^\sigma)&F_\alpha^\beta\end{pmatrix}\qquad(1.8)$$

with respect to the coordinates $(x^{\bar\alpha},x^{\alpha},x^{\bar{\bar\alpha}})$ on $t^{*}(M_n)$ We call ${}^{cc}F$ the complete lift of the tensor field F of type (1,1) to $t^{*}(M_n)$ (Yıldırım 2015).

Theorem 1.2.1. If $X\in\Im_0^1(T(M_n))$, $\omega\in\Im_1^0(M_n)$ and $F\in\Im_1^1(T(M_n))$, then

i) ${}^{cc}F^{cc}X={}^{cc}(FX)+\gamma(L_XF)$,

ii) ${}^{cc}F^{vv}\omega={}^{vv}(\omega\circ F)$,

where L_X the operator of Lie derivation with respect to X (Yıldırım 2015).

Theorem 1.2.2. For any $F\in\Im_1^1(T(M_n)), F^2=-I$ (Yıldırım 2015),

$({}^{cc}F)^2=-I-\gamma(N_F)$.

Theorem 1.2.3. Let $X\in\Im_0^1(T(M_n))$ and $F\in\Im_1^1(T(M_n))$. Then

$L_{ccX}{}^{cc}F=0$

if $L_XF = 0$ (Yıldırım 2015).

2. Results

2.1. Integrability Conditions of Almost Complex Structure on Semi Cotangent Bundle

Definition 2.1.1. Let F be an almost complex structure on $T(M_n)$, i.e., $F^2 = -I$. We say that F is integrable if the Nijenhuis tensor N_F of F is identically equal to zero. The Nijenhuis tensor N_F is defined by

$$N_F=[FX,FY]-F[X,FY]-F[FX,Y]+F^2[X,Y]$$

for any $X,Y\in\Im_0^1(T(M_n))$ (Salimov 2013, Salimov and Çayır 2013).

Theorem 2.1.1. Let N_{ccF} the Nijenhuis tensor of almost complex structure ${}^{cc}F\in\Im_1^1(t^*(M_n))$. Then

$$N_{ccF}=N({}^{cc}X,{}^{cc}Y)=0$$

if and only if $N_F = 0$, where $X,Y\in\Im_0^1(T(M_n))$, N_F the Nijenhuis tensor of $F\in\Im_1^1(T(M_n))$

Proof.

$$\begin{aligned}N_{ccF}&=N({}^{cc}X,{}^{cc}Y)=[{}^{cc}F^{cc}X,{}^{cc}F^{cc}Y]-{}^{cc}F[{}^{cc}F^{cc}X,{}^{cc}Y]-\\&{}^{cc}F[{}^{cc}X,{}^{cc}F^{cc}Y]-[{}^{cc}X,{}^{cc}Y]\\&=[{}^{cc}(FX)+\gamma(L_XF),{}^{cc}(FY)+\gamma(L_YF)]-\\&{}^{cc}F[{}^{cc}(FX)+\gamma(L_XF),{}^{cc}Y]\\&-{}^{cc}F[{}^{cc}X,{}^{cc}(FY)+\gamma(L_YF)]-[{}^{cc}X,{}^{cc}Y]\end{aligned}$$

From Theorem 1.1.1, Theorem 1.1.2, Theorem 1.2.2 and Definition 2.1.1, we have

$$\begin{aligned}&={}^{cc}[(FX),(FY)]+\gamma L_{FX}(L_YF)-\gamma L_{FY}(L_XF)\\&+\gamma\{(L_XF)(L_YF)-(L_YF)(L_XF)\}\\&-{}^{cc}(F[(FX),Y])-\gamma L_{[(FX),Y]}F+{}^{cc}F\gamma(L_Y(L_XF))\\&-{}^{cc}(F[X,FY])-\gamma L_{[X,FY]}F-{}^{cc}F\gamma(L_X(L_YF))-{}^{cc}[X,Y].\end{aligned}$$

If we put the relation of $L_X(L_YF)-L_Y(L_XF)=L_{[X,Y]}F$, then we have

$$\begin{aligned}N({}^{cc}X,{}^{ccY})&={}^{cc}([(FX),(FY)]-(F[(FX),Y])-(F[X,FY])-\\&[X,Y])+\gamma(L_{FX}(L_YF))-L_{FY}(L_XF)+(L_XF)(L_YF)-\\&(L_YF)(L_XF)-L_{[FX,Y]}F-L_{[X,FY]}F+L_{F[X,Y]}F.\end{aligned}$$

For $N_F=[(FX),(FY)]-F[(FX),Y]-F[X,FY]-[X,Y]$ and

$$P=\begin{pmatrix}L_{FX}(L_YF)-L_{FY}(L_XF)+(L_XF)(L_YF)-\\(L_YF)(L_XF)-L_{[FX,Y]}F-L_{[X,FY]}F+L_{F[X,Y]}F\end{pmatrix},$$

we get

$$N_{ccF}={}^{cc}N_F+\gamma P,$$

where P is the tensor field of type (1,1) on $T(M_n)$.

Since N_{ccF} is zero, we get ${}^{cc}N_F+\gamma P=0$. Because of $N_F = 0$, the Theorem 2.1.1 is proved.

Theorem 2.1.2. Let F be an almost complex structure on $T(M_n)$. Then the comple lift ${}^{cc}F$ of F on $t^*(M_n)$ is an almost complex structure in $t^*(M_n)$ if and only if F is integrable.

Proof. We can infer from the Theorem 2.1.1

Theorem 2.1.3. Let M_n be an n-dimensional diffrentiable manifold of class C^∞. We now define a tensor field J of type (1,1) on $t^*(M_n)$ by

$$J={}^{cc}F-\frac{1}{2}\gamma(NF),$$

where ${}^{cc}F\in\Im_1^1(t^*(M_n)),F\in\Im_1^1(T(M_n)),N\in\Im_2^1(T(M_n))$. Then J is an almost complex structure on $t^*(M_n)$, i.e. $J^2 = -I$.

Proof.

$$\begin{aligned}J^2&=\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)^2=({}^{cc}F)^2-\frac{1}{2}{}^{cc}F\gamma(NF)\\&-\frac{1}{2}\gamma(NF)^{cc}F+\frac{1}{4}\gamma(NF)\gamma(NF)\\&=-I-\gamma N_F-{}^{cc}F\gamma(NF)\\&=-I-\gamma N_F-\gamma(NF^2)\\&=-I-\gamma(N+NF^2)\\&=-I\end{aligned}$$

where F is an almost complex structure on $T(M_n)$, so $F^2= -I$ and $({}^{cc}F)^2=-I-\gamma N_F$.

Theorem 2.1.4. Let $N({}^{cc}X,{}^{cc}Y)$ be the Nijenhuis tensor of almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ on $t^*(M_n)$. The almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ is integrable if and only if the almost complex structure F on $T(M_n)$ is integrable.

Proof. Let F be integrable. Then $N_F = 0$ and so ${}^{cc}F-\frac{1}{2}\gamma(NF)={}^{cc}F$. From Theorem 2.1.2, ${}^{cc}F$ is also integrable.

Suppose conversely that ${}^{cc}F-\frac{1}{2}\gamma(NF)$ is integrable. Then the Nijenhuis tensor $N({}^{cc}X,{}^{cc}Y)$ of ${}^{cc}F-\frac{1}{2}\gamma(NF)$ is zero on $t^*(M_n)$.

$$N({}^{cc}X,{}^{cc}Y)=\Big[\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{cc}X,\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{cc}Y\Big]$$
$$-\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)\Big[\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{cc}X,{}^{cc}Y\Big]$$
$$-\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)\Big[{}^{cc}X,\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{cc}Y\Big]-[{}^{cc}X,{}^{cc}Y]$$

From Theorem 1.1.1, Theorem 1.1.2, Theorem 1.2.2, Theorem 2.1.3 and Definition 2.1.1, we get

$$={}^{cc}[(FX),(FY)]+\gamma L_{FX}\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)-$$
$$\gamma L_{FY}\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)$$
$$+\gamma\Big(\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)\Big)$$
$$-\gamma\Big(\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big)$$
$$-{}^{CC}(F[(FX),Y])-\gamma L_{[FX,Y]}F+\gamma\Big(\Big(L_Y(L_XF-\tfrac{1}{2}(NF)_X\Big)F\Big)+$$
$$\tfrac{1}{2}\gamma(NF)_{[FX,Y]}$$
$$-{}^{CC}(F[X,FY])-\gamma L_{[X,FY]}F-\gamma\Big(\Big(L_X(L_YF-\tfrac{1}{2}(NF)_Y\Big)F\Big)$$
$$+\tfrac{1}{2}\gamma(NF)_{[X,FY]}-{}^{CC}[X,Y]$$

where $L_X(L_YF)-L_Y(L_XF)=L_{[X,Y]}F$, then we have

$$N({}^{cc}X,{}^{cc}Y)={}^{cc}([(FX),(FY)])-(F[(FX),Y])-$$
$$(F[X,FY])-[X,Y])$$
$$+\gamma\Big(L_{FX}\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)-L_{FY}\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big)$$
$$+\Big(\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)\Big)$$
$$-\Big(\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big)$$
$$-L_{[FX,Y]}F+\Big(\Big(L_Y\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)F+\tfrac{1}{2}(NF)_{[FX,Y]}\Big)\Big)$$
$$-L_{[X,FY]}F+\Big(\Big(L_X\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)F+\tfrac{1}{2}(NF)_{[X,FY]}\Big)\Big).$$

For

$$N_F=[(FX),(FY)]-F[(FX),Y]-F[X,FY]-[X,Y]$$

and

$$P=\Big(L_{FX}\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)-L_{FY}\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big)$$
$$+\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)$$
$$-\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)$$
$$-L_{[FX,Y]}F+\Big(\Big(L_Y\Big(L_XF-\tfrac{1}{2}(NF)_X\Big)F\Big)+\tfrac{1}{2}(NF)_{[FX,Y]}\Big)$$
$$-L_{[X,FY]}F+\Big(\Big(L_X\Big(L_YF-\tfrac{1}{2}(NF)_Y\Big)F\Big)+\tfrac{1}{2}(NF)_{[X,FY]}\Big),$$

we get

$$N_{{}^{cc}F}={}^{cc}N_F+\gamma P,$$

where P is the tensor field of type (1,1) on $T(M_n)$.

Since $N_{{}^{cc}F}$ is zero, we get ${}^{cc}N_F+\gamma P=0$. Because of, the Theorem 2.1.4 is proved.

Theorem 2.1.5. Let $N({}^{vv}\omega,{}^{vv}\theta)$ be the Nijenhuis tensor of almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ on $t^*(M_n)$. Then the almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ is integrable, where $\omega,\theta\in\Im_1^0(T(M_n)),N_F$ the Nijenhuis tensor of $F\in\Im_1^1(T(M_n))$.

Proof.

$$N({}^{vv}\omega,{}^{vv}\theta)=\Big[\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{vv}\omega,\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{vv}\theta\Big]$$
$$-\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)\Big[\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{vv}\omega,{}^{vv}\theta\Big]$$
$$-\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)\Big[{}^{vv}\omega,\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)^{vv}\theta\Big]-[{}^{vv}\omega,{}^{vv}\theta]$$

From Theorem 1.1.1, Theorem 1.1.2, Theorem 1.2.1, Theorem 2.1.3 and Definition 2.1.1, we get

$$-\big({}^{cc}F-\tfrac{1}{2}\gamma(NF)\big)[{}^{vv}\omega,{}^{vv}(\theta\circ F)]-[{}^{vv}\omega,{}^{vv}\theta]$$
$$=0$$

where $\omega\circ F,\theta\circ F\in\Im_1^0(T(M_n)),[{}^{vv}\omega,{}^{vv}\theta]=0.$

Theorem 2.1.6. Let $N({}^{cc}X,{}^{vv}\omega)$ be the Nijenhuis tensor of almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ on $t^*(M_n)$. Then almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ is integrable if and only if $LF = 0$ and $L\omega = 0$.

Proof.

$$N({}^{cc}X,{}^{vv}\omega)=\left[\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X,\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega\right]$$
$$-\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\left[\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X,{}^{vv}\omega\right]$$
$$-\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\left[{}^{cc}X,\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega\right]-[{}^{cc}X,{}^{vv}\omega]$$

From Theorem 1.1.1, Theorem 1.1.2, Theorem 1.2.1, Theorem 2.1.3 and Definition 2.1.1, we get

$$={}^{vv}(\omega\circ(L_{FX}F))-(L_X(\omega\circ F)\circ F)-(L_X\omega)).$$

For $LF=0$ and $L\omega=0$, we have $N({}^{cc}X,{}^{vv}\omega)=0$. The Theorem 2.1.6 is proved.

2.2. Tachibana operators applied to ${}^{cc}X$ and ${}^{vv}\omega$ with respect to an almost complex structure ${}^{cc}F-\frac{1}{2}\gamma(NF)$ on $t^*(M_n)$

Definition 2.2.1. Let $\varphi\in\Im_1^1(M_n)$ and $\Im(M_n)=\sum_{r,s=0}^{\infty}\Im_s^r(M_n)$ be a tensor alebra over R. A map $\phi_\varphi|_{r+s>0}:\overset{*}{\Im}(M_n)\to\Im(M_n)$ is called a Tachibana operator or ϕ_φ operator on M_n if

a) ϕ_φ is linear with respect to constant coefficient,

b) $\phi_\varphi:\overset{*}{\Im}(M_n)\to\Im_{s+1}^r(M_n)$ for all r and s,

c) $\phi_\varphi\left(K\overset{C}{\otimes}L\right)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M_n)$

d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M_n)$, where L_Y is the Lie derivation with respect to Y.

(e) $(\phi_{\varphi X}\eta)Y=(d(l_Y\eta))(\varphi X)-(d(l_Y(\eta\circ\varphi)))X+\eta((L_Y\varphi)X)$
$=\phi X(l_Y\eta)-X(l_{\varphi Y}\eta)+\eta((L_Y\varphi)X)$

for all $\eta\in\Im_1^0(M_n)$ and $X,Y\in\Im_0^1(M_n)$, where $l_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\Im_s^r(M_n)$ the module of all pure tensor fields of type (r,s) on M_n with respect to the affinor field [6].

Theorem 2.2.1. Let ${}^{cc}F-\frac{1}{2}\gamma(NF)$ be an almost complex structure on $t^*(M_n)$ and $X,Y\in\Im_0^1(T(M_n))$. Then we get the following results.

i) $\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X}{}^{cc}Y={}^{cc}(\phi_{FX}Y)+\gamma P$

where $P\in\Im_1^1T(M_n)$ and

$$P=L_Y\left(L_XF-\frac{1}{2}(NF)_X\right)+L_{[Y,X]}F-\frac{1}{2}(NF)_{[Y,X]}.$$

ii) $\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X}{}^{vv}\omega={}^{vv}(\phi_{FX}\omega)$

iii) $\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega}{}^{cc}X={}^{vv}(\phi_{F\omega}X)$

iv) $\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega}{}^{vv}\theta=0,$

where $\omega,\theta\in\Im_1^0(T(M_n)),X,Y\in\Im_0^1(T(M_n)),N_F$ the Nijenhuis tensor of $F\in\Im_1^1(T(M_n))$.

Proof.

i) $$\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X}{}^{cc}Y=-\left(L_{{}^{cc}Y}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\right){}^{cc}X$$
$$=-L_{{}^{cc}Y}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X+\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)L_{{}^{cc}Y}{}^{cc}X$$

From Theorem 1.1.1, Theorem 1.1.2, Theorem 1.2.1, Theorem 2.1.3 and Definition 2.2.1, we get

$$=-{}^{cc}((L_YF)X)+\gamma P$$
$$={}^{cc}(\phi_{FX}Y)+\gamma P$$

ii) $$\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X}{}^{vv}\omega=-\left(L_{{}^{vv}\omega}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\right){}^{cc}X$$
$$=-L_{{}^{vv}\omega}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{cc}X+\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)[{}^{vv}\omega,{}^{cc}X]$$
$$=-\left[{}^{vv}\omega,{}^{cc}(FX)+\gamma\left(L_XF-\frac{1}{2}(NF)_X\right)\right]$$
$$-{}^{cc}F[{}^{cc}X,{}^{vv}\omega]+\frac{1}{2}\gamma(NF))[{}^{cc}X,{}^{vv}\omega]$$
$$=-[{}^{vv}\omega,{}^{cc}(FX)]-{}^{vv}\omega\gamma\left(L_XF-\frac{1}{2}(NF)_X\right)$$
$$-{}^{cc}F^{vv}(L_X\omega)+\frac{1}{2}\gamma(NF)^{vv}(L_X\omega)$$
$$={}^{vv}(L_{FX}\omega)-{}^{cc}F^{vv}[X,\omega]$$
$$=-{}^{vv}[\omega,FX]+{}^{vv}(([\omega,X])F)$$
$$=-{}^{vv}((L_XF)X)-{}^{vv}((L_\omega X)F)+{}^{vv}((L_\omega X)F)$$
$$=-{}^{vv}((L_XF)X)$$
$$={}^{vv}(\phi_{FX}\omega)$$

iii) $$\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega}{}^{cc}X=-\left(L_{{}^{cc}X}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\right){}^{vv}\omega$$
$$=-\left[{}^{cc}X,\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega\right]+\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)[{}^{cc}X,{}^{vv}\omega]$$
$$=-[{}^{cc}X,{}^{vv}(\omega\circ F)]+{}^{cc}F[{}^{cc}X,{}^{vv}\omega]-\frac{1}{2}\gamma(NF)^{vv}[X,\omega]$$
$$=-{}^{vv}(L_X(\omega\circ F))+{}^{cc}F\,{}^{vv}[X,\omega]$$
$$=-{}^{vv}((L_X\omega)F)-{}^{vv}((L_XF)\omega)+{}^{vv}((L_X\omega)F)$$
$$=-{}^{vv}((L_XF)\omega)$$
$$={}^{vv}(\phi_{F\omega}X)$$

iv) $$\phi_{\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega}{}^{vv}\theta=-\left(L_{{}^{vv}\theta}\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)\right){}^{vv}\omega$$
$$=-\left[{}^{vv}\theta,\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right){}^{vv}\omega\right]$$
$$+\left({}^{cc}F-\frac{1}{2}\gamma(NF)\right)[{}^{vv}\theta,{}^{vv}\omega]$$
$$=-[{}^{vv}\theta,{}^{vv}(\omega\circ F)]$$
$$=0$$

The Theorem 2.2.1 is proved.

3. Discussion

By putting the tangent bundle instead of the fiber bundle it is introduced a new class of semi-cotangent bundle, which has explicit formulas for a projectable tensor fields. Firstly, we get the integrability conditions of the Nijenhuis Tensors $N({}^{cc}X, {}^{cc}Y), N({}^{cc}X, {}^{vv}\omega), N({}^{vv}\omega, {}^{vv}\theta)$ of almost complex structure ${}^{cc}F - \frac{1}{2}\gamma(NF)$. Later, it is demonstrated the results of Tachibana operators applied ${}^{cc}X$ and ${}^{vv}\omega$ according to structure ${}^{cc}F - \frac{1}{2}\gamma(NF)$ in semi cotangent bundle $t^*(M_n)$.

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