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Research Article

Padovan-Type Sequences via The Hurwitz Matrices

Hurwitz Matrisleri Yardımıyla Elde Edilen Padovan-Tipli Diziler

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Abstract

In this paper, we define the recurrence sequences by using the Hurwitz matrices which are obtained from the characteristic polynomials of the Padovan, the Pell-Padovan and the Jacobsthal-Padovan sequences and then, we give miscellaneous properties of these sequences. Also, we study these sequences modulo m and we obtain the cyclic groups which are generated by the generating matrices when read modulo m. Then, we derive the relationships among the orders of the obtained cyclic groups and the periods of the defined sequences according to modulo m. Furthermore, we extend the Padovan-Hurwitz and the Pell-Padovan-Hurwitz sequences to groups. Finally, we obtain the lengths of the periods of the extended sequences in the semihedral SD_{2m} and the modular maximal-cyclic group $M\_m(2)$ for $m \geq 4$ as applications of the results obtained.

Keywords: Hurwitz matrix, Sequence, Group, Length

Öz

Bu makalede, Padovan, Pell-Padovan ve Jacobsthal-Padovan dizilerinin karakteristik polinomlarından elde edilen Hurwitz matrisleri kullanılarak indirgemeli dizler tanımlanmış ve bu dizilerin çeşitli özellikleri verilmiştir. Ayrıca, bu diziler m modülünde çalışılmış ve dizilerin üreteç matrisleri m modülüne indirgenerek devirli grupların üreteçleri olarak kabul edilip devirli gruplar elde edilmiştir. Bunun sonucu olarak, elde edilen devirli grupların mertebeleri ile tanımlanan dizilerin m modülüne göre periyotları arasında bağıntılar üretilmiştir. Buna ek olarak, Padovan-Hurwitz ve Pell-Padovan-Hurwitz dizileri gruplara genişletilmiştir. En sonunda, elde edilen sonuçların uygulaması olarak, $m \geq 4$ için SD_{2m} semidihedral grup ve $M\_m(2)$ modular maximal-cyclic grubun genişletilmiş dizilerinin periyotlarının uzunlukları elde edilmişitir.

Anahtar Kelimeler: Hurwitz matrisi, Dizi, Grup, Uzunluk

1. Introduction

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors (Coxeter and Greitzer 1967, Deveci and Aküzüm 2015, Falcon and Plaza 2009, Freyy Sellers 2000, Gogin and Myllari 2007, Kalman 1982, Kilic and Tasci 2006, Stakhov and Rozin 2006, Yilmaz and Bozkurt, 2009). In this paper, we develop properties of the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences which are obtained from the Hurwitz matrices of the characteristic polynomials of the Padovan, the Pell-Padovan and the Jacobsthal-Padovan sequences.

In (Deveci 2015, Deveci and Aküzüm 2015, Deveci and Aküzüm 2014, Deveci and Avci 2015, Deveci and Karaduman 2012, Lü and Wang 2007, Tas and Karaduman 2014), the authors obtained the cyclic groups via some special matrices. In this paper, we consider the multiplicative orders of the matrices M_1, M_2 and M_3 working modulo m and then, we obtain the cyclic groups. Also, we study the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences modulo m. Then we derive the relationships among the orders of the obtained cyclic groups and the periods of the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences according to modulo m.

The study of recurrence sequences in groups began with the earlier work of Wall (Wall, 1960) where the ordinary Fibonacci sequences in cyclic groups were investigated. The theory was expanded to some special linear recurrence

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sequences by several authors; see for example, (Aydin and Smith 1994, Campbell and Campbell 2009, Deveci 2015, Deveci and Aküzüm 2015, Deveci and Avci 2015, Doostie and Hashemi 2006, Dikici and Smith 1997, Knox 1992, Ozkan et.al 2003, Tas and Karaduman 2014). In this paper, we define the Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits of groups then we study these sequences in finite groups. Also, we obtain the lengths of the periods of the Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits of the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ for $m \geq 4$ as applications of the results.

2. Material and Methods

Let P be a nth degree real polynomial given by

$P(x)=a_0x^n + a_1x^{n-1} +\ldots+a_{n-1}x + a_n$.

In (Hurwitz 1895), the Hurwitz matrix $H_n=[h_{ij}]_{n\times n}$ associated to P was defined as follows:

$$H_n=\begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}.$$

The Padovan sequence $\{P(n)\}$ is defined by a third-order recurrence equation:

$P(n+3) = P(n+1) + P(n)$

for $n \geq 0$, where $P(0) = P(1) = P(2) = 1$.

For more information on this sequence, see (http://mathworld.wolfram.com/Padovan Sequence.html)

The Pell-Padovan sequence $\{P_p(n)\}$ is defined (Shannon et al. 2006a, Shannon et. Al 2006b) by a third-order recurrence equation:

$P_p(n+3) = 2P_p(n+1) + P_p(n)$

for $n \geq 0$, where $P_p(0) = P_p(1) = P_p(2) = 1$.

The Jacobsthal-Padovan sequence $\{J(n)\}$ is defined (Deveci 2015) by a third-order recurrence equation:

$J(n+2) = J(n) + 2J(n-1)$

for $n \geq 0$, where $J(-1) = 0$ and $J(0) = J(1) = 1$.

It is easy to see that the characteristic polynomials of the Padovan, the Pell-Padovan and the Jacobsthal-Padovan sequences are as follows, respectively:

$f_1(x) = x^3-x-1$,

$f_2(x) = x^3-2x-1$

and

$f_3(x) = x^3-x-2$.

For a given matrix $A = [a_{ij}]$ of integers, $A \pmod m$ means that the entries of A are reduced modulo m. Let $\langle A \rangle_m = \{(A)^n \pmod m) \mid n \geq 0\}$. If $(\det A, m) = 1$, $\langle A \rangle_m$ is a cyclic group. We denote cardinal of the set $\langle A \rangle_m$ by $|\langle A \rangle_m|$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the *period* of the sequence. For example, the sequence *a,b,c,d,b,c,d,b,c,d,...* is periodic after the initial element *a* and has period 3. *A sequence is simply periodic with period k* if the first *k* elements in the sequence form a repeating subsequence. For example, the sequence *a,b,c,d,a,b,c,d,a,b,c,d,...* is simply periodic with period 4.

The groups SD_{2^m} and $M\_m(2)$ are defined as follows:

$$SD_{2^m} = \langle x,y \mid x^{2^{m-1}} = y^2 = e, yxy = x^{2^{m-2}-1} \rangle$$

and

$$M\_m(2) = \langle x,y \mid x^{2^{m-1}} = y^2 = e, yxy = x^{2^{m-2}+1} \rangle$$

where m is an integer such that $m \geq 4$.

For more information on these groups see (Dummit and Foote 2004, Gorenstein 1980, Huppert 1967).

3. Results

3.1. Padovan-Hurwitz, Pell-Padovan-Hurwitz and Jacobsthal-Padovan-Hurwitz Numbers

We can write the following Hurwitz matrices for the polynomials f_1, f_2 and f_3, respectively:

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$P_p = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$J=\begin{bmatrix}0 & -2 & 0\\ 1 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

Let the notation A^T denote the transpose of the matrix A. Now we define the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences by using the matrices P^T, P_p^T and J^T as follows, respectively:

$$x_n^1=-x_{n-3}^1-x_{n-4}^1, \tag{1}$$

$$x_n^2=-2x_{n-3}^2-x_{n-4}^2 \tag{2}$$

and

$$x_n^3=-x_{n-3}^3-2x_{n-4}^3 \tag{3}$$

for $n > 4$, where $x_1^k=x_2^k=x_3^k=0$ and $x_4^k=1$ for $k = 1,2,3$.

Note that the generating functions of the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences are as follows, respectively:

$$f^{(1)}(x)=\frac{x^3}{x^4+x^3+1},$$

$$f^{(2)}(x)=\frac{x^3}{x^4+2x^3+1}$$

and

$$f^{(3)}(x)=\frac{x^3}{2x^4+x^3+1}.$$

By (1), (2) and (3), we can write the following companion matrices, respectively:

$$M_1=\begin{bmatrix}0 & 0 & -1 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

$$M_2=\begin{bmatrix}0 & 0 & -2 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

and

$$M_3=\begin{bmatrix}0 & 0 & -1 & -2\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

The matrices M_1, M_2 and M_3 are said to be the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz matrices, respectively. Note that det M_1 = det M_2 = 1 and det M_3 = 2.

By an inductive argument, we may write

$$(M_i)^n=\begin{bmatrix}x_{n+4}^i & x_{n+5}^i & x_{n+6}^i & -x_{n+3}^i\\ x_{n+3}^i & x_{n+4}^i & x_{n+5}^i & -x_{n+2}^i\\ x_{n+2}^i & x_{n+3}^i & x_{n+4}^i & -x_{n+1}^i\\ x_{n+1}^i & x_{n+2}^i & x_{n+3}^i & -x_n^i\end{bmatrix}, \tag{4}$$

for $i = 1,2$ and

$$(M_3)^n=\begin{bmatrix}x_{n+4}^3 & x_{n+5}^3 & x_{n+6}^3 & -2x_{n+3}^3\\ x_{n+3}^3 & x_{n+4}^3 & x_{n+5}^3 & -2x_{n+2}^3\\ x_{n+2}^3 & x_{n+3}^3 & x_{n+4}^3 & -2x_{n+1}^3\\ x_{n+1}^3 & x_{n+2}^3 & x_{n+3}^3 & -2x_n^3\end{bmatrix}. \tag{5}$$

It is well-known that the Simpson formula for a recurrence sequence can be obtained from the determinant of its generating matrix, so that we can write the Simpson formulas for the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences

as:

$$\begin{aligned}&x_n^i\begin{bmatrix}x_{n+2}^i x_{n+4}^i x_{n+6}^i-x_{n+2}^i(x_{n+5}^i)^2+2x_{n+3}^i x_{n+4}^i x_{n+5}^i-\\ (x_{n+3}^i)^2 x_{n+6}^i-(x_{n+4}^i)^3\end{bmatrix}+\\ &x_{n+1}^i\begin{bmatrix}-x_{n+1}^i x_{n+4}^i x_{n+6}^i+x_{n+1}^i(x_{n+5}^i)^2-2x_{n+2}^i x_{n+4}^i x_{n+5}^i+\\ 2x_{n+2}^i x_{n+3}^i x_{n+6}^i+2x_{n+3}^i(x_{n+4}^i)^2-2(x_{n+3}^i)^2 x_{n+5}^i\end{bmatrix}+\\ &x_{n+2}^i\begin{bmatrix}2x_{n+2}^i x_{n+3}^i x_{n+5}^i+x_{n+2}^i(x_{n+4}^i)^2-(x_{n+2}^i)^2 x_{n+6}^i-\\ 3(x_{n+3}^i)^2 x_{n+4}^i\end{bmatrix}+\\ &(x_{n+3}^i)^4=1\end{aligned}$$

for $i = 1,2$ and

$$\begin{aligned}&x_n^3\begin{bmatrix}x_{n+2}^3 x_{n+4}^3 x_{n+6}^3-x_{n+2}^3(x_{n+5}^3)^2+2x_{n+3}^3 x_{n+4}^3 x_{n+5}^3-\\ (x_{n+3}^3)^2 x_{n+6}^3-(x_{n+4}^3)^3\end{bmatrix}+\\ &x_{n+1}^3\begin{bmatrix}-x_{n+1}^3 x_{n+4}^3 x_{n+6}^3+x_{n+1}^3(x_{n+5}^3)^2-2x_{n+2}^3 x_{n+4}^3 x_{n+5}^3+\\ 2x_{n+2}^3 x_{n+3}^3 x_{n+6}^3+2x_{n+3}^3(x_{n+4}^3)^2-2(x_{n+3}^3)^2 x_{n+5}^3\end{bmatrix}+\\ &x_{n+2}^3\begin{bmatrix}2x_{n+2}^3 x_{n+3}^3 x_{n+5}^3+x_{n+2}^3(x_{n+4}^3)^2-(x_{n+2}^3)^2 x_{n+6}^3-\\ 3(x_{n+3}^3)^2 x_{n+4}^3\end{bmatrix}+\\ &(x_{n+3}^3)^4=2^{n-1}\end{aligned}$$

For more information on the Simpson formula of a recurrence sequence, see (Coxeter and Greitzer 1967).

Let $\{\alpha_1^{(1)},\alpha_2^{(1)},\alpha_3^{(1)},\alpha_4^{(1)}\}$, $\{\alpha_1^{(2)},\alpha_2^{(2)},\alpha_3^{(2)},\alpha_4^{(2)}\}$ and $\{\alpha_1^{(3)},\alpha_2^{(3)},\alpha_3^{(3)},\alpha_4^{(3)}\}$ be the sets of the eingevalues of the matrices M_1, M_2 and M_3, respectively and let $V^{(k)}$ be a 4x4

Vandermonde matrix as follows:

$$V^{(k)}=\begin{bmatrix}(\alpha_1^{(k)})^3 & (\alpha_2^{(k)})^3 & (\alpha_3^{(k)})^3 & (\alpha_4^{(k)})^3\\ (\alpha_1^{(k)})^2 & (\alpha_2^{(k)})^2 & (\alpha_3^{(k)})^2 & (\alpha_4^{(k)})^2\\ (\alpha_1^{(k)}) & (\alpha_2^{(k)}) & (\alpha_3^{(k)}) & (\alpha_4^{(k)})\\ 1 & 1 & 1 & 1\end{bmatrix}$$

where k = 1,2,3. Suppose now that

$$W_k^i=\begin{bmatrix}(\alpha_1^{(k)})^{u+4-i}\\ (\alpha_2^{(k)})^{u+4-i}\\ (\alpha_3^{(k)})^{u+4-i}\\ (\alpha_4^{(k)})^{u+4-i}\end{bmatrix}$$

and $V_j^{(k,i)}$ is a 4x4 matrix obtained from $V^{(k)}$ by replacing the jth column of $V^{(k)}$ by W_k^i.

We can now the Binet-type formulas for the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences with the following Theorem.

Theorem 3.1.1. Let $(M_k)^u=[m_{ij}^{(k,u)}]$ for k = 1,2,3. Then

$$m_{ij}^{(k,u)}=\frac{\det V_j^{(k,i)}}{\det V^{(k)}}\ .$$

Proof. Since the eigenvalues of the matrices M_1, M_2 and M_3 are distinct, the matrices M_1, M_2 and M_3 are diagonalizable. Let

$$D^{(k)}=diag(\alpha_1^{(k)},\alpha_2^{(k)},\alpha_3^{(k)},\alpha_4^{(k)}),$$

then it is readily seen that $M_kV^{(k)} = V^{(k)}D^{(k)}$ for k = 1,2,3. Since the matrix $V^{(k)}$ is invertible,

$(V^{(k)})^{-1}M_kV^{(k)} = D^{(k)}$.

Thus, the matrix M_k is similar to $D^{(k)}$ for k = 1,2,3. So we get for $n \geq 1$ that

$(M^{(k)})^uV^{(k)} = V^{(k)}(D^{(k)})^u$.

Then we can write the following linear system of equations for $n \geq 1$:

$$\begin{cases}m_{i1}^{(k,u)}(\alpha_1^{(k)})^3+m_{i2}^{(k,u)}(\alpha_1^{(k)})^2+m_{i3}^{(k,u)}(\alpha_1^{(k)})+m_{i4}^{(k,n)}=(\alpha_1^{(k)})^{u+4-i}\\ m_{i1}^{(k,u)}(\alpha_2^{(k)})^3+m_{i2}^{(k,u)}(\alpha_2^{(k)})^2+m_{i3}^{(k,u)}(\alpha_2^{(k)})+m_{i4}^{(k,n)}=(\alpha_2^{(k)})^{u+4-i}\\ m_{i1}^{(k,u)}(\alpha_3^{(k)})^3+m_{i2}^{(k,u)}(\alpha_3^{(k)})^2+m_{i3}^{(k,u)}(\alpha_3^{(k)})+m_{i4}^{(k,n)}=(\alpha_3^{(k)})^{u+4-i}\\ m_{i1}^{(k,u)}(\alpha_4^{(k)})^3+m_{i2}^{(k,u)}(\alpha_4^{(k)})^2+m_{i3}^{(k,u)}(\alpha_4^{(k)})+m_{i4}^{(k,n)}=(\alpha_4^{(k)})^{u+4-i}\end{cases}$$

from which we obtain

$$m_{ij}^{(k,u)}=\frac{\det V_j^{(k,i)}}{\det V^{(k)}}\ \text{for } k = 1,2,3 \text{ and } i,j = 1,2,3,4.$$

Theorem 3.1.1 gives immediately:

Corollary 3.1.1. Let x_u^k be the uth term of the sequence $\{x_n^k\}$ for k = 1,2,3. Then

$$x_u^k=\frac{\det V_4^{(k,4)}}{\det V^{(k)}}\ \text{for } k = 1,2$$

and

$$x_u^3=-\frac{\det V_4^{(3,4)}}{2\det V^{(3)}}\ .$$

3.2. The Cyclic Groups via The Matrices M_1, M_2 and M_3

Since det M_1 = detM_2 = 1, it is clear that the sets $\langle M_1\rangle_m$ and $\langle M_2\rangle_m$ are cyclic groups for every positive integer m. Moreover, the set $\langle M_3\rangle_m$ is a cyclic group for every positive odd integer m.

We next consider the orders of the cyclic groups which are generated by the matrices M_1, M_2 and M_3.

Theorem 3.2.1. Let M be any of the matrices M_1, M_2 and M_3. Suppose also that u is the largest positive integer and p is a prime such that (det M, p) = 1 and $|\langle M\rangle_p|=|\langle M\rangle_{p^u}|$. Then $|\langle M\rangle_{p^v}|=p^{v-u}.|\langle M\rangle_p|$ for every $v \geq u$.

Proof. Let us consider the cyclic group $\langle M^3\rangle_{p^v}$, then p ≠ 2. Suppose also that α is a positive integer and $|\langle M_3\rangle_p|$ is denoted by $k(p)$. If $(M_3)^{k(p^{a+1})} \equiv I(\text{mod } p^{a+1})$, then $(M_3)^{k(p^{a+1})} \equiv I(\text{mod } p^a)$, where I is the 4x4 identity matrix. Thus we get $k(p^a)$ that divides $k(p^{a+1})$. Also, writing $(M_3)^{k(p^a)} = I + (m_{ij}^{(a)}.p^a)$ we obtain

$$(M_3)^{k(p^a).p}=(I+(m_{ij}^{(a)}.p^a))^p=\sum_{i=0}^{p}\binom{p}{i}(m_{ij}^{(a)}.p^a)^i\equiv,$$
$$I(\text{mod } p^{a+1})$$

by the binomial expansion. This yields that $k(p^{a+1})$ divides $k(p^a).p$. Thus $k(p^{a+1})= k(p^a)$ or $k(p^{a+1})= k(p^a).p$. It is clear that $k(p^{a+1})= k(p^a).p$ holds if and only if there exists an integer $m_{ij}^{(a)}$ which is not divisible by p. Since u is the largest positive integer such that $k(p)= k(p^u)$, $k(p^u)\neq k(p^{u+1})$. There is an $m_{ij}^{(u+1)}$ which is not divisible by p. So we get that $k(p^{u+1})\neq k(p^{u+2})$. To complete the proof we may use an inductive method on u.

There are similar proofs for the cyclic groups $\langle M_1\rangle_{p^v}$ and $\langle M_2\rangle_{p^v}$.

Theorem 3.2.2. Let $\langle G\rangle_m$ be any of the cyclic groups $\langle M_1\rangle_m$, $\langle M_2\rangle_m$, and $\langle M_{3k}\rangle_m$. Suppose also that m has the prime factorization $m=\prod_{i=1} p_i^{e_i},(k\geq 1)$. Then $|\langle G\rangle_m|$ equals to the least common multiple of the $|\langle G\rangle_{p_i^{e_i}}|$'s.

Proof. Let us consider the cyclic group $\langle M_1\rangle_m$, then m is a positive integer. Let $|\langle M_1\rangle_{p_i^{e_i}}|=\lambda_i$ for 1 ≤ i ≤ k and let $|\langle M_1\rangle_m|=\lambda$. Then by (4), we have

$x^1_{\lambda_i+4} \equiv 1 \bmod p_i^{e_i}$,
$-x^1_{\lambda_i} \equiv 1 \bmod p_i^{e_i}$,
$x^1_{\lambda_i+\alpha} \equiv 0 \bmod p_i^{e_i}$ *for* $\alpha = 1,2,3,5,6$
and
$x^1_{\lambda+4} \equiv 1 \bmod m$,
$-x^1_{\lambda} \equiv 1 \bmod m$,
$x^1_{\lambda+\alpha} \equiv 0 \bmod m$ *for* $\alpha = 1,2,3,5,6$.

Which implies that $x^1_{\lambda+\beta} = a.x^1_{\lambda_i+\beta}$, $(a \in N)$, for $0 \leq \beta \leq 6$, that is, $(M_1)^{\lambda}$ is of the form $a.(M_1)^{\lambda_i}$ for all values of i. Thus it is verified that

$$|\langle M_1 \rangle_m| = \mathrm{lcm}[\langle M_1 \rangle p_1^{e_1}, \langle M_1 \rangle p_2^{e_2}, ..., \langle M_1 \rangle p_k^{e_k}].$$

There are similar proofs for the cyclic groups $\langle M_2 \rangle_m$ and $\langle M_3 \rangle_m$.

Reducing the Padovan-Hurwitz, the Pell-Padovan-Hurwitz and the Jacobsthal-Padovan-Hurwitz sequences by a modulus m, we can get a repeating sequences, denoted by

$$\{x^k_n(m)\} = \{x^k_1(m), x^k_2(m), x^k_3(m), ...x^k_i(m), ...\},$$

where k = 1,2,3 and $x^k_i(m) = x^k_i(\bmod m)$. They have the same recurrences relation as in (1), (2) and (3), respectively.

Theorem 3.2.3. The cases of the sequences $\{x^1_n(m)\}, \{x^2_n(m)\}$ *and* $\{x^3_n(m)\}$, are:

- The sequences $\{x^1_n(m)\}$ and $\{x^2_n(m)\}$ are simply periodic for every positive integer m.
- The sequence $\{x^3_n(m)\}$ is periodic for every positive integer m. In particular, if m is a positive odd integer, then the sequence $\{x^3_n(m)\}$ will be simply periodic.

Proof. Let us consider the sequence $\{x^2_n(m)\}$ as an example. Suppose also that $X = \{(x^2_1, x^2_2, x^2_3, x^2_4) \mid x^2_1, x^2_2, x^2_3, x^2_4$ are integers such that $0 \leqslant x^2_1, x^2_2, x^2_3, x^2_4 \leq m-1\}$, then we have $|X| = m^4$. Since there are m^4 distinct 4-tuples of elements of Z_m, at least one of the 4-tuples appears twice in the sequence $\{x^2_n(m)\}$. Thus, the subsequence following this 4-tuple repeats; that is the sequence is periodic. So if $x^2_{i+4}(m) \equiv x^2_{j+4}(m), ..., x^2_{i+1}(m) \equiv x^2_{j+1}(m)$ and $i>j$, then $i \equiv j \bmod 4$. From (2), we can easily derive that

$$x^2_{n-4} = -x^2_n - 2x^2_{n-3}.$$

Thus we obtain

$$x^2_i(m) \equiv x^2_j(m), x^2_{i-1}(m) \equiv x^2_{j-1}(m), ...x^2_{i-j+1}(m) \equiv x^2_1(m)$$

which implies that the $\{x^2_n(m)\}$ is a simply periodic sequence.

There are similar proofs for the sequences $\{x^2_n(m)\}$ and $\{x^3_n(m)\}$.

We next denote the periods of the sequences $\{x^1_n(m)\}, \{x^2_n(m)\}$ and $\{x^3_n(m)\}$ by $P_1(m)$, $P_2(m)$ and $P_3(m)$, respectively, and we present the relationships among the periods $P_1(m)$, $P_2(m)$ and $P_3(m)$ and the orders $|\langle M_1 \rangle_m|, |\langle M_2 \rangle_m|, |\langle M_3 \rangle_m|$, respectively, in the following result.

Corollary 3.2.1-(i). If p is a prime, then $P_k(p) = |\langle M_k \rangle_p|$ for k = 1,2.

(ii). If p is a prime such that $p \neq 2$, then $P_3(p) = |\langle M_3 \rangle_p|$.

Proof. This follows directly from (4) and (5).

3.3. The Padovan-Hurwitz and The Pell-Padovan-Hurwitz Sequences in Groups

Let G be a finite j-generator group and let

$$X = \{(x_1, x_2, ..., x_j) \in \underbrace{G \times G \times ... \times G}_{j} \mid < \{x_1, x_2, ..., x_j\} >= G\}.$$

We call $(x_1, x_2, ..., x_j)$ a generating j-tuple for .

Definition 3.3.1. Let $G = \langle X \rangle$ be a finitely generated group such that $X = \{x_1, x_2, ..., x_j\}$. Then we denote the Padovan-Hurwitz orbit by means of: $a_n = (a_{n-4})^{-1} (a_{n-3})^{-1}$

for $n \geq 5$, with initial conditions

$$\begin{cases} a_1 = (x_1)^{-1}, a_2 = x_2, a_3 = x_3, a_4 = x_4 & \text{if } j = 4, \\ a_1 = x_1, a_2 = (x_1)^{-1}, a_3 = x_2, a_4 = x_3 & \text{if } j = 3, \\ a_1 = (x_1)^{-1}, a_2 = x_1, a_3 = (x_1)^{-1}, a_4 = x_2 & \text{if } j = 2. \end{cases}$$

For a j-tuple $(x_1, x_2, ..., x_j) \in X$, the Padovan-Hurwitz orbit is denoted by $O^1_{(x_1, x_2, ..., x_j)}(G)$.

Definition 3.3.2. Let $G = \langle X \rangle$ be a finitely generated group such that $X = \{x_1, x_2, ..., x_j\}$. Then we denote the Pell-Padovan-Hurwitz orbit by means of:

$b_n = (b_{n-4})^{-1} (b_{n-3})^{-2}$

for $n \geq 5$, with initial conditions

$$\begin{cases} b_1 = (x_1)^{-1}, b_2 = x_2, b_3 = x_3, b_4 = x_4 & \text{if } j = 4, \\ b_1 = (x_1)^{2}, b_2 = (x_1)^{-1}, b_3 = x_2, b_4 = x_3 & \text{if } j = 3, \\ b_1 = (x_1)^{-4}, b_2 = (x_1)^{2}, b_3 = (x_1)^{-1}, b_4 = x_2 & \text{if } j = 2. \end{cases}$$

For a j-tuple $(x_1,x_2,...,xj) \in X$, the Pell-Padovan-Hurwitz orbit is denoted by $O^2_{(x_1,x_2,...,x_j)}(G)$.

Theorem 3.3.1. The Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits of a finite group are simply periodic.

Proof. Let us consider the Padovan- Hurwitz orbit $O^1_{(x_1,x_2,x_3,x_4)}(G)$ and let n be the order of G. Since there n^4 distinct 4-tuples of elements of G, at least one of the 4-tuples appears twice in the sequence $O^1_{(x_1,x_2,x_3,x_4)}(G)$. Thus, consider the subsequence following this 4-tuple. Because of the repetition, the sequence is periodic. Since the orbit $O^1_{(x_1,x_2,x_3,x_4)}(G)$ is periodic, there exist natural numbers u and v, with $u \geq v$, such that

$a_{u+1} = a_{v+1}, a_{u+2} = a_{v+2}, a_{u+3} = a_{v+3}$ and $a_{u+4} = a_{v+4}$.

By Definition 3.3.1, we know that

$$a_{n-4} = (a_{n-3})^{-1}(a_n)^{-1}.$$

Therefore, we obtain $a_u = a_v$, and hence,

$$a_{u-(v-4)} = a_{v-(v-4)} = a_4, a_{u-(v-3)} = a_{v-(v-3)} = a_3, a_{u-(v-2)} = a_{v-(v-2)} = a_2$$

and

$$a_{u-(v-1)} = a_{v-(v-1)} = a_1,$$

which implies that the sequence $O^1_{(x_1,x_2,x_3,x_4)}(G)$ is a simply periodic.

The proofs for other orbits are similar to the above are omitted.

We denote the lengths of the periods of the orbits $O^1_{(x_1,x_2,...,x_j)}(G)$ and $O^2_{(x_1,x_2,...,x_j)}(G)$ by $L^1_{(x_1,x_2,...,x_j)}(G)$ and $L^1_{(x_1,x_2,...,x_j)}(G)$, respectively. From the definitions of the Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits it is clear that the periods of these sequences in a finite group depend on the chosen generating set and the order in which the assignments of $x_1,x_2,...,x_j$ are made.

We will now address the lengths of the periods of the Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits of the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ which are metacyclic groups of order 2^m.

Theorem 3.3.2. For generating pairs (x,y) and (y,x), the lengths the periods of the Padovan-Hurwitz and the Pell-Padovan-Hurwitz orbits of the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ are as follows:

$$L^1_{(x,y)}(SD_{2^m}) = L^1_{(y,x)}(SD_{2^m}) = L^1_{(x,y)}(M\_m(2)) = L^1_{(y,x)}(M\_m(2)) = P_1(2^{m-1})$$

and

$$L^2_{(x,y)}(SD_{2^m}) = L^2_{(y,x)}(SD_{2^m}) = L^2_{(x,y)}(M\_m(2)) = L^2_{(y,x)}(M\_m(2)) = P_2(2^{m-1})$$

Proof. We first note that $P_1(2)=15$ and $P_2(2)=4$. Let us consider the Padovan-Hurwitz orbit $O^1_{(x,y)}(SD_{2^m})$. Then we have the sequence

$$x^{-1},x,x^{-1},y,e,e,xy,y,e,yx^{-1},x^{-2^{m-2}+1},y,xy,x^2y,yx,x^{-1},$$
$$x^{2^{m-2}-1},x^{-2^{m-2}+3},x^{2^{m-2}-2}y,x^{-2^{m-2}+2},x^{-2},yx^{2^{2m-4}-5.2^{m-2}+5},y,x^{2^{m-2}},$$
$$x^{-2^{2m-4}+5.2^{m-2}-3}y,x^{-2^{2m-4}+5.2^{m-2}-5},yx^{-2^{m-2}},yx^3,yx^{2^{2m-3}-10.2^{m-2}+8},$$
$$x^{2^{2m-4}-2^m+5}y,...$$

Using the above, the sequence becomes:

$$a_1 = x^{-1}, a_2 = x, a_3 = x^{-1}, a_4 = y, ...,$$
$$a_{31} = x^3, a_{32} = x^5, a_{33} = x^3, a_{34} = yx^8, ...,$$
$$a_{61} = x^7, a_{62} = x^9, a_{63} = x^7, a_{64} = yx^{16}, ...,$$
$$a_{30i+1} = x^{4i-1}, a_{30i+2} = x^{4i+1}, a_{30i+3} = x^{4i-1}, a_{30i+4} = yx^{8i}, ...,$$

where $i \geq 1$. So we need the smallest integer i such that $4i = 2^{m-1}. k$ for $k \in$ N. If we choose $i = 2^{m-3}$, then we obtain $L^1_{(x,y)}(SD_{2^m}) = P_1(2^{m-1})$.

Now we consider the Pell-Padovan-Hurwitz orbit $O^2_{(y,x)}(M\_m(2))$. Since the sequence $O^2_{(y,x)}(M\_m(2))$ is

$$e,e,y^{-1},x,e,e,yx^{-2},x^{-1},e,x^4,x^4y,x,x^{-8},x^{2^{2m-3}+2^{m-1}-2^m-12},y,x^{-6},x^{15},...,$$

we obtain

$$b_1 = e, b_2 = e, b_3 = y\text{-}1, b_4 = x, ...,$$
$$b_{16i+1} = x^{16i}, b_{16i+2} = x^{8i}, b_{16i+3} = x^{8i}y, b_{16i+4} = x, ...,$$

where $i \geq 1$. Thus we need the smallest integer i such that $8i = 2^{m-1}.k$ for $k \in N$. If we choose $i = 2^{m-4}$, then we obtain $L^2_{(y,x)}(M\_m(2)) = P_2(2^{m-1})$.

The proofs for other orbits are similar to the above and are omitted.

4. Discussion

Since $\langle M_k \rangle_{p^\alpha}$, ($k$ = 1,2,3) is an element of the general linear group $GL(4,Z_{p^\alpha})$ and $|GL(4,Z_{p^\alpha})| = \prod_{i=0}^{3}(p^4 - p^i)$ it is readily seen that $|\langle M_k \rangle_{p^\alpha}|$ is a factor $\prod_{i=0}^{3}(p^4 - p^i)$.

Conjecture 4.1. If $p \geq 5$ is a prime, then for k = 1,2,3, there exists an i with $0 \leq i \leq 3$ such that $|\langle M_k \rangle_{p^\alpha}|$ divides $(p^4\text{-}p^i)$.

Table 1-3 list some primes for which the conjecture is true.

Table 1. The order of $\langle M_1 \rangle_p$

p	$\lvert \langle M_1 \rangle_p \rvert$	result
5	124	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p$
23	528	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-1$
83	571786	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p$
173	29928	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p^2$
193	192	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p^3$
3719	3674095997	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p$
7213	52027368	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p^2$
9137	41742384	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-1$
11149	62150100	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p^2$
47147	370473268	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-1$
215459	476295677797320	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-1$
652541	12903325960	$\lvert\langle M_1 \rangle_p\rvert \,\vert\, p^4-p^2$

Table 2. The order of $\langle M_2 \rangle_p$

p	$\lvert \langle M_2 \rangle_p \rvert$	result
11	110	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p^3$
37	938	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p$
79	780	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-1$
103	102	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p^3$
631	265862	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p$
1009	1018080	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p^2$
5441	9868160	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p^2$
22229	1729371742	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p$
31147	970135608	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-1$
131149	382223560	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p^2$
331147	73105777838	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p$
611411	747648044666	$\lvert\langle M_2 \rangle_p\rvert \,\vert\, p^4-p$

Table 3. The order of $\langle M_3 \rangle_p$

p	$\lvert \langle M_3 \rangle_p \rvert$	result
7	1200	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-1$
17	4912	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p$
53	148876	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p$
449	448	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p^3$
2459	6046680	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p^2$
4111	285620833002720	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p^2$
10007	33380016	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-1$
20173	203474964	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-1$
31237	10159836009684	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p$
40037	1602961368	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p^2$
254389	14906692068977268	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p$
700279	2990187060	$\lvert\langle M_3 \rangle_p\rvert \,\vert\, p^4-p^2$

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