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A STUDY ON NON-LINEAR DISCRETE-TIME STATE-SPACE MODELS AND ADAPTIVE EXTENDED KALMAN FILTER APPLICATION ON OSCILLATION OF AN OBJECT TIED TO THE END OF SPRING

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ABSTRACT. In this work, Adaptive Extended Kalman Filter (AEKF) is introduced and its use for oscillation of an object connected to the end of a spring is shown. As a new approach, an AEKF is used as a nonlinear estimation tool for online estimation of the states and parameters of an oscillating object attached to the end of a spring model. Parameter states that do not change with time were examined. The simulation results revealed that with proper selection of initial values of AEKF, AEKF is a very useful tool for this particular application.

1. Introduction

The optimum linear filtering and estimation methods were introduced by Kalman [1]. The Kalman filter (KF) solves the problem of estimating instantaneous states of a linear dynamic system that is distorted by Gaussian white noise using measurements that are linear functions of system state but distorted by additional white noise. KF is widely used in many areas of signal processing, control and optimization [2-10]. The KF essentially represents a recursive solution to Gauss's original least squares problem [11-15]. Discrete-time state-space models have been developed during the 1960s for applications such as tracking and controlling the location of satellites, guided missiles, space vehicles and targets which have mobility. Moreover, the state-space models have several fields of application for modelling the physical, neurological, physiological and economic processes. The parameter estimation problem is encountered in different areas. The problem of

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estimating a model parameter, which occurs as the coefficient of a dynamic system state variable, is an important problem to be solved in system identification. When the problem of estimating the parameters in a linear system is solved simultaneously with the problem of estimating the state variable, the linear model becomes nonlinear [10]. EKF is very sensitive to initiation and tends to filter bias if system matrices are not properly selected. There are several research papers that address these issues and analyze the stability and robustness of the filter [19–20].

2. Discrete-Time State-Space Model And Adaptive KF

Discrete-time linear state space models have been employed in 1960’s mostly in controlling and signaling processes in defence industry. The extension and application of such models in other fields have taken place in the beginning of 1990s. A general state space model takes the following form:

$$x_{k+1} = \Phi_k x_k + G_k w_k \tag{1}$$

$$y_k = H_k x_k + v_k \tag{2}$$

Here, $x_k \in \Re^n$ represents the state vector while $y_k \in \Re^m$ represents the observation vector. Φ_k is the nxn system transition matrix, H_k is the mxn observation matrix. $w_k \in \Re^n$ and $v_k \in \Re^m$ are white noises with zero mean, for which the following assumptions can be made for each k, j values:

$$E\left[v_k\right] = 0 \tag{3}$$

$$E\left[w_k\right] = 0 \tag{4}$$

$$E\left[v_k v_j'\right] = R_k \delta_{kj} \tag{5}$$

$$E\left[w_k w_j'\right] = Q_k \delta_{kj} \tag{6}$$

$$E\left[v_k w_j'\right] = 0 \tag{7}$$

$$E\left[x_0\right] = \bar{x}_0 \tag{8}$$

$$E\left[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'\right] = P_0 \tag{9}$$

$$E\left[x_0 w_k'\right] = 0 \tag{10}$$

$$E\left[x_0 v_k'\right] = 0 \tag{11}$$

Moreover, for $k = 0,1,2,\ldots \Phi_k$, H_k, G_k, Q_k and R_k are assumed to be known. As introduced in [3], the filtering problem is to estimate the state vector x_k, given the observation vector $Y_k = \left\{y_0, y_1, \ldots, y_k\right\}$, which can be denoted as:

$$\hat{x}_{k|k} = E\left[x_k \middle| y_0, y_1, \ldots, y_k\right] = E\left[x_k \middle| Y_k\right]$$

with the covariance matrix:

$$P_{k|k} = E\left[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})' \middle| Y_k\right]$$

Let the observation matrix take the form: $Y_{k-1} = \left\{y_0, y_1, \ldots, y_{k-1}\right\}$, then estimating the state vector x_k will be as $\hat{x}_{k|k-1} = E\left[x_k \middle| y_0, y_1, \ldots, y_{k-1}\right] = E\left[x_k \middle| Y_{k-1}\right]$ with the covariance matrix

$$P_{k|k-1} = E\left[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})' \middle| Y_{k-1}\right]$$

In this case, the Kalman Filter, depending on the starting values

$$P_{0|-1} = P_0$$

$$\hat{x}_{0|-1} = \bar{x}_0$$

is characterized by the following algorithms:

$$\hat{x}_{k|k-1} = \Phi_{k-1}\hat{x}_{k-1|k-1} \tag{12}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\left[y_k - H_k\hat{x}_{k|k-1}\right] \tag{13}$$

$$K_k = P_{k|k-1}H_k'\left[H_k P_{k|k-1}H_k' + R_k\right]^{-1} \tag{14}$$

$$P_{k|k} = \left[I - K_k H_k\right]P_{k|k-1} \tag{15}$$

$$P_{k|k-1} = \Phi_{k-1}P_{k-1|k-1}\Phi_{k-1}' + G_{k-1}Q_{k-1}G_{k-1}' \tag{16}$$

As described in [4-5], equation (14) is also known as the “Kalman Gain”.

2.1 **Non-Linear State Space Models and EKF**

A non-linear state space model takes the form of

$$x_{k+1} = f_k(x_k) + H_k(x_k)\xi_k \tag{17}$$

$$y_k = g_k(x_k) + \eta_k \tag{18}$$

Here, f_k and g_k are vector-valued functions, while ξ_k and η_k represent white noise processes with the covariance matrices, Q_k and R_k, respectively. The starting values for the EKF algorithm are:

$$P_0 = \mathrm{cov}(x_0)$$

$$\hat{x}_0 = E(x_0)$$

As mentioned in [5-9], for $k = 1, 2, \ldots$

$$P_{k|k-1} = \left[\frac{\partial f_{k-1}}{\partial x_{k-1}}(\hat{x}_{k-1})\right] P_{k-1} \left[\frac{\partial f_{k-1}}{\partial x_{k-1}}(\hat{x}_{k-1})\right]' + H_{k-1}(\hat{x}_{k-1}) Q_{k-1} H'_{k-1}(\hat{x}_{k-1}) \tag{19}$$

$$\hat{x}_{k|k-1} = f_{k-1}(\hat{x}_{k-1}) \tag{20}$$

$$K_k = P_{k|k-1}\left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1})\right]\left[\left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1})\right] P_{k|k-1}\left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1})\right]' + R_k\right]^{-1} \tag{21}$$

$$P_k = \left[I - K_k\left[\frac{\partial g_k}{\partial x_k}(\hat{x}_{k|k-1})\right]\right] P_{k|k-1} \tag{22}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\left[y_k - g_k(\hat{x}_{k|k-1})\right] \tag{23}$$

represent the EKF updating equations.

In order to apply EKF, the matrices in the state space model above should be written as the functions, which depend on the unknown parameter vector, θ. That is, let the matrices be represented as $\Phi_k(\theta)$, $G_k(\theta)$, $H_k(\theta)$. Furthermore, let θ be a random walk process. In this case the following equations,

$$x_{k+1} = \Phi_k(\theta_k)x_k + G_k(\theta_k)w_k \tag{24}$$

$$y_k = H_k(\theta_k)x_k + v_k \tag{25}$$

and the parameter vector

$$\theta_{k+1} = \theta_k + \zeta_k \tag{26}$$

form the new state space model:

$$\begin{bmatrix} x_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k(\theta_k)x_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} G_k(\theta_k)w_k \\ \zeta_k \end{bmatrix} \tag{27}$$

$$y_k = \begin{bmatrix} H_k(\theta_k) & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \theta_k \end{bmatrix} + v_k \tag{28}$$

The above model is non-linear for which EKF can be readily applied. ζ_k in equation (26) shows the white noise process for which the covariance matrix is assumed to be $\mathrm{cov}(\zeta_k) = S_k = S > 0$. In the particular case where $S = 0$, the parameter vector is assumed to be time-invariant, where EKF cannot be operative. If EKF algorithm is applied to equations (27)-(28), depending on the following starting values

$$\begin{bmatrix} \hat{x}_0 \\ \hat{\theta}_0 \end{bmatrix} = \begin{bmatrix} E(x_0) \\ E(\theta_0) \end{bmatrix} \text{ and } P_0 = \begin{bmatrix} \mathrm{cov}(x_0) & 0 \\ 0 & S_0 \end{bmatrix}$$

for $k = 1, 2, \ldots$ we get:

$$\begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\theta}_{k|k-1} \end{bmatrix} = \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1})\hat{x}_{k-1} \\ \hat{\theta}_{k-1} \end{bmatrix} \tag{29}$$

$$P_{k|k-1} = \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1}) & \frac{\partial}{\partial\theta}\left(\Phi_{k-1}(\hat{\theta}_{k-1})\right)\hat{x}_{k-1} \\ 0 & I \end{bmatrix} P_{k-1} \begin{bmatrix} \Phi_{k-1}(\hat{\theta}_{k-1}) & \frac{\partial}{\partial\theta}\left(\Phi_{k-1}(\hat{\theta}_{k-1})\right)\hat{x}_{k-1} \\ 0 & I \end{bmatrix}'$$

$$+ \begin{bmatrix} G_{k-1}(\hat{\theta}_{k-1}) Q_{k-1} G'_{k-1}(\hat{\theta}_{k-1}) & 0 \\ 0 & S_{k-1} \end{bmatrix} \tag{30}$$

$$K_k = P_{k|k-1}\left[H_k(\hat{\theta}_{k-1}) \quad 0\right]' \left[\left[H_k(\hat{\theta}_{k-1}) \quad 0\right] P_{k|k-1} \left[H_k(\hat{\theta}_{k-1}) \quad 0\right]' + R_k\right]^{-1} \tag{31}$$

$$P_k = \left[I - K_k\left[H_k(\hat{\theta}_{k-1}) \quad 0\right]\right] P_{k|k-1} \tag{32}$$

$$\begin{bmatrix} \hat{x}_k \\ \hat{\theta}_k \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\theta}_{k|k-1} \end{bmatrix} + K_k\left[y_k - \left[H_k(\hat{\theta}_{k-1})\hat{x}_{k|k-1}\right]\right] \tag{33}$$

In order to eliminate divergence in the KF, adaptive methods are used [16-17] forgetting factor is proposed by Özbek and Aliev [17].

$$P_{t|t-1} = \alpha\left(F_{t-1}P_{t-1|t-1}F'_{t-1} + G_{t-1}Q_{t-1}G'_{t-1}\right) \tag{34}$$

3. Oscillation Of An Object Tied To The End Of Spring

According to the Hook's Law [14], the amount of changes in the length of a spring is proportional with the size of the affecting force. Proportion coefficient is called "spring constant". Let an object with m mass be tied to the other end of an L -long vertical spring of which one end is fixed and the spring stretched as l. Based on our experiences, the object oscillates (resonance) when pulled slightly and released afterwards. There may be friction with surrounding environment (air, water) and forces which affect the object temporally. When the spring which is $L+l$ long during the time $t=0$ is pulled and released as y_0, it starts oscillating. Oscillation continues depending on the friction with the surroundings and external forces.

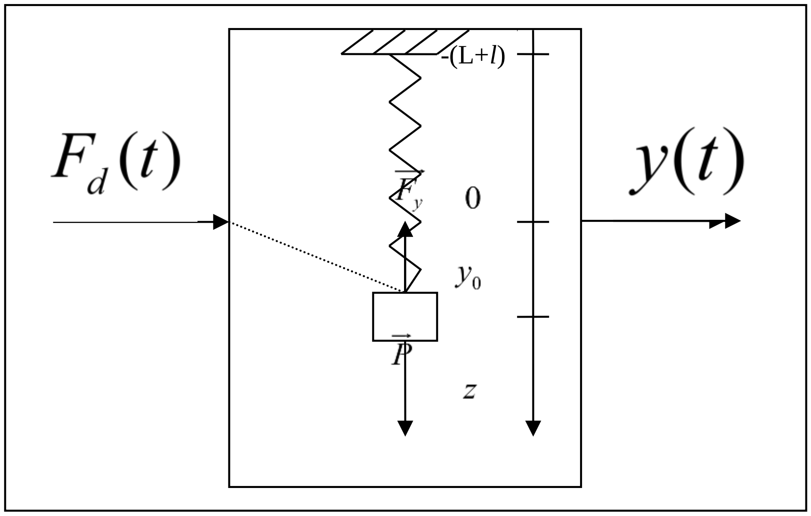

FIGURE 1. Diagram of the model.

Let the size of friction force be presumed as proportional with velocity, which is

$$F_s = s.v(t)$$

For the forces which affect the object at a certain time t,

$$\vec{F} = \vec{P} + \vec{F}_y + \vec{F}_s + \vec{F}_d$$

could be written (Figure 1).

$\vec{P}$ is the gravitational force which affects the object,

$\vec{F}_y$ is the force affected by the spring to the object,

$\vec{F}_s$ is the friction force, and

$\vec{F}_d$ is the external force which affects the object.

Considering that the directions of the forces $\vec{F}_y$ and $\vec{F}_s$ are backwards the direction of movement, differential equation

$$my''(t) = mg - k(l + y(t)) - sy(t) + F_d(t)$$

could be generated (for the size of these forces) on an y- axis of which the initial point is the same as the point and direction on which the mass is hanging, the sense is the same as the gravitational force. Considering that

$$F_y = P$$

the model

$$my''(t) + sy'(t) + ky(t) = F_d(t)$$
$$y(0) = y_0 \ , \quad y'(0) = v_0$$

is obtained. Here, s is the friction coefficient regarding the environment, k is the spring constant, $\vec{F}_d$ is the external force affecting the object, y_0 is the position of the object at starting time and v_0 is the magnitude of the velocity of the object at starting time. The model given as differential equation being

$$y''(t) + \tfrac{s}{m} y'(t) + \tfrac{k}{m} y(t) = \tfrac{1}{m} F_d(t)$$
$$y(0) = y_0 \ , \ y'(0) = v_0$$

resulting from the change of variable

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{x}_1(t) + \tfrac{s}{m} z(t) = z'(t) + \tfrac{s}{m} z(t) \end{cases}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) - s.x_1(t) \\ \dot{x}_2(t) = y''(t) + \tfrac{s}{m} y'(t) = \tfrac{1}{m} F_d(t) - \tfrac{k}{m} y(t) = \tfrac{1}{m} F_d(t) - \tfrac{k}{m} x_1(t) \end{cases}$$

$$\begin{cases} \dot{x}_1(t) = -\tfrac{s}{m} x_1(t) + x_2(t) \\ \dot{x}_2(t) = -\tfrac{k}{m} x_1(t) + \tfrac{1}{m} F_d(t) \end{cases}$$

State-space model:

$$\underline{\dot{x}}(t) = \begin{bmatrix} -\tfrac{s}{m} & 1 \\ -\tfrac{k}{m} & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ \tfrac{1}{m} \end{bmatrix} F_d(t) \qquad (35)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t) \qquad (36)$$

$$\underline{x}(0) = \begin{bmatrix} y_0 \\ v_0 + \tfrac{s}{m} y_0 \end{bmatrix} \qquad (37)$$

is obtained. The purpose of a system, depending on the observed outputs, may be to determine the system's parameters in cases where these parameters are unknown, to check the system by giving appropriate inputs and to estimate the future behavior of the system. Let us consider the movement of the object tied to the spring. In mks measurement system, let $m=1, s=1, k=1$, external force

$$F_d(t)=0,\ t\geq 0$$

position of the object at starting moment

$$y(0)=1$$

and the velocity

$$y'(t)=v_0=0$$

Given that the differential equation expressing the movement of the object is

$$y''(t)+y'(t)+y(t)=0$$
$$y(0)=1,\ y'(0)=0$$

the solution is

$$y(t)=e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \tag{38}$$

The diagram of the solution for $t\in[0,10]$ is on the left in Figure 2. Given that the continuous-time state space model is

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{39}$$

$$y(t)=x_1(t) \tag{40}$$

$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix} \tag{41}$$

the discrete-time state space model obtained for $\Delta t=0.1$, for

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix}=\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+\begin{bmatrix}-0.1 & 0.1\\ -0.1 & 0\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} \tag{42}$$

$$y(k)=[1\ 0]\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} \tag{43}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{44}$$

$k = 1, 2, ..., 100$ ($t \in [0,10]$).

The diagram of the numerical solution obtained from the reduction relation in the discrete-time state space model, which means the diagram of $y(k)$ in return for k is Figure 3. In Figure 4, given that $v_k \sim N(0, 0.02)$ the diagram of noisy observations $y(k) + v_k$ in return for k is seen.

Let us consider the model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1-0.1x_3(k) & 0.1 & 0 \\ -0.1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + w_k \tag{45}$$

$$y(k) = [1\ 0\ 0] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + v_k \tag{46}$$

$$w_k \sim N(0, Q_k) \quad v_k \sim N(0, R_k)$$

in which friction coefficient is unknown and regarded as the third state variable; and there are noises for state and observation variables. The state equation in this model is a nonlinear equation according to the state variables. Let the purpose be to estimation of the friction coefficient with EKF by observing the $y(k)$ position of the oscillating mass.

4. Simulation Study

A simulation may be generated to evaluate the operability of the AKF. Given that the variable $y(k)$ which was observed as the system's output is a noise-added total of the first state variable. The actual values and estimated values of the first state variable $x_1(k)$ during the simulation have been obtained as in Figure 5. AEKF could track the state variable closely. Friction coefficient which is the unknown parameter of the system is given as the third state variable in the model, and the estimation obtained with AEKF shown in Figure 6. The estimation improves as time progresses with new observations and achieves steady state.

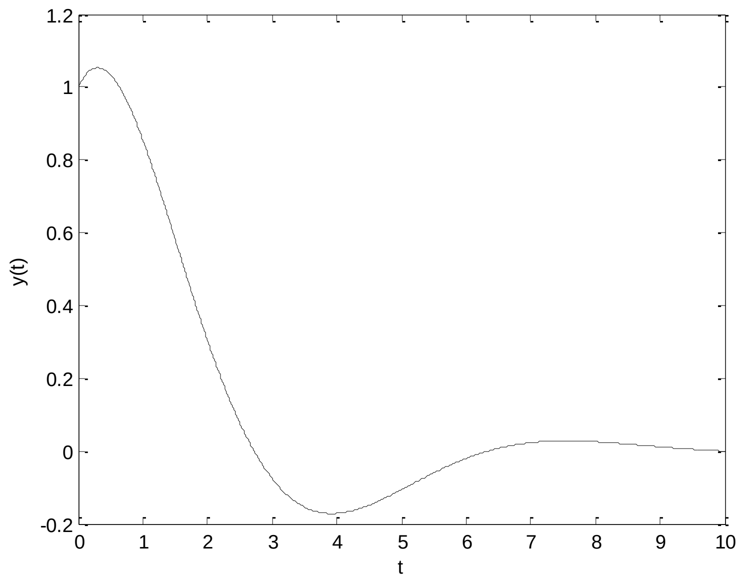

FIGURE 2. Solution of differential equation.

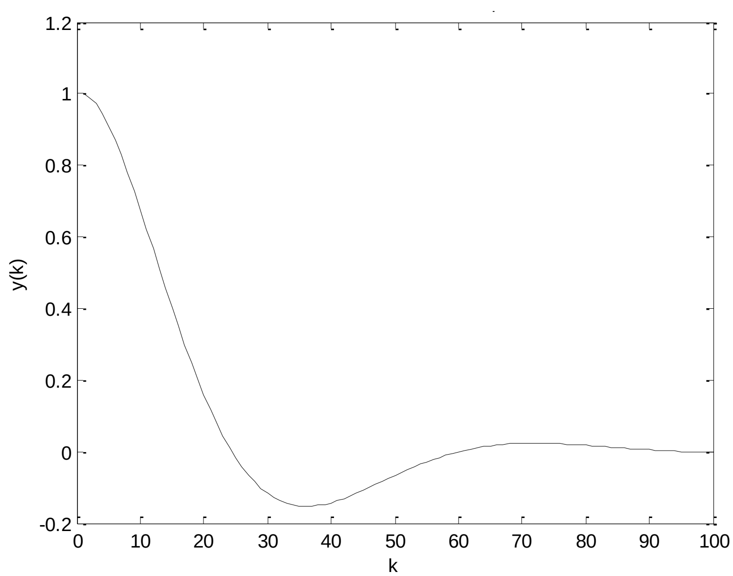

FIGURE 3. Numerical solution of the state-space model.

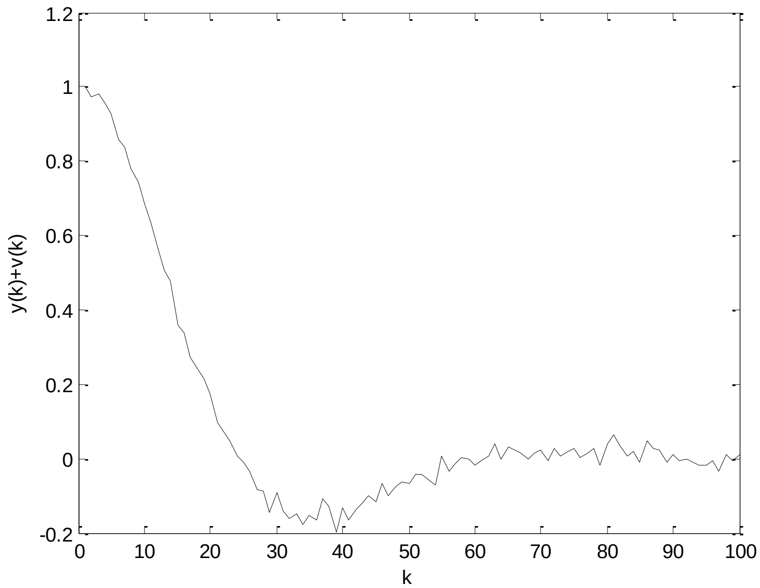

FIGURE 4. Observation+noise.

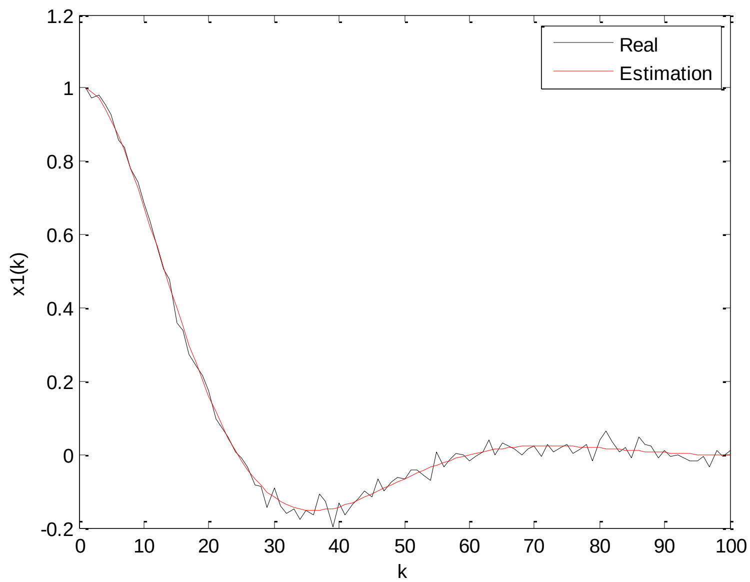

FIGURE 5. EKF result.

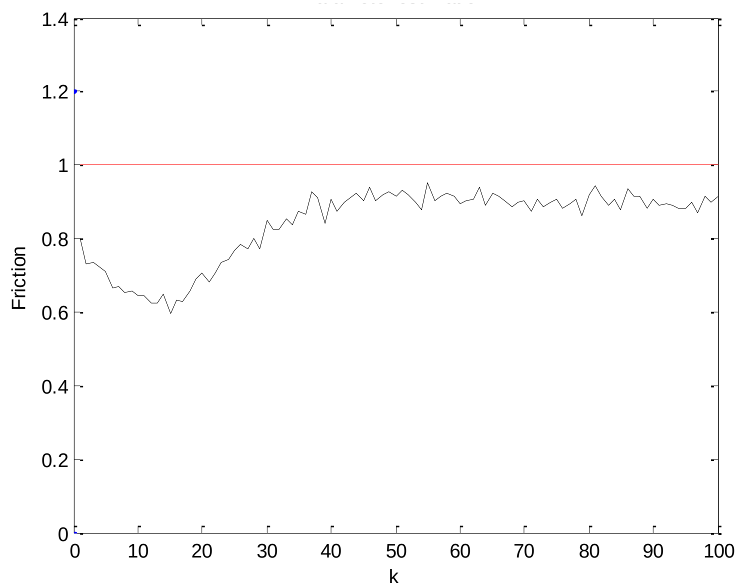

FIGURE 6. Parameter estimation of friction.

5. CONCLUSIONS

In this work, EKF is introduced and its use for oscillating an object connected to the end of a spring is shown. As a new approach, an adaptive AEKF is used as a nonlinear estimation tool for online estimation of states and parameters at the same time as the oscillation of an object attached to the end of an arc pattern. Parameter states that do not change with time were examined. The simulation results show that, with proper selection of the initial value of EKF, AEKF is a very useful tool for this particular application. For the time-varying parameters in the state-space model, some adjustments can be made in AEKF to make the estimation even more powerful. Studies on AEKF are still up-to-date and ongoing.

Authors Contribution Statement The authors contributed equally to the article.

Declaration of Competing Interest The authors declares that there is no conflict of interest.

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