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Research Article

On New Inequalities of Hermite-Hadamard-Fejér Type for Harmonically s-Convex Functions via Fractional Integrals

Kesirli İntegraller Yolu ile Harmonik s-Konveks Fonksiyonlar için Hermite-Hadamard-Fejér Tipli Yeni Eşitsizlikler Üzerine

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Abstract

In this paper, some Hermite-Hadamard-Fejér type integral inequalities for harmonically s-convex functions in fractional integral forms are obtained.

Keywords: Harmonically s-convex functions, Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integrals

Öz

Bu çalışmada, kesirli integraller yolu ile harmonik s-konveks fonksiyonlar için bazı Hermite-Hadamard-Fejér tipli yeni eşitsizlikler elde edilmiştir.

Anahtar Kelimeler: Harmonik s-konveks fonksiyonlar, Hermite-Hadamard eşitsizliği, Hermite-Hadamard-Fejér eşitsizliği, Riemann-Liouville kesirli integraller

1. Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a<b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality (Hadamard 1893).

The most well-known inequalities related to the integral mean of a convex function f are the Hermite-Hadamard inequalities or their weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In (Fejér 1906), Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality :

Theorem 1. Let $f:[a,b] \to \mathbb{R}$ be a convex function. Then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+1(b)}{2}\int_a^b g(x)\,dx \tag{1.2}$$

holds, where $g:[a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve and extend the inequalities and see (Bombardelli and Varošanec 2009, İşcan 2013, İşcan 2014, Sarıkaya 2012, Tseng et al. 2011).

Definition 1. (Kilbas et al. 2006). Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{a-}f$ of order $\alpha>0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\ x>a$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$ and

$$J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x).$$

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Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see (Dahmani 2010, İşcan 2013-a., İşcan 2014-a, Sarıkaya et al. 2013, Wang et al. 2012, Wang et al. 2013).

Definition 2. (İşcan 2015). Let $I \subset (0,\infty)$ be a real interval. A function $f{:}I \to \mathbb{R}$ is said to be harmonically s-convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y) + (1-t)^s f(x)$$

for all $x, y \in I$, $t \in [0,1]$, and for some fixed $s \in (0,1]$.

In (İşcan 2014-b), İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

Definition 3. Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $f{:}I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 2. (İşcan 2014-b). Let $f{:}I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a<b$. If $f \in L[a,b]$ then the following inequalities holds:

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

In (Latif et al. 2015) Latif et al. gave the following definition:

Definition 4. A function $g{:}[a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

In (Chen and Wu 2014) Chan and Wu presented Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows:

Theorem 3. Let $f{:}I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a<b$. If $f \in L[a,b]$ and $g{:}[a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \le \int_a^b \frac{f(x)g(x)}{x^2}dx \le \\ &\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \end{aligned} \tag{1.5}$$

In (Kunt et al. 2016) Kunt et al. presented, respectively, Hermite-Hadamard inequality in fractional integral forms for harmonically convex functions, Hermite-Hadamard-Fejér inequality in fractional integral forms for harmonically convex functions as follows:

Theorem 4. Let $f{:}I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a<b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals holds:

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha} \\ &\left\{J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/b)\right\} \le \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.6}$$

with $\alpha>0$ and $h(x)=1/x$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Theorem 5. Let $f{:}[a,b] \to \mathbb{R}$ be harmonically convex function with $a<b$ and $f \in L[a, b]$. If $g{:}[a,b] \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequalities for fractional integrals holds:

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right] \\ &\le \left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-}(fg\circ h)(1/b)\right] \\ &\le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right] \end{aligned} \tag{1.7}$$

with $\alpha>0$ and $h(x)=1/x$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Lemma 1. (Kunt et al. 2016). Let $f{:}I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ and $a<b$. If $g{:}[a,b] \to \mathbb{R}$ is integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following equality for fractional integrals holds:

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right] \\ &-\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-}(fg\circ h)(1/b)\right] \\ &= \frac{1}{\Gamma(\alpha)}\left[\begin{aligned} &\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{\frac{1}{b}}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right)(f\circ h)'(d)dt \\ &-\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds\right)(f\circ h)'(d)dt \end{aligned}\right] \end{aligned} \tag{1.8}$$

with α>0 and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

In this paper, we obtain some new inequalities connected with the left-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically *s*-convex function in fractional integrals.

2. Results

Throughout this section, we take $\| g \|_{\infty}=\sup_{t\in[a,b]}|g(t)|$, for the continuous function $g:[a,b]\rightarrow\mathbb{R}$.

Theorem 6. Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable function on I^{o} such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|$ is harmonically *s*-convex on $[a,b]$, g: $[a,b]\rightarrow\mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for fractional integrals holds:

$$\begin{vmatrix} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/b)\right]\end{vmatrix} \tag{2.1}$$
$$\le\frac{\| g \|\infty^{ab(b-a)}}{\Gamma(\alpha+1}\left(\frac{b-a}{ab}\right)^{\alpha}[C_1(\alpha)|f'(a)|+C_2(\alpha)|f'(b)|]$$

where

$$C_1(\alpha)=\int_0^{\frac{1}{2}}\frac{u^{\alpha+s}}{(ub+(1-u)a)^2}du+\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}u^{s}}{(ub+(1-u)a)^2}du \tag{2.2}$$

$$C_2(\alpha)=\int_0^{\frac{1}{2}}\frac{u^{\alpha}(1-u)^{s}}{(ub+(1-u)a)^2}du+\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha+s}}{(ub+(1-u)a)^2}du \tag{2.3}$$

with 0 < α ≤ 1 and $h(x) = 1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. From Lemma 1 we have

$$\begin{vmatrix} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(fg\circ h)(1/b)\right]\end{vmatrix}$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{\frac{1}{b}}^{t}(s-\frac{1}{b})^{\alpha-1}|(g\circ h)(s)|ds\right)|(f\circ h)'(t)|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}(\frac{1}{a}-s)^{\alpha-1}|(g\circ h)(s)|ds\right)|(f\circ h)'(t)|dt\end{array}\right]$$
$$\le\frac{\| g \|\infty}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{\frac{1}{b}}^{t}(s-\frac{1}{b})^{\alpha-1}ds\right)|(f\circ h)'(t)|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}(\frac{1}{a}-s)^{\alpha-1}ds\right)|(f\circ h)'(t)|dt\end{array}\right]$$
$$=\frac{\| g \|\infty}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{(t-\frac{1}{b})^{\alpha}}{\alpha}\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{(\frac{1}{a}-t)^{\alpha}}{\alpha}\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt\end{array}\right]$$

Setting $t=\frac{ub+(1-u)a}{ab}$, and $dt=\left(\frac{b-a}{ab}\right)du$ gives

$$\begin{vmatrix} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/b)\right]\end{vmatrix}$$
$$\le\frac{\| g \|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{l}\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du \\ +\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\end{array}\right] \tag{2.4}$$

Since $|f'|$ is harmonically *s*-convex on $[a,b]$, we have

$$\left|f'\left(\frac{ab}{ub+(1-u)at}\right)\right|\le u^{s}|f'(a)|+(1-u)^{s}|f'(b)| \tag{2.5}$$

If we use (2.5) in (2.4), we have

$$\begin{vmatrix} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/b)\right]\end{vmatrix}$$
$$\le\frac{\| g \|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{l}\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}[u^{s}|f'(a)|+(1-u)^{s}|f'(b)|]du \\ +\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}[u^{s}|f'(a)|+(1-u)^{s}|f'(b)|]du\end{array}\right] \tag{2.6}$$

If we use (2.2) and (2.3) in (2.6), we have (2.1). This completes the proof.

Corollary 1. In Theorem 6:

(1) If we take α =1 we have the following Hermite-Hadamard-Fejér inequality for harmonically *s*-convex functions which is related to the left-hand side of (1.5):

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|$$
$$\le\| g \|_{\infty}(b-a)^2[C_1(1)|f'(a)|+C_2(1)|f'(b)|],$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard inequality for harmonically *s*-convex functions in fractional integral forms which is related to the left-hand side of (1.6):

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{l}J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a) \\ +J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ h)(1/b)\end{array}\right\}\right|$$
$$\le\frac{ab(b-a)}{2^{1-\alpha}}[C_1(\alpha)|f'(a)|+C_2(\alpha)|f'(b)|],$$

(3) If we take α = 1 and $g(x) = 1$ we have the following Hermite-Hadamard inequality for harmonically *s*-convex functions which is related to the left-hand side of :(1.4)

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le ab(b-a)$$
$$[C_1(1)|f'(a)|+C_2(1)|f'(b)|]$$

Theorem 7. Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I^{o} such that $f' \in L[a,b]$, where $a,b \in I$ and $a<b$. If $|f'|^{q}, q \geq 1$, is harmonically *s*-convex on $[a,b]$, $g:[a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for fractional integrals holds:

$$\begin{vmatrix} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{vmatrix}$$
$$\leq \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\begin{bmatrix} C_3^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix} C_4(\alpha)\,|f'(a)|^{q} \\ +C_5(\alpha)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix} C_7(\alpha)\,|f'(a)|^{q} \\ +C_8(\alpha)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}}\end{bmatrix} \tag{2.7}$$

where

$$C_3(\alpha)=\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}du, \quad C_4(\alpha)=\int_0^{\frac{1}{2}}\frac{u^{\alpha+s}}{(ub+(1-u)a)^2}du \tag{2.8}$$

$$C_5(\alpha)=\int_0^{\frac{1}{2}}\frac{u^{\alpha}(1-u)^{s}}{(ub+(1-u)a)^2}du, \quad C_6(\alpha)=\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du \tag{2.9}$$

$$C_7(\alpha)=\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}u^{s}}{(ub+(1-u)a)^2}du, \quad C_8(\alpha)=\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha+s}}{(ub+(1-u)a)^2}du \tag{2.10}$$

with $\alpha>1$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Using (2.4), power mean inequality and the harmonically *s*-convexity of $|f'|^{q}$, it follows that

$$\begin{vmatrix} f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(fg\circ h)(1/b)\right]\end{vmatrix}$$
$$\leq \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\begin{bmatrix}\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du \\ +\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\end{bmatrix}$$
$$\leq \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\begin{bmatrix}\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^{q}du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^{q}du\right)^{\frac{1}{q}}\end{bmatrix}$$
$$\leq \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\begin{bmatrix}\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}\left[u^{s}|f'(a)|^{q}+(1-u)^{s}|f'(b)|^{q}\right]du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left[u^{s}|f'(a)|^{q}+(1-u)^{s}|f'(b)|^{q}\right]du\right)^{\frac{1}{q}}\end{bmatrix}$$
$$= \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\begin{bmatrix}\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\begin{pmatrix}\int_0^{\frac{1}{2}}\frac{u^{\alpha+s}}{(ub+(1-u)a)^2}du\,|f'(a)|^{q} \\ +\int_0^{\frac{1}{2}}\frac{u^{\alpha}(1-u)^{s}}{(ub+(1-u)a)^2}du\,|f'(b)|^{q}\end{pmatrix}^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\begin{pmatrix}\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha}u^{s}}{(ub+(1-u)a)^2}du\,|f'(a)|^{q} \\ +\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\alpha+s}}{(ub+(1-u)a)^2}du\,|f'(b)|^{q}\end{pmatrix}^{\frac{1}{q}}\end{bmatrix} \tag{2.11}$$

If we use (2.8), (2.9) and (2.10) in (2.11), we have (2.7). This completes the proof.

Corollary 2. In Theorem 7:

(1) If we take α =1 we have the following Hermite-Hadamard-Fejér inequality for harmonically *s*-convex functions which is related to the left-hand side of (1.5):

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx-\right|$$
$$\leq \|g\|_{\infty}(b-a)^2\begin{bmatrix}C_3^{1-\frac{1}{q}}(1)\left[\begin{pmatrix}C_4(1)\,|f'(a)|^{q} \\ +C_5(1)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(1)\left[\begin{pmatrix}C_7(1)\,|f'(a)|^{q} \\ +C_8(1)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}}\end{bmatrix}$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard inequality for harmonically *s*-convex functions in fractional integral forms which is related to the left-hand side of (1.6):

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\begin{Bmatrix}J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a) \\ +J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ h)(1/b)\end{Bmatrix}\right|$$
$$\leq \frac{ab(b-a)}{2^{1-\alpha}}\begin{bmatrix}C_3^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix}C_4(\alpha)\,|f'(a)|^{q} \\ +C_5(\alpha)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha)\left[\begin{pmatrix}C_7(\alpha)\,|f'(a)|^{q} \\ +C_8(\alpha)\,|f'(b)|^{q}\end{pmatrix}\right]^{\frac{1}{q}}\end{bmatrix}$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard inequality for harmonically s-convex functions which is related to the left-hand side of (1.4):

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le ab(b-a)\left[\begin{array}{l}C_3^{1-\frac{1}{q}}(1)\left[\begin{array}{l}C_4(1)|f'(a)|^q\\+C_5(1)|f'(b)|^q\end{array}\right]^{\frac{1}{q}}\\+C_6^{1-\frac{1}{q}}(1)\left[\begin{array}{l}C_7(1)|f'(a)|^q\\+C_8(1)|f'(b)|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

We can state another inequality for $q>1$ as follows:

Theorem 8. Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I^o such that $f'\in L[a,b]$, where $a, b\in I$ and $a<b$. If $|f'|^q, q>1$, is harmonically s-convex on $[a, b]$, $g:[a,b]\to\mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality for fractional integrals holds:

$$\left|\begin{array}{l}\frac{f(a)+f(b)}{2}\left[J^\alpha_{1/b+}(g\circ h)(1/a)+J^\alpha_{1/a-}(g\circ h)(1/b)\right]\\-\left[J^\alpha_{1/b+}(fg\circ h)(1/a)+J^\alpha_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right|$$
$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\left[\begin{array}{l}C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right]\tag{2.12}$$

where

$$C_9(\alpha)=\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}},C_{10}(\alpha)=\left(\int_{\frac{1}{2}}^1\frac{(1-u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\tag{2.13}$$

with $\alpha>1$, $h(x) = 1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$ and $1/p + 1/q = 1$.

Proof. Using (2.4), Hölder's inequality and the harmonically s-convexity of $|f'|^q$, it follows that

$$\left|\begin{array}{l}f\left(\frac{2ab}{a+b}\right)\left[J^\alpha_{\frac{a+b}{2ab}+}(g\circ h)(1/a)+J^\alpha_{\frac{a+b}{2ab}-}(g\circ h)(1/b)\right]\\-\left[J^\alpha_{\frac{a+b}{2ab}+}(fg\circ h)(1/a)+J^\alpha_{\frac{a+b}{2ab}-}(fg\circ h)(1/b)\right]\end{array}\right|$$
$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha\left[\begin{array}{l}\int_0^{\frac{1}{2}}\frac{u^\alpha}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\\+\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du\end{array}\right]$$
$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|^qdu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\frac{(1-u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|^qdu\right)^{\frac{1}{q}}\right]$$
$$\le\frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}u^s|f'(a)|^q+(1-u)^s|f'(b)|^qdu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\frac{(1-u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}u^s|f'(a)|^q+(1-u)^s|f'(b)|^qdu\right)^{\frac{1}{q}}\right]\tag{2.14}$$

Calculating following integrals, we have

$$\int_0^{\frac{1}{2}}u^s|f'(a)|^q+(1-u)^s|f'(b)|^qdu=\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\tag{2.15}$$

$$\int_{\frac{1}{2}}^1u^s|f'(a)|^q+(1-u)^s|f'(b)|^qdu=\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\tag{2.16}$$

If we use (2.13), (2.15) and (2.16) in (2.14), we have (2.12). This completes the proof.

Corollary 3 In Theorem 8:

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejér inequality for harmonically s-convex functions which is related to the left-hand side of (1.5):

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|$$
$$\le\|2\|_\infty(b-a)^2\left[\begin{array}{l}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(1)\left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right],$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard inequality for harmonically s-convex functions in fractional integral forms which is related to the left-hand side of (1.6):

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^\alpha\left\{\begin{array}{l}J^\alpha_{\frac{a+b}{2ab}+}(f\circ h)(1/a)\\+J^\alpha_{\frac{a+b}{2ab}-}(f\circ h)(1/b)\end{array}\right\}\right|$$
$$\le\frac{ab(b-a)}{2^{1-\alpha}}\left[\begin{array}{l}C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right],$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard inequality for s-harmonically convex functions which is related to the left-hand side of (1.4):

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq ab(b-a)\left[\begin{array}{l}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\\+C_{10}^{\frac{1}{p}}(1)\left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right].$$

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