HSJG
Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 2, No. 2, 38-47, 2020
Research Article

Semi-Symmetry Properties of $\mathcal{S}$-Manifolds Admitting a Quarter-Symmetric Metric Connection

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Abstract
In this study $\mathcal{S}$-manifolds admitting a quarter-symmetric metric connection naturally related with the $\mathcal{S}$-structure are considered and some general results concerning the curvature of such a connection is given. In addition, we prove that an $\mathcal{S}$-manifold has constant f-sectional curvature with respect to this quarter-symmetric metric connection if and only if has the same constant f-sectional curvature with respect to the Riemannian connection. In particular, the conditions of semi-symmetry, Ricci semi-symmetry, and projective semi-symmetry of this quarter-symmetric metric connection are investigated.

Keywords and 2010 Mathematics Subject Classification
Keywords: $\mathcal{S}$-manifold — quarter-symmetric metric connection — semi-symmetry properties.
MSC: 53C05, 53C25

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Article History: Received 6 October 2020; Accepted 12 November 2020

1. Introduction

The idea of metric connection with torsion tensor in a Riemannian manifold was introduced by Hayden [8]. Later, Yano [17] studied some properties of semi symmetric metric connection on a Riemannian manifold. The semi-symmetric metric connection has important physical application such as the displacement on the earth surface following a fixed point is metric and semi-symmetric. Golab [5] defined semi-symmetric non-metric connections on a Riemannian manifold (M,g) and studied some of its properties. More precisely, if ∇ is a linear connection in a differentiable manifold M, the torsion tensor T of ∇ is given by $T(Z,W)=\nabla_Z W-\nabla_W Z-[Z,W]$, for any vector fields Z and W on M. The connection ∇ is said to be symmetric if the torsion tensor T vanishes, otherwise it is said to be non-symmetric. In this case, ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form $T(Z,W)=\eta(W)Z-\eta(Z)W$, for any Z,W, where η is a 1-form on M. Moreover, ∇ is called a metric connection if $\nabla g=0$, otherwise it is called non-metric. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. In [12] and [13] some kinds of quarter symmetric metric connection were studied. On the other hand, given a Riemannian manifold (M,g) of dimension $n\geq 3$ endowed with a linear connection ∇ whose curvature tensor field is denoted by R, for any $(0,k)$-tensor field $\tilde{W}$ on M, $k\geq 1$, the $(0,k+2)$-tensor field $R.\tilde{W}$ is defined by

$$(R.\tilde{W})(Z_1\ldots,Z_k,Z,Y)=-\sum_{i=1}^{k}\tilde{W}(Z_1,\ldots,Z_{i-1},R(Z,Y)Z_i,Z_{i+1},\ldots,Z_k), \tag{1}$$

for any $Z,Y,Z_1,\ldots,Z_k\in\mathscr{X}(M)$. In this context, M is called semi-symmetric respect to ∇ if $R.R=0$ and Ricci semi-symmetric if $R.S=0$, where S is denoting the Ricci tensor field of ∇. Moreover, M is said to be projectively semi-symmetric if $R.P=0$, being P the Weyl projective curvature tensor field of ∇, defined by

$$P(V,U)Z=R(V,U)Z-\frac{1}{n-1}\{S(U,Z)V-S(V,Z)U\} \tag{2}$$

(alternatively, $P(V,U,Z,W) = g(P(V,U)Z,W))$, for any U, V, $Z,W \in \mathscr{X}(M)$. For the Riemannian connection it is known that the semi-symmetry implies the Ricci semi-symmetry (for more details, [4, 14] and references therein can be consulted; specifically, for the contact geometry case we recommend the papers [9, 11, 15]).

In 1963, Yano [16] introduced the notion of f-structure on a C^∞ m-dimensional manifold M, as a non-vanishing tensor field φ of type $(1,1)$ on M which satisfies $\varphi^3+\varphi = 0$ and has constant rank r. It is known that r is even, say $r = 2n$. Moreover, TM splits into two complementary subbundles $\mathrm{Im}\varphi$ and $\ker\varphi$ and the restriction of φ to $\mathrm{Im}\varphi$ determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ [1], where $s = m-2n$. In 1970, Goldberg and Yano [6] introduced globally frame f-manifolds (also called metric f- manifolds and f.pk-manifolds). A wide class of globally frame f-manifolds was introduced in [1] by Blair according to the following definition: a metric f-structure is said to be a K-structure if the fundamental 2-form Φ, defined usually as $\Phi(X,Y) = g(X,\varphi Y)$, for any vector fields X and Y on M, is closed and the normality condition holds, that is, $[\varphi,\varphi]+2\sum_{i=1}^{s} d\eta^i\otimes\xi_i = 0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ. A K-manifold is called an S-manifold if $d\eta^k = \Phi$, for all $k = 1,\dots,s$. The S-manifolds have been studied by several authors (see, for instance, [2, 3, 7, 10]).

The purpose of this paper is to link the three notions commented above by investigating semi-symmetry properties of S-manifolds endowed with certain quarter-symmetric metric connection naturally related with the S-structure. To this end, in Section 2 we give a brief introduction about S-manifolds. Section 3 is devoted to obtaining results on the curvature properties of S-manifold with Riemannian connection. In Section 4 we define a quarter-symmetric metric connection on an S-manifold, obtaining some general results and, in Section 5, we investigate the curvature and the Ricci tensor fields of such connection. Specially, we prove that an S-manifold has constant f-sectional curvature with respect to this quarter-symmetric metric connection if and only if has the same constant f-sectional curvature with respect to the Riemannian connection. Consequently, the curvature of the quarter-symmetric metric connection is completely determined by its f-sectional curvature. Finally, in Section 6 we present the results concerning the semi-symmetry properties of the quarter-symmetric metric connection.

2. Preliminaries

A $(2n+s)-$ dimensional differentiable manifold M is called a *metric f-manifold* if there exist an $(1,1)$ type tensor field φ, s vector fields $\xi_1,\dots,\xi_s$, s 1-forms $\eta^1,\dots,\eta^s$ and a Riemannian metric g on M such that

$$\varphi^2 = -I+\sum_{i=1}^{s}\eta^i\otimes\xi_i,\ \eta^i(\xi_j) = \delta_{ij}, \tag{3}$$

$$g(\varphi U,\varphi V) = g(U,V)-\sum_{i=1}^{s}\eta^i(U)\eta^i(V) \tag{4}$$

for any $U,V\in\mathscr{X}(M)$, $i,j\in\{1,\dots,s\}$. In addition we have:

$$\varphi\xi_i = 0,\ \eta^i\circ\varphi = 0,\ \eta^i(U) = g(U,\xi_i). \tag{5}$$

Then, a 2-form Φ is defined by $\Phi(U,V) = g(U,\varphi V)$ for any $U,V\in\mathscr{X}(M)$ called the *fundamental 2-form*. In what follows, we denote by $\mathscr{M}$ the distribution spanned by the structure vector fields $\xi_1,\dots,\xi_s$ and by $\mathscr{L}$ its orthogonal complementary distribution. Then, $\mathscr{X}(M) = \mathscr{L}\oplus\mathscr{M}$. If $U\in\mathscr{M}$ we have $\varphi U = 0$ and if $U\in\mathscr{L}$ we have $\eta^i(U) = 0$, for any $i\in\{1,\dots,s\}$, that is, $\varphi^2 U = -U$.

Moreover, a metric f-manifold is *normal* if

$$[\varphi,\varphi]+2\sum_{i=1}^{s}d\eta^i\otimes\xi_i = 0$$

where $[\varphi,\varphi]$ is denoting the Nijenhuis tensor field associated to φ. A metric f-manifold is said to be an *S-manifold* if it is normal and

$$\eta^1\wedge\cdots\wedge\eta^s\wedge(d\eta^i)^n \neq 0 \text{ and } \Phi = d\eta^i,\ 1\le i\le s.$$

Examples of S-manifolds can be found in [1, 2, 7].

Theorem 1. *An $S-$manifold $(M,\varphi,\xi_i,\eta^i,g)$ satisfies the condition*

$$(\nabla^*_U\varphi)V = \sum_{i=1}^{s}\{g(\varphi U,\varphi V)\xi_i+\eta^i(V)\varphi^2 U\} \tag{6}$$

for all $U,V\in\mathscr{X}(M)$, where ∇^* denotes the Riemannian connection with respect to g [2].

From (6), we have

$$\nabla^*_U \xi_i = -\varphi U \tag{7}$$

for any $U \in \mathscr{X}(M)$, $i \in \{1,\dots,s\}$.

Theorem 2. *Let $(M,\varphi,\xi_i,\eta^i,g)$ be a $(2n+s)$-dimensional $S-$manifold. Then*

$$R^*(U,V)\xi_i = \sum_{j=1}^{s}\{\eta^j(U)\varphi^2 V - \eta^j(V)\varphi^2 U\}, \tag{8}$$

$$R^*(U,\xi_i)V = -\sum_{j=1}^{s}\{g(\varphi U,\varphi V)\xi_j + \eta^j(V)\varphi^2 U\} \tag{9}$$

for all $U,V \in \mathscr{X}(M)$, $i \in \{1,\dots,s\}$, where R^* denotes the curvature of the Riemannian connection [3].

Corollary 3. *Let $(M,\varphi,\xi_i,\eta^i,g)$ be a $(2n+s)$-dimensional S-manifold. Then*

$$R^*(\xi_i,U,\xi_j,V) = -g(\varphi U,\varphi V), \tag{10}$$

$$K^*(\xi_i,U) = g(\varphi U,\varphi U), \tag{11}$$

$$S^*(U,\xi_i) = 2n\sum_{i=1}^{s}\eta^i(U) \tag{12}$$

for all $U,V \in \mathscr{X}(M)$, $i,j \in \{1,\dots,s\}$, where K^* and S^* denote respectively the sectional curvature and the Ricci tensor field of the Riemannian connection [3].

Since, from (11), we have that $K^*(\xi_i,\xi_j) = 0$, for any $i,j \in \{1,\dots,s\}$, an S-manifold can not have constant sectional curvature. For this reason, it is necessary to introduce a more restrictive curvature. In general, a plane section π on a metric f-manifold $(M,\varphi,\xi_i,\eta^i,g)$ is said to be an *f-section* if it is determined by a unit vector U, normal to the structure vector fields and φU. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c and then, it is denoted by $M(c)$. In such case, the curvature tensor field R^* of $M(c)$ satisfies [10]:

$$\begin{aligned} R^*(U,V,K,L) = &\sum_{i,j=1}^{s}\{g(\varphi U,\varphi L)\eta^i(V)\eta^j(K) - g(\varphi U,\varphi K)\eta^i(V)\eta^j(L) \\ &+g(\varphi V,\varphi K)\eta^i(U)\eta^j(L) - g(\varphi V,\varphi L)\eta^i(U)\eta^j(K)\} \\ &+\frac{c+3s}{4}\{g(\varphi U,\varphi L)g(\varphi V,\varphi K) - g(\varphi U,\varphi K)g(\varphi V,\varphi L)\} \\ &+\frac{c-s}{4}\{\Phi(U,L)\Phi(V,K) - \Phi(U,K)\Phi(V,L) - 2\Phi(U,V)\Phi(K,L)\} \end{aligned} \tag{13}$$

for any $U,V,K,L \in \mathscr{X}(M)$.

3. Semi-Symmetry Properties of S-Manifolds Respect to the Riemannian Connection

With respect to the Riemannian connection ∇^* of an S-manifold $(M,\varphi,\xi_i,\eta^i,g)$, we can prove:

Theorem 4. *Any semi-symmetric S-manifold $(M,\varphi,\xi_i,\eta^i,g)$ is an S-space-form of constant f-sectional curvature equal to s.*

Proof. Let $U \in \mathscr{L}$ be a unit vector field. Since $(M,\varphi,\xi_i,\eta^i,g)$ is semi-symmetric, then,

$$(R^*.R^*)(U,\xi_i,U,\varphi U,\varphi U,\xi_j) = 0$$

for any $i,j \in \{1,\dots,s\}$. Expanding this formula from (1) and taking into account (9), we get $R^*(U,\varphi U,\varphi U,U) = s$, which completes the proof. ■

Observe that, in the case $s=1$, by using (10) we obtain that a semi-symmetric Sasakian manifold is of constant curvature equal to 1. This result was firstly proved by Takahashi (see [15]).

Theorem 5. *Let $(M,\varphi,\xi_i,\eta^i,g)$ be a Ricci semi-symmetric S-manifold. Then, its Ricci tensor field S^* respect the Riemannian connection satisfies*

$$S^*(U,V)=2n\{sg(\varphi U,\varphi V)+\sum_{i,j=1}^{s}\eta^i(U)\eta^j(V)\} \tag{14}$$

for any $U,V\in\mathscr{X}(M)$.

Proof. Since $(M,\varphi,\xi_i,\eta^i,g)$ is Ricci semi-symmetric, then, by using (1),

$$S^*(R^*(U,\xi_i)\xi_j,V)+S^*(\xi_j,R^*(U,\xi_i)V)=0$$

for any $U,V\in\mathscr{X}(M)$ and $i,j\in\{1,\dots,s\}$. Now, from (9) and (12) we get the desired result. ■

Corollary 6. *Any Ricci semi-symmetric Sasakian manifold is an Einstein manifold.*

Proof. Considering $s=1$ in (14), we deduce $S^*(U,V)=2ng(\varphi U,\varphi V)+\eta(U)\eta(V)=2ng(U,V)$ for any $U,V\in\mathscr{X}(M)$. ■

For the Weyl projective curvature tensor field, we have the following theorem:

Theorem 7. *Any projectively semi-symmetric S-manifold $(M,\varphi,\xi_i,\eta^i,g)$ is an S-space-form of constant f-sectional curvature equal to s.*

Proof. Let $U\in\mathscr{L}$ a unit vector field. Then, from (2) and taking into account (9) and (10), we have

$$(R^*.P^*)(U,\xi_i,U,\varphi U,\varphi U,\xi_j)=(R^*.R^*)(U,\xi_i,U,\varphi U,\varphi U,\xi_j)=s-R^*(U,\varphi U,\varphi U,U)$$

for any $i,j=1,\dots,s$ and this completes the proof. ■

4. A Quarter-Symmetric Metric Connection on $\mathcal{S}$-Manifolds

From now on, let M denote a $(2n+s)$-dimensional manifold $(M,\varphi,\xi_i,\eta^i,g)$. We define a new connection on M given by

$$\nabla_U V=\nabla^*_U V-\sum_{j=1}^{s}\eta^j(U)\varphi V \tag{15}$$

for any $U,V\in\mathscr{X}(M)$. It is easy to show that ∇ is a linear connection on M. Moreover, we can prove:

Theorem 8. *Let M be an S-manifold. The linear connection ∇ defined in (15) is a quarter-symmetric metric connection on M.*

Using (15) and taking into account that the Riemannian connection is free-torsion, the torsion tensor T of the connection ∇ is given by

$$T(U,V)=\sum_{j=1}^{s}\{\eta^j(V)\varphi U-\eta^j(U)\varphi V\} \tag{16}$$

for any $U,V\in\mathscr{X}(M)$. Moreover, by using (15) again, we have, for all $U,V,Z\in\mathscr{X}(M)$ and since ∇^* is a metric connection, that:

$$(\nabla_U g)(V,Z)=\sum_{j=1}^{s}\eta^j(U)\{g(\varphi V,Z)+g(V,\varphi Z)\}. \tag{17}$$

Proof. From (16) and (17) we conclude that the linear connection ∇ is a quarter-symmetric metric connection on M. ■

Example 9. *Let us consider $\mathbf{R}^{2n+s}$ with its standard S-structure given by [7]*

$$\eta^\alpha = \frac{1}{2}\left(dz^\alpha - \sum_{i=1}^{n} y^i dx^i\right),\ \xi_\alpha = 2\frac{\partial}{\partial z^\alpha},$$

$$g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\left(\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)\right),$$

$$\varphi\left(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + \sum_{\alpha} Z_\alpha \frac{\partial}{\partial z^\alpha}\right) = \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}) + \sum_{\alpha=1}^{s}\sum_{i=1}^{n} Y_i y^i \frac{\partial}{\partial z^\alpha}$$

where (x^i, y^i, z^α), $i = 1,\ldots,n$ and $\alpha = 1,\ldots,s$, are the cartesian coordinates. It is known that, with this structure, $\mathbf{R}^{2n+s}$ is an S-space-form of constant f-sectional curvature $c = -3s$. If, following [7], we denote

$$(x^1,\ldots,x^n,y^1,\ldots,y^n,z^1,\ldots,z^s) = (x^1,\ldots,x^{2n+s})$$

the Christoffel symbols of the quarter-symmetric metric connection defined in (15) are given by

$$\Gamma^b_{ai} = \Gamma^{*b}_{ai} - \frac{1}{2}sy_i\delta_{ab};\ \Gamma^b_{a\alpha} = \Gamma^{*b}_{a\alpha} + \frac{1}{2}\delta_{ab}$$

for any $a,b \in \{1,\ldots,2n+s\}$, $i \in \{1,\ldots,n\}$ and $\alpha \in \{1,\ldots,s\}$, where Γ^{*b}_{ai} and $\Gamma^{*b}_{a\alpha}$ are denoting the Christoffel symbols of the Riemannian connection of $\mathbf{R}^{2n+s}$ and the not-written symbols are the same as the Riemannian connection ones (see [7] for the details concerning them).

Corollary 10. *Let M be an S-manifold. Then we have*

$$\nabla_U \xi_i = -\varphi U \tag{18}$$

$$(\nabla_U \eta^i)W = g(U,\varphi W) = \Phi(U,W) \tag{19}$$

for any $U,W \in \mathscr{X}(M)$, $i \in \{1,\ldots,s\}$.

Proof. First, taking $W = \xi_i$ in (15), from (7) we have

$$\nabla_U \xi_i = \nabla^*_U \xi_i - \sum_{j=1}^{s} \eta^j(U)\varphi\xi_i = -\varphi U.$$

Now, by using (5), (7) and (15) again:

$$\begin{aligned}(\nabla_U \eta^i)(W) &= U\eta^i(W) - \eta^i(\nabla_U W)\\ &= g(\nabla^*_U W, \xi_i) + g(W, \nabla^*_U \xi_i) - \eta^i(\nabla_U W)\\ &= g(\varphi W, U).\end{aligned}$$

■

Theorem 11. *Let M be an $S-$manifold. Then, we have*

$$(\nabla_U \varphi)V = \sum_{i=1}^{s}\{g(\varphi U, \varphi V)\xi_i + \eta^i(V)\varphi^2 U\} \tag{20}$$

for all $U,V \in \mathscr{X}(M)$.

Proof. From (15), we get:

$$(\nabla_U \varphi)V = (\nabla^*_U \varphi)V - \sum_{i=1}^{s} \eta^i(V)\varphi U.$$

Therefore, we obtain the result from (6). ■

By using (3) and (20), we easily prove:

Corollary 12. *Let M be an S-manifold. Then we have*

$$(\nabla_U \varphi)\xi_i = -\varphi\nabla_U \xi_i = \varphi^2 U, \tag{21}$$

$$\nabla_{\xi_i}\varphi U = \varphi\nabla_{\xi_i} U \tag{22}$$

for all $U \in \mathscr{X}(M)$, $i \in \{1,\dots,s\}$.

5. The Curvature of ∇

Let M be an S-manifold endowed with the quarter-symmetric metric connection ∇ defined in (15). From the Formula (3.2) in [1], denoting by R and R^* the curvature tensor fields of ∇ and ∇^*, respectively, we have that

$$\begin{aligned} R(U,L)W &= R^*(U,L)W + \sum_{i=1}^{s}\eta^i(U)\{(\nabla_L\varphi)W\} - \sum_{i=1}^{s}\eta^i(L)\{(\nabla_U\varphi)W\} \\ &\quad +2sg(U,\varphi L)\varphi W \end{aligned} \tag{23}$$

for all $U,L,W \in \mathscr{X}(M)$. From (8), (9) and (23), we get:

Corollary 13. *Let M be an S-manifold. Then we have*

$$R(U,V)\xi_i = 2\sum_{j=1}^{s}\{\eta^j(U)\varphi^2 V - \eta^j(V)\varphi^2 U\} = 2R^*(U,V)\xi_i, \tag{24}$$

$$R(U,\xi_i)V = -2\sum_{j=1}^{s}\left\{g(\varphi U,\varphi V)\xi_j + \eta^j(V)\varphi^2 U\right\} = -2R^*(U,\xi_i)V, \tag{25}$$

$$R(U,\xi_j)\xi_i = R^*(U,\xi_j)\xi_i - \varphi^2 U = -2\varphi^2 U, \tag{26}$$

$$R(\xi_i,\xi_j)U = R^*(\xi_i,\xi_j)U = 0 \tag{27}$$

$$R(\xi_k,\xi_j)\xi_i = 0, \tag{28}$$

for all $U,V \in \mathscr{X}(M)$, $i,j,k \in \{1,\dots,s\}$.

Corollary 14. *Let M be an S-manifold. Then*

$$\begin{aligned} R(U,V,L,K) &= -R(V,U,L,K), \\ R(U,V,L,K) &= -R(U,V,K,L), \\ R(U,V,L,K) &= R(L,K,U,V) \end{aligned}$$

for any $U,V,K.L \in \mathscr{X}(M)$.

Corollary 15. *Let M be an S-manifold. Then*

$$R(\varphi U,\varphi V,\varphi L,\varphi K) = R^*(U,V,L,K) + 2sg(\varphi U,V)g(L,\varphi K) \tag{29}$$

for any $U,V,L,W \in \mathscr{L}$.

Proof. It is a direct computation from (23) taking into account that [2]

$$R^*(\varphi U,\varphi V,\varphi L,\varphi K)=R^*(U,V,L,K)$$

for any $U,V,L,K\in\mathscr{L}$. ■

To consider the sectional curvature of the quarter-symmetric metric connection ∇ has no sense because, from (24) we have that $R(\xi_i,U,U,\xi_i)=g(R(\xi_i,U)U,\xi_i)=2$, while from (26), $R(U,\xi_i,\xi_i,U)=g(R(U,\xi_i)\xi_i,U)=1$, for any unit vector field $U\in\mathscr{L}$ and any $i\in\{1,\dots,s\}$. However, the f-sectional curvature of ∇ is well defined, since, by using (23), we obtain that, for any unit vector field $U\in\mathscr{L}$:

$$R(U,\varphi U,\varphi U,U)=R^*(U,\varphi U,\varphi U,U)+2s[g(U,U)]^2.$$

Consequently, taking into account (13), from (23) we prove the following theorem.

Theorem 16. *Let M be an S-manifold. Then, the f-sectional curvature associated with the quarter-symmetric metric connection ∇ is constant if and only if the f-sectional curvature associated with the Riemannian connection is constant too. In this case, both constants are the same and the curvature of ∇ is given by*

$$R(U,V,Z,W)=\sum_{i,j=1}^{s}\{g(\varphi^2V,W)\eta^i(U)\eta^j(Z)-g(\varphi^2U,W)\eta^i(V)\eta^j(Z) \tag{30}$$

$$\begin{aligned}&+2g(\varphi V,\varphi Z)\eta^i(U)\eta^j(W)-2g(\varphi U,\varphi Z)\eta^i(V)\eta^j(W)\}\\&+\sum_{i,k=1}^{s}\{g(\varphi V,W)\eta^i(U)\eta^k(Z)-g(\varphi U,W)\eta^k(Z)\eta^i(V)\}\\&+2sg(U,\varphi V)g(\varphi Z,W)\\&+\frac{c+3s}{4}\{g(\varphi U,\varphi W)g(\varphi V,\varphi Z)-g(\varphi U,\varphi Z)g(\varphi V,\varphi W)\}\\&+\frac{c-s}{4}\{\Phi(U,W)\Phi(V,Z)-\Phi(U,Z)\Phi(V,W)-2\Phi(U,V)\Phi(Z,W)\}\end{aligned}$$

for any $U,V,Z,W\in\mathscr{X}(M)$.

With respect to the Ricci tensor field S of the connection ∇ we know that it is a symmetric tensor field. In fact, since $d\eta^i=\Phi$, for any $i\in\{1,\dots,s\}$, from Formulas (3.4) and (3.14) in [1] we deduce that

$$S(K,L)=S(L,K) \tag{31}$$

for any $K,L\in\mathscr{X}(M)$, where $\dim(M)=2n+s$. Moreover,

$$S(U,V)=S^*(U,V)+2s\sum_{k=1}^{2n}\{g(\varphi U,E_k)g(\varphi V,E_k)\}+sg(\varphi V,\varphi U) \tag{32}$$

for any $U,V\in\mathscr{X}(M)$. Therefore, by using (12):

Proposition 17. *Let M be an S-manifold. Then, we have*

$$S(U,\xi_i)=s\eta^i(U)+2n\sum_{i=1}^{s}\eta^i(U) \tag{33}$$

for any $U\in\mathscr{X}(M)$, $i\in\{1,\dots,s\}$.

Corollary 18. *Let M be an S-manifold. Then we have*

$$S(\xi_j,\xi_i)=s+2n \tag{34}$$

for any $i,j=\{1,\dots,s\}$.

Moreover, we can prove:

Proposition 19. *Let M be an S-manifold. Then*

$$S(\varphi U,\varphi V)=S(U,V)-4s\sum_{i=1}^{s}g(U,\varphi E_i)g(V,\varphi E_i) \tag{35}$$

$$+2s\sum_{i=1}^{s}g(E_i,U)g(V,E_i)-2n\sum_{i,j=1}^{s}\eta^i(U)\eta^j(V)$$

for any $U,V\in\mathscr{X}(M)$.

Proof. Let $\{E_1,\dots,E_n,\varphi E_1,\dots,\varphi E_n,\xi_1,\dots,\xi_s\}$ be an f-basis. Then, since from (25),

$$R(\xi_j,\varphi U,\varphi V,\xi_j)=R^*(\xi_j,\varphi U,\varphi V,\xi_j)+g(\varphi U,\varphi V)$$

for any $j\in\{1,\dots,s\}$, then, by using (29), (14) and (32) taking into account that $U,V\in\mathscr{L}$, we deduce:

$$\begin{aligned}
S(\varphi U,\varphi V) &= \sum_{i=1}^{n}\{R(E_i,\varphi U,\varphi V,E_i)+R(\varphi E_i,\varphi U,\varphi V,\varphi E_i)\}+\sum_{j=1}^{s}R(\xi_j,\varphi U,\varphi V,\xi_j)\\
&= \sum_{i=1}^{s}\{R^*(E_i,\varphi U,\varphi V,E_i)+R^*(\varphi E_i,\varphi U,\varphi V,\varphi E_i)\}+\sum_{j=1}^{s}R^*(\xi_j,\varphi U,\varphi V,\xi_j)\\
&\quad +2s\sum_{i=1}^{s}\{g(E_i,U)g(V,E_i)+g(\varphi E_i,U)g(V,\varphi E_i)\}+\sum_{j=1}^{s}g(\varphi U,\varphi V)\\
&= S^*(\varphi U,\varphi V)+2s\sum_{i=1}^{s}\{g(E_i,U)g(V,E_i)+g(\varphi E_i,U)g(V,\varphi E_i)\}+sg(\varphi U,\varphi V)\\
&= S(U,V)-4s\sum_{i=1}^{s}g(U,\varphi E_i)g(V,\varphi E_i)+2s\sum_{i=1}^{s}g(E_i,U)g(V,E_i)-2n\sum_{i,j=1}^{s}\eta^i(U)\eta^j(V).
\end{aligned}$$

But, from (25) again,

$$R(\xi_j,U,V,\xi_j)=2(\nabla_U\varphi)V$$

for any $j\in\{1,\dots,s\}$ and this completes the proof. ∎

Corollary 20. *Let M be an S-manifold. Then we have*

$$S(X,Y)=S(\varphi X,\varphi Y)+4n\sum_{i=1}^{s}\eta^i(X)\eta^i(Y) \tag{36}$$

for all $U,V\in\mathscr{X}(M)$.

Proof. We can put

$$U=U_0+\sum_{i=1}^{s}\eta^i(U)\xi_i \text{ and } V=V_0+\sum_{j=1}^{s}\eta^j(V)\xi_j$$

where $U_0,V_0\in\mathscr{L}$. Then, from (33) and (34):

$$S(U,V)=S(U_0,V_0)+4n\sum_{i,j=1}^{s}\eta^i(U)\eta^j(V). \tag{37}$$

Now, by using (35), $S(U_0,V_0)=S(\varphi U_0,\varphi V_0)=S(\varphi U,\varphi V)$ and the proof is completed. ∎

6. Semi-Symmetry Properties of an $\mathcal{S}$-Manifold with Respect to ∇

For the quarter-symmetric metric connection defined in (15) on an S-manifold M we can prove:

Theorem 21. *Let M be a $(2n+s)$-dimensional S-manifold, $n \geq 1$.If M is semi-symmetric respect to the quarter-symmetric metric connection ∇ then*

$$R(U,\varphi U,\varphi U,U) = -2s$$

for any U is unit vector field on M.

Proof. If $R.R = 0$, then, from (1) we deduce that

$$R(R(U,\xi_i)U,\varphi U,\varphi U,\xi_j)+R(U,R(U,\xi_i)\varphi U,\varphi U,\xi_j) \tag{38}$$

$$+R(U,\varphi U,R(U,\xi_i)\varphi U,\xi_j)+R(U,\varphi U,\varphi U,R(U,\xi_i)\xi_j) = 0$$

for any unit vector field $U \in \mathscr{X}(M)$ and any $i,j = 1,\ldots,s$. By using (25) and (26), a direct expansion of (38) gives $R(U,\varphi U,\varphi U,U) = -2s$. ■

Moreover, we have:

Theorem 22. *Let M be a $(2n+s)$-dimensional S-manifold, $n \geq 1$. If M is Ricci semi-symmetric respect to the quarter-symmetric metric connection ∇ then*

$$S(Y,V) = 2nsg(\varphi Y,\varphi V)+4n\sum_{\alpha,\beta=1}^{s}\eta^{\alpha}(Y)\eta^{\beta}(V)$$

for all $Y,V \in \mathscr{X}(M)$.

Proof. We take $X = \xi_i$ and $U = \xi_j$.. Then, from (1) we have that

$$R(\xi_i,Y).S = S(R(\xi_i,Y)\xi_j,V)+S(\xi_j,R(\xi_i,Y)V) \tag{39}$$

for any unit vector field $Y \in \mathscr{X}(M)$ and any $i,j \in \{1,\ldots,s\}$. Now, by using (26) and (33)

$$S(R(\xi_i,Y)\xi_j,V) = -2S(Y,V)+4n\sum_{\alpha,\beta=1}^{s}\eta^{\alpha}(Y)\eta^{\beta}(V) \tag{40}$$

for all $V \in \mathscr{X}(M)$. Next, from (25) and (33) we have

$$S(\xi_j,R(\xi_i,Y)V) = 4nsg(\varphi Y,\varphi V). \tag{41}$$

The proof is completed. ■

Due to the above results, it is natural to consider the Weyl projective curvature tensor field of ∇ (see (2)). For this tensor field we obtain the following theorem.

Theorem 23. *Let M be a $(2n+s)$-dimensional S-manifold M with $n \geq 1$. If M is projectively semi-symmetric respect to the quarter-symmetric metric connection ∇ then*

$$S(Y,V) = 2nsg(\varphi Y,\varphi V)+4n\sum_{\alpha,\beta=1}^{s}\eta^{\alpha}(Y)\eta^{\beta}(V)$$

for all $Y,V \in \mathscr{X}(M)$.

Proof. From (1) we have that

$$(P(X,Y).S)(U,V) = P(X,Y).S(U,V) - S(P(X,Y)U,V) - S(U,P(X,Y)V) \tag{42}$$

If $P(X,Y).S = 0$, then we have

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0.$$

Therefore, if we calculate the latter equation we can obtain;

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0.$$

The proof is completed from the Theorem 6.2. ■

7. Conclusions

A quarter-symmetric metric connection is defined on $S-$manifolds. Some properties of the curvature and the Ricci tensor fields of such connection are obtained. In addition, an S-manifold has constant f-sectional curvature with respect to this quarter-symmetric metric connection if and only if has the same constant f-sectional curvature with respect to the Riemannian connection. Consequently, the curvature of the quarter-symmetric metric connection is completely determined by its f-sectional curvature. This topic is open and there are many issues to work on.

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