Karaelmas Fen ve Mühendislik Dergisi
Journal home page: http://fbd.beun.edu.tr

Research Article
Received / Geliş tarihi : 20.04.2016
Accepted / Kabul tarihi : 01.02.2017

On Some Congruences with the Terms of Second Order Sequence and Harmonic Numbers

İkinci Mertebeden Dizinin Terimlerini ve Harmonik Sayıları İçeren Bazı Kongrüanslar

Neşe Ömür, Sibel Koparal*

Kocaeli University, Faculty of Arts and Sciences, Department of Mathematics, Kocaeli, Turkey

Abstract

In this paper, we give the generalization of the congruences in (1.2) and (1.3). For example, for b∈Z\{0}, we have
$\sum_{k=0}^{p-1}\frac{U_{k+\varepsilon}(1,b^2)}{b^k}H_k \equiv 0 \,(mod\, p)$
where p is a prime such that $p \nmid b\Delta$, $\Delta = 1-4b^2$ and $\varepsilon = \left(1-\left(\frac{\Delta}{p}\right)\right)/2$

Keywords: Congruence, Harmonic numbers, The second order sequences

Öz

Bu makalede, (1.2) ve (1.3) deki kongrüansların genellemesi verildi. Örneğin b∈Z\{0} için $p \nmid b\Delta$ olacak şekilde p asal sayısı, $\Delta = 1-4b^2$ ve $\varepsilon = \left(1-\left(\frac{\Delta}{p}\right)\right)/2$ olmak üzere $\sum_{k=0}^{p-1}\frac{U_{k+\varepsilon}(1,b^2)}{b^k}H_k \equiv 0 \,(mod\, p)$

Anahtar Kelimeler: Kongrüans, Harmonik sayılar, İkinci mertebeden diziler

1. Introduction

The second order sequences $\{U_n(A,B)\}$ and $\{V_n(A,B)\}$ are defined for $n>0$ by

$$U_{n+1}(A,B) = AU_n(A,B) - BU_{n-1}(A,B)$$

and

$$V_{n+1}(A,B) = AV_n(A,B) - BV_{n-1}(A,B)$$

in which $U_0(A,B) = 0$, $U_1(A,B) = 1$ and $V_0(A,B) = 2$, $V_1(A,B) = A$, respectively, where A and B are arbitrary integers.

The Binet formulae of sequences $\{U_n(A,B)\}$ and $\{V_n(A,B)\}$ are

$$U_n(A,B) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(A,B) = \alpha^n + \beta^n,$$

respectively, where $\alpha, \beta = \left(A \pm \sqrt{A^2 - 4B}\right)/2$. If $A = 1$ and $B = -1$, then $U_n(1, -1) = F_n$ (nth Fibonacci number) and $V_n(1, -1) = L_n$ (nth Lucas number).

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, harmonic numbers are those rational numbers given by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n}\frac{1}{k}.$$

Wolstenholme proved that if $p > 3$ is a prime, then

$$H_{p-1} \equiv 0 (mod\, p^2) \tag{1.1}$$

(Wolstenholme 1862). For an odd prime p and an integer a, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Sun showed the congruences involving harmonic numbers and Lucas sequences (Sun 2012). For example, let $p > 3$ be a prime. For $A,B \in \mathbb{Z}$ with $p \nmid A$,

$$\sum_{k=1}^{p-1}\frac{V_k(A,B)}{kA^k}H_k \equiv 0 \,(mod\ p),$$

$$\sum_{k=1}^{p-1}\frac{U_k(A,B)}{kA^k}H_k \equiv \frac{2}{p}\sum_{k=1}^{p-1}\frac{U_k(A,B)}{kA^k} (mod\ p),$$

and for a prime $p > 5$, if $\left(\frac{p}{15}\right) = 1$,

$$\sum_{k=1}^{p-1}\frac{U_k(1,4)}{2^k}H_k \equiv 0 \,(mod\ p), \tag{1.2}$$

*Corresponding Author: sibel.koparal@kocaeli.edu.tr

Neşe Ömür orcid.org/0000-0002-3972-9910
Sibel Koparal orcid.org/0000-0001-9574-9652

if $\left(\frac{p}{15}\right)=-1$,

$$\sum_{k=1}^{p-1}\frac{U_{k+1}(1,4)}{2^k}H_k\equiv 0\,(mod\ p). \tag{1.3}$$

The author clearly gave that for any odd prime p and $k\in\{1,2,...,p-1\}$

$$(-1)^k\binom{p-1}{k}=\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\equiv 1-pH_k\,(mod\ p^2). \tag{1.4}$$

In this paper, we give the generalization of the congruences in (1.2) and (1.3). For example, for $b\in\mathbb{Z}\setminus\{0\}$,

$$\sum_{k=0}^{p-1}\frac{U_{k+\varepsilon}(1,b^2)}{b^k}H_k\equiv 0\,(mod\ p),$$

where p is a prime such that $p\nmid b\Delta, \Delta=1-4b^2$ and $\varepsilon=\left(1-\left(\frac{\Delta}{p}\right)\right)/2$.

2. Some Congruences Involving Harmonic Numbers

In this section, we will give the congruences involving harmonic numbers and the terms of the second order sequences $\{U_n(A,B)\}$ and $\{V_n(A,B)\}$. For this, we remember the following Lemma given by (Sun 2003).

Lemma1. Let $A,B\in\mathbb{Z}$ and p be an odd prime with $\left(\frac{B}{p}\right)=1$. For $m\in\mathbb{Z}$ with $m^2\equiv B(mod\ p)$,

$$U_{(p-1)/2}(A,B)\equiv\begin{cases}0\,(mod\ p), & if\left(\frac{A^2-4B}{p}\right)=1,\\ \frac{1}{m}\left(\frac{A-2m}{p}\right)(mod\ p), & if\left(\frac{A^2-4B}{p}\right)=-1,\end{cases}$$

and

$$U_{(p+1)/2}(A,B)\equiv\begin{cases}\left(\frac{A-2m}{p}\right)(mod\ p) & if\left(\frac{A^2-4B}{p}\right)=1,\\ 0\,(mod\ p), & if\left(\frac{A^2-4B}{p}\right)=-1.\end{cases}$$

Firstly, we state the following theorem.

Theorem1. For $b\in\mathbb{Z}\setminus\{0\}$, then

$$U_p(1,b^2)-b^{p-1}\left(\frac{\Delta}{p}\right)\equiv\frac{1}{2}b^{\left(\frac{\Delta}{p}\right)-1}U_{p-\left(\frac{\Delta}{p}\right)}(1,b^2)(mod\ p^2), \tag{2.1}$$

where p is a prime such that $p\nmid b\Delta$ and $\Delta=1-4b^2$.

Proof.

It is known that

$$\delta=\frac{1+\sqrt{\Delta}}{2}\text{ and }\gamma=\frac{1-\sqrt{\Delta}}{2}$$

are the roots of the characteristic equation $x^2 - x + b^2 = 0$. Using Binet formula of the sequence $\{U_n(1,b^2)\}$ and with help of the congruence $(\delta^p-\gamma^p)\equiv(\delta-\gamma)^p(mod\ p)$, we have

$$\Delta U_p(1,b^2)=(\delta-\gamma)^2U_p(1,b^2)=(\delta-\gamma)(\delta^p-\gamma^p)$$
$$\equiv(\delta-\gamma)^{p+1}=\Delta^{(p+1)/2}(mod\ p).$$

Hence, for $p\nmid\Delta$, we write

$$U_p(1,b^2)\equiv\Delta^{(p-1)/2}\equiv\left(\frac{\Delta}{p}\right)(mod\ p).$$

Similarly, we get

$$V_p(1,b^2)=\delta^p+\gamma^p\equiv(\delta+\gamma)^p=1\,(mod\ p).$$

It is clearly given that for any prime number p,

$$U_p(1,b^2)+V_p(1,b^2)=2U_{p+1}(1,b^2). \tag{2.2}$$

For $\left(\frac{\Delta}{p}\right)=1$ and $p\nmid b$, using recurrence relation of the sequence $\{U_n(1,b^2)\}$ and (2.2), we have

$$b^2U_{p-1}(1,b^2)=U_p(1,b^2)-U_{p+1}(1,b^2)$$
$$=U_p(1,b^2)-\frac{1}{2}(U_p(1,b^2)+V_p(1,b^2))$$
$$=\frac{1}{2}(U_p(1,b^2)-V_p(1,b^2))\equiv\frac{\left(\frac{\Delta}{p}\right)-1}{2}=0\,(mod\ p)$$

and by the little Fermat Theorem, we get

$$V_{p-1}(1,b^2)=2U_p(1,b^2)-U_{p-1}(1,b^2)$$
$$\equiv 2\equiv 2b^{p-1}(mod\ p).$$

Since the congruence

$$(V_{p-1}(1,b^2)-2b^{p-1})(V_{p-1}(1,b^2)+2b^{p-1})$$
$$=(\delta^{p-1}+\gamma^{p-1})^2-4(\delta\gamma)^{p-1}$$
$$=(\delta^{p-1}-\gamma^{p-1})^2=\Delta U_{p-1}^2(1,b^2)\equiv 0\,(mod\ p^2),$$

we have $V_{p-1}(1,b^2)\equiv 2b^{p-1}(mod\,p^2)$. Thus

$$2U_p(1,b^2)=U_{p-1}(1,b^2)+V_{p-1}(1,b^2)$$
$$\equiv U_{p-1}(1,b^2)+2b^{p-1}(mod\ p^2). \tag{2.3}$$

For $\left(\frac{\Delta}{p}\right)=-1$, by (2.2), we have

$$2U_{p+1}(1,b^2)=U_p(1,b^2)+V_p(1,b^2)\equiv 0\,(mod\ p)$$

and with the help of recurrence relation of the sequence $\{U_n(1,b^2)\}$, the little Fermat Theorem and (2.2), we get

$$V_{p+1}(1,b^2)=2U_{p+2}(1,b^2)-U_{p+1}(1,b^2)$$
$$=-2b^2U_p(1,b^2)+U_{p+1}(1,b^2)$$
$$\equiv 2b^2\equiv 2b^{p+1}(mod\ p).$$

Considering the congruence

$$(V_{p+1}(1,b^2)-2b^{p+1})(V_{p+1}(1,b^2)+2b^{p+1})$$
$$=(\delta^{p+1}+\gamma^{p+1})^2-4(\delta\gamma)^{p+1}$$
$$=(\delta^{p+1}-\gamma^{p+1})^2=\Delta U_{p+1}^2(1,b^2)\equiv 0\,(mod\ p^2),$$

we have $V_{p+1}(1,b^2)\equiv 2b^{p+1}(mod\,p^2)$. Hence

$$\begin{aligned}2b^2U_p(1,b^2) &= 2(U_{p+1}(1,b^2)-U_{p+2}(1,b^2)) \\ &= 2U_{p+1}(1,b^2)-(U_{p+1}(1,b^2)+V_{p+1}(1,b^2)) \\ &= U_{p+1}(1,b^2)-V_{p+1}(1,b^2) \\ &\equiv U_{p+1}(1,b^2)-2b^{p+1}\ (mod\ p^2).\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), the proof is completed.

Secondly, we give theorem involving the generalization of the congruences in (1.2) and (1.3).

Theorem2. For $b \in \mathbb{Z}/\{0\}$,

$$\sum_{k=0}^{p-1} \frac{U_{k+\varepsilon}(1,b^2)}{b^k} H_k \equiv 0\ (mod\ p), \tag{2.5}$$

where p is a prime such that $p \nmid b\Delta$ and $\varepsilon = \left(1-\left(\frac{\Delta}{p}\right)\right)/2$.

Proof.

With the help of (1.4), we get

$$\sum_{k=0}^{p-1} \frac{U_{k+\varepsilon}(1,b^2)}{b^k} H_k \equiv \sum_{k=0}^{p-1} \frac{U_{k+\varepsilon}(1,b^2)}{b^k} \frac{\left(1-(-1)^k\binom{p-1}{k}\right)}{p} (mod\ p),$$

where $p \nmid b\Delta$.

For $\varepsilon = 0, 1$, it is enough to show Theorem 2 that

$$\sum_{k=0}^{p-1} b^{p-1-k} U_{k+\varepsilon}(1,b^2) \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} (-b)^{p-1-k} U_{k+\varepsilon}(1,b^2) (mod\ p^2).$$

Using Binet formula of the sequence $\{U_{k+\varepsilon}(1,b^2)\}$, we write

$$\sum_{k=0}^{p-1} b^{p-1-k} \frac{\delta^{k+\varepsilon}-\gamma^{k+\varepsilon}}{\delta-\gamma} \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} (-b)^{p-1-k} \frac{\delta^{k+\varepsilon}-\gamma^{k+\varepsilon}}{\delta-\gamma} (mod\ p^2).$$

By the sums

$$\sum_{k=0}^{p-1} x^k y^{p-1-k} = \frac{x^p-y^p}{x-y} \text{ and } \sum_{k=0}^{p-1} \binom{p-1}{k} x^k y^{p-1-k} = (x+y)^{p-1},$$

we write

$$\frac{1}{\delta-\gamma}\left(\delta^\varepsilon \frac{\delta^p-b^p}{\delta-b} - \gamma^\varepsilon \frac{\gamma^p-b^p}{\gamma-b}\right) \equiv \frac{\delta^\varepsilon(\delta-b)^{p-1}-\gamma^\varepsilon(\gamma-b)^{p-1}}{\delta-\gamma} (mod\ p^2). \tag{2.6}$$

It is known that

$$(\delta-b)(\gamma-b) = \delta\gamma - b(\delta+\gamma) + b^2 = 2b^2 - b$$

and

$$\begin{aligned}&\delta^\varepsilon(\gamma-b)(\delta^p-b^p) - \gamma^\varepsilon(\delta-b)(\gamma^p-b^p) \\ &= (\delta-\gamma)(b^{p+\varepsilon} - bU_{p+\varepsilon}(1,b^2) + b^2U_{p+\varepsilon-1}(1,b^2)).\end{aligned}$$

So the congruence in (2.6) can rewritten

$$\frac{b^{p+\varepsilon-1} - U_{p+\varepsilon}(1,b^2) + bU_{p+\varepsilon-1}(1,b^2)}{2b-1} \equiv \frac{\delta^\varepsilon(\delta-b)^{p-1}-\gamma^\varepsilon(\gamma-b)^{p-1}}{\delta-\gamma} (mod\ p^2).$$

By the equalities $(\delta-b)^2 = (1-2b)\delta$ and $(\gamma-b)^2 = (1-2b)\gamma$, we have

$$\begin{aligned}&\frac{\delta^\varepsilon(\delta-b)^{p-1}-\gamma^\varepsilon(\gamma-b)^{p-1}}{\delta-\gamma} = \\ &\frac{\delta^\varepsilon((1-2b)\delta)^{(p-1)/2}-\gamma^\varepsilon((1-2b)\gamma)^{(p-1)/2}}{\delta-\gamma} \\ &= \frac{(1-2b)^{(p-1)/2}(\delta^{(p-1)/2+\varepsilon}-\gamma^{(p-1)/2+\varepsilon})}{\delta-\gamma} \\ &= (1-2b)^{(p-1)/2} U_{\left(p-\left(\frac{\Delta}{p}\right)\right)/2}(1,b^2).\end{aligned}$$

Taking A = 1 and B = b^2 in Lemma 1, we write

$$U_{\left(p-\left(\frac{\Delta}{p}\right)\right)/2}(1,b^2) \equiv 0\ (mod\ p), \tag{2.7}$$

$$U_{\left(p+\left(\frac{\Delta}{p}\right)\right)/2}(1,b^2) \equiv \left(\frac{1-2b}{p}\right) b^{\left(\left(\frac{\Delta}{p}\right)-1\right)/2} (mod\ p). \tag{2.8}$$

For $\left(\frac{\Delta}{p}\right) = 1$, by (2.7) and (2.8), we have

$$\begin{aligned}&U_{(p-1)/2}(1,b^2) \equiv 0\ (mod\ p), \\ &V_{(p-1)/2}(1,b^2) = 2U_{(p+1)/2}(1,b^2) - U_{(p-1)/2}(1,b^2) \equiv \\ &2\left(\frac{1-2b}{p}\right)(mod\ p)\end{aligned}$$

and

$$\begin{aligned}U_{p-1}(1,b^2) &= U_{(p-1)/2}(1,b^2)V_{(p-1)/2}(1,b^2) \\ &\equiv 2\left(\frac{1-2b}{p}\right)U_{(p-1)/2}(1,b^2) \\ &\equiv 2(1-2b)^{(p-1)/2}U_{(p-1)/2}(1,b^2)(mod\ p^2).\end{aligned}$$

For $\left(\frac{\Delta}{p}\right) = -1$, by (2.7), we have

$$\begin{aligned}&U_{(p+1)/2}(1,b^2) \equiv 0(mod\ p), \\ &V_{(p+1)/2}(1,b^2) = 2U_{(p+3)/2}(1,b^2) - U_{(p+1)/2}(1,b^2) \\ &= U_{(p+1)/2}(1,b^2) - 2b^2U_{(p-1)/2}(1,b^2) \\ &\equiv -2b^2\frac{1}{b}\left(\frac{1-2b}{p}\right) = -2b\left(\frac{1-2b}{p}\right)(mod\ p)\end{aligned}$$

and

$$\begin{aligned}U_{p+1}(1,b^2) &= U_{(p+1)/2}(1,b^2)V_{(p+1)/2}(1,b^2) \\ &\equiv -2b\left(\frac{1-2b}{p}\right)U_{(p+1)/2}(1,b^2) \\ &\equiv -2b(1-2b)^{(p-1)/2}U_{(p=1)/2}(1,b^2)(mod\ p^2).\end{aligned}$$

Thus, the right-hand side of (2.6) is congruent to $U_{p-\left(\frac{\Delta}{p}\right)}(1,b^2)/(2(-b)^\varepsilon)(mod\ p^2)$.

Thus (2.6) is equivalent to the congruence

$$\frac{b^{p+\varepsilon-1} - U_{p+\varepsilon}(1,b^2) + bU_{p+\varepsilon-1}(1,b^2)}{2b-1} \equiv \frac{U_{p-\left(\frac{\Delta}{p}\right)}(1,b^2)}{2(-b)^\varepsilon}(mod\ p^2) \tag{2.9}$$

For $\left(\frac{\Delta}{p}\right)=1$, from (2.1), we have $\varepsilon=0$, and (2.9) reduces to the congruence

$$2(b^{p-1}-U_p(1,b^2)+bU_{p-1}(1,b^2))\equiv(2b-1)U_{p-1}(1,b^2)(mod\ p^2).$$

For $\left(\frac{\Delta}{p}\right)=-1$, from (2.1), we have $\varepsilon=1$, and (2.9) can be rewritten as

$$-2b(b^p-U_{p+1}(1,b^2)+bU_p(1,b^2))\equiv(2b-1)U_{p+1}(1,b^2)(mod\ p^2).$$

Thus, we have completed the proof of Theorem 2.

For example, if we take b = 2 in Theorem 2, we get the congruences in (1.2) and (1.3).

3. References

Sun, ZH. 2003. Values of Lucas sequences modulo primes, *Rocky Mount. J. Math.* 33: 1123-1145.

Sun, ZW. 2012. On harmonic numbers and Lucas sequences. *Pub. Math. Debrecen,* 80(2): 25-41.

Wolstenholme, J. 1862. On certain properties of prime numbers. *Quart. J. Math.,* 5: 35-39.