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ON THE RELIABILITY CHARACTERISTICS OF THE STANDARD TWO-SIDED POWER DISTRIBUTION

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Abstract. In this study, the standard two-sided power (STSP) distribution is considered with regard to statistical reliability analysis in detail. For this purpose, along with the reliability and hazard functions of the distribution, particular reliability indices that are useful in maintenance and replacement policies are obtained and they are evaluated with their plots. The STSP distribution is classified based on aging according to various cases of its parameters. Then, we studied the classical and Bayesian estimations of the reliability and hazard functions. In Bayesian estimation, symmetric and different asymmetric loss functions are considered. For obtaining the Bayes estimates, Monte Carlo Markov Chain simulation using the Gibbs algorithm is performed. Various simulation schemes are performed for comparing the performances of the estimators. Further, the Bayesian predictions of the future observations based on the observed samples are obtained. A real data example is used to illustrate the theoretical outcomes.

1. Introduction

Lifetime, survival time or failure time data is encountered in many study fields such as reliability assesment in engineering, clinical trial studies in medicine, biomedical engineering, social studies and etc. In this purpose; lifetimes of peoples, components, patients, industrial robots, animals, plants, cogs, softwares and etc. are considered with probability distributions. In statistical literature, there are many different probability distributions for modelling lifetime data. In reliability theory, a finite upper limit to the lifetime data does not frequently consider and thereby many lifetime distributions are defined over the range $(0, \infty)$ [14]. The commonly used lifetime distributions are the exponential, Weibull, lognormal, gamma and pareto

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etc. distributions. On the other hand, in many cases, the lifetime distributions are needed to consider on a finite range. For example, the pressure, strength, length, temperature, weight, or voltage of material can take any value on a finite range (e.g. $150 - 250$ MPa). Also, the existence of the censoring or truncation causes to reduce lifetimes on a finite range. In these cases, finite range distributions could be considered for modelling them. In the reliability studies, distributions on finite ranges are considered for failure data [1] in various studies. As a special case, finite ranges can be occur over the range $[0, 1]$ and used for modeling uncertainty about the probability of success of an experiment. In these cases, beta distributions could be considered as the most used lifetime distribution. The Beta distributions are quite useful to modeling many uncertainties since their versatile structure [10]. On the other hand, the standard two-sided power distribution, denoted by STSP, is introduced by van Dorp and Kotz [21] and it has the following probability density function (pdf) and the reliability function

$$f(x|\alpha,\beta) = \begin{cases} \alpha(\frac{x}{\beta})^{\alpha-1} & ,0 < x \leq \beta \\ \alpha(\frac{1-x}{1-\beta})^{\alpha-1} & ,\beta \leq x < 1 \end{cases} \tag{1}$$

$$R(x) = P(X > x) = \begin{cases} 1-\beta(\frac{x}{\beta})^{\alpha} & ,0 < x \leq \beta \\ (1-\beta)(\frac{1-x}{1-\beta})^{\alpha} & ,\beta \leq x < 1 \end{cases} \tag{2}$$

while the hazard(failure rate, hazard rate or force of mortality) function is given by

$$\lambda(x) = \frac{f(x)}{R(x)} = \begin{cases} \alpha/\{\left(\frac{\beta}{x}\right)^{\alpha-1} - x\} & ,0 < x \leq \beta \\ \alpha/\{1-x\} & ,\beta \leq x < 1 \end{cases} \tag{3}$$

where $\alpha > 0$ is the shape and $0 < \beta < 1$ is the reflection parameters. The STSP distribution is proposed as a peaked alternative of beta distribution by Kotz and van Dorp [12]. Since the STSP distribution is defined on a finite range and has similar flexibility, the STSP distribution is a beta-like distribution. The parameters of the distribution determine the shapes of the distribution and similar to the beta case. For example, the STSP distribution is unimodal in the case of $0 < \beta < 1$ & $\alpha > 0$ and U shaped for $0 < \beta < 1$ & $0 < \alpha < 1$. It has relations with some other distributions according to its special cases. For instance; the uniform distribution on $(0, 1)$ for $\alpha = 1$ and the triangular model for $\alpha = 2$ are obtained. In the case of $\beta = 0.5$, the STSP distribution is symmetric and the left-skewed and right-skewed distributions occurs when $\beta > 0.5$ and $\beta < 0.5$, respectively, for $\alpha > 1$. The STSP distribution is intelligibly more flexible than the power function distribution which is a special case of the distribution in the case of $\beta = 1$ (see Fig. 1). In this way, the STSP distribution can be used in reliability and life testing experiments on $[0, 1]$ range of finite-range datasets. Particularly, when these types of lifetime data have any threshold point, they are convenient for modelling by a two-sided distribution. Mance, Barker and Chimka [13] studied some features of two-sided power distribution (TSP) which is an extension of the STSP distribution in reliability analysis, firstly. They introduced the reliability and hazard functions of

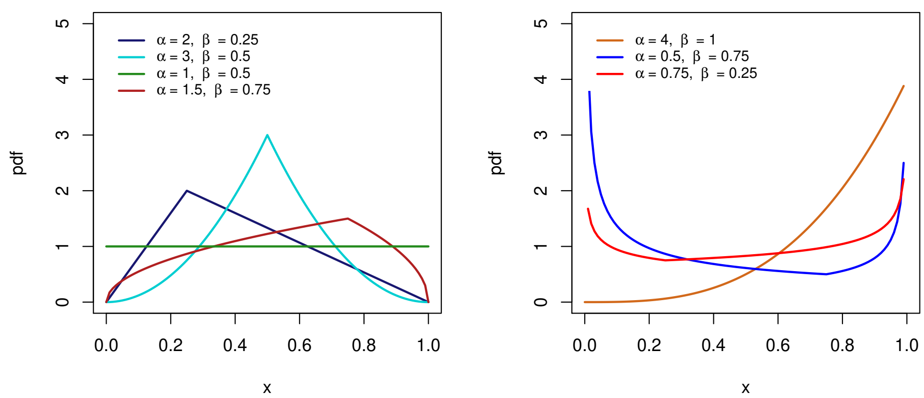

FIGURE 1. Plots of probability denstiy function of the STSP distribution for various choices of its parameters.

the TSP distribution and presented their plots with usefulness in engineering. Using analytical estimation procedure, they obtained the TSP parameters and compared the distribution with the Weibull distribution. Recently, Çetinkaya and Genç [8], [9] studied the STSP distribution under moments of order statistics and stress-strength reliability.

As a further study, we consider the STSP distribution under statistical reliability context. Fundamental reliability indices such as reliability and hazard functions are given and their plots are interpreted according to changing in parameters of the distribution. Following, some reliability indices which are useful in maintenance and replacement policies in engineering are given. Further, we considered the classifying of the STSP distribution based on notions of aging according to various cases of its parameters. Otherwise, as a diagnostics test if a data comes from the STSP distribution, we examined the hazard plot. After these main reliability indices, we obtanined the classical and Bayesian estimations of the reliability and hazard functions based on the symmetrical and asymmetrical loss functions. A real dataset is used to illustrate the outcomes and all estimates are compared. In the last section, Bayes prediction of a future sample based on current available sample is obtained.

2. Reliability characteristics

The STSP distribution is a two-sided distribution and quite useful on the finite range. The reliability graph of the STSP distribution is both convex and concave,or likely S-shaped, depending on different cases of its parameters (see Fig. 2-3). In

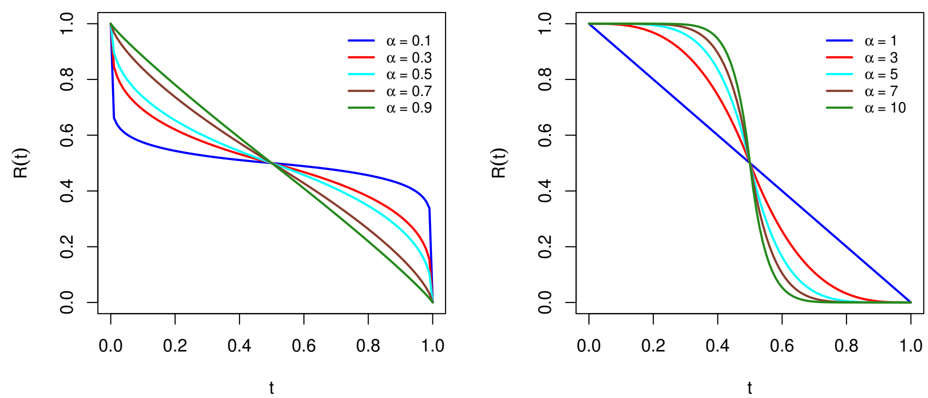

FIGURE 2. Reliability function plots of the symmetrical STSP distribution ($\beta = 0.5$) for different shape parameters.

symmetrical case, that is if $\beta = 0.5$, in the case of $\alpha < 1$, it is convex for the smaller values than β and concave for bigger values than β. On the conversely, in the case of $\alpha > 1$, it is concave for the smaller values than β and convex for bigger values than β. If the STSP distribution is not in symmetrical case, that is if $\beta \neq 0.5$, it is convex for small β values and it turns to concave with increasing β for $\alpha > 1$. On the other hand, it is convex for large β values and it turns to concave with decreasing β for $\alpha < 1$. While $\alpha = 1$, the STSP distribution has constant decreasing reliability.

Concave reliability curve imply low failure in early and useful life along with rapid increase in later life. On the contrary, convex reliability curve imply high failure in early and useful life along with rapid decrease in later life, the convexity or concavity of a reliability curve is depend on environmental conditions and genetic structure of the observations.

In parallel to its reliability function, the STSP distribution has both increasing and decreasing failure rate based on different cases of its parameters (see Fig. 4). On the other hand, the hazard function (Eq.3) shows that for any case of parameters the STSP distribution does not have constant hazard where imminent risk of failure does not change with time. It is clearly seen that, the failure rate of the STSP distribution is increasing for $\alpha > 1$ values and in the form of bathtube curve for $\alpha < 1$. Also, $\lambda(t)$ is not differantiable in the $t = \beta$ point so there is a cusp as seen in Fig. 4. Detailed comments about behaviour of the hazard function are given in the next section.

In statistical reliability studies, there are some indices to compare survival random

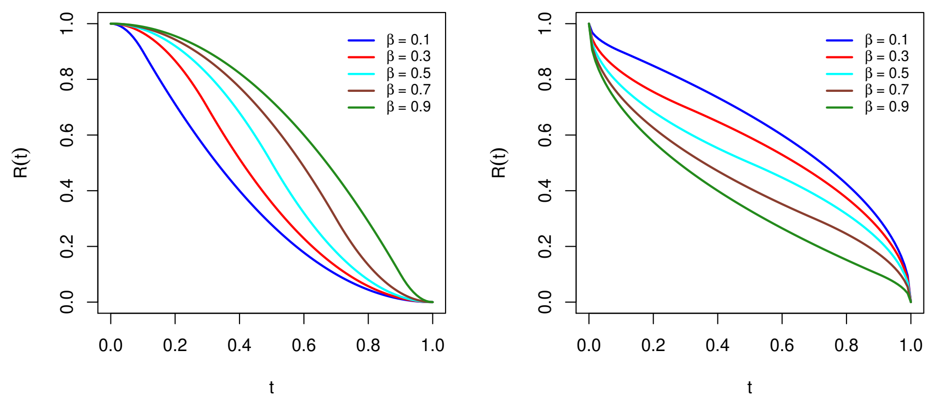

FIGURE 3. Reliability function plots of the STSP distribution for different reflection parameters in the case of $\alpha > 1 (\alpha = 2)$ on left and $\alpha < 1 (\alpha = 0.5)$ on right.

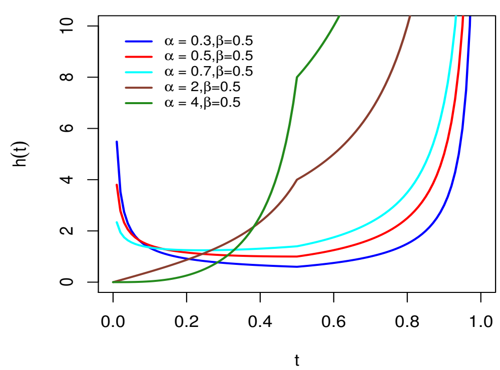

FIGURE 4. Hazard function plots of the symmetrical STSP distribution ($\beta = 0.5$) for different shape parameters.

variables. Also, these indices are quite useful for maintenence and replacement policies.

Firstly, mean time to failure (MTTF) is the length of lifetime a component is expected to failure. MTTF is one of various methods to assess the reliability of a component. The mean time to failure (MTTF) of the STSP distribution can be obtained by using the pdf (1) of the distribution as in the following.

$$MTTF = E(X) = \frac{\beta(\alpha-1)+1}{\alpha+1}$$

Mean residual life time (MRL) at age-t can be considered as another reliability index. Resiual lifetime at age t is about the question of a component how much life does it have left in on avarage while the experimental component still alive and under observation at time t [18]. Mean time to failure for the STSP distribution can be easily obtained as in the following. Firstly, the conditional density of the X given $X > t$ is obtained by

$$f(X|X>t) = \begin{cases} \frac{\alpha(\frac{x}{\beta})^{\alpha-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,0<t<x<\beta \ (t<\beta, CaseI) \\ \frac{\alpha(\frac{1-x}{1-\beta})^{\alpha-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,0<t<\beta<x<1 \ (t<\beta, CaseI) \\ \frac{\alpha}{1-x}\left(\frac{1-x}{1-t}\right)^{\alpha} & ,\beta<t<x<1(t>\beta, CaseII) \end{cases}$$

Then, mean residual lifetime at age-t can be obtained by using

$r(t) = E(X-t|X>t) = \int (x-t)f(x|x>t)dx$ and equally

$E(X-t|X>t) = \frac{\int_t^1 R(x)dx}{R(t)} = \frac{\int_t^{\beta} R_1(x)dx + \int_{\beta}^1 R_2(x)dx}{R(t)}$ for $t \le \beta$

$E(X-t|X>t) = \frac{\int_t^1 R(x)dx}{R(t)} = \frac{\int_t^1 R_2(x)dx}{R(t)}$ for $t > \beta$.

where $R_1(x)$ and $R_2(x)$ are the two sides of the reliability function (2), respectively. Thus, under the STSP distribution mean residual lifetime at age-t is obtained as in the following

$$r(t) = E(X-t|X>t) = \begin{cases} \frac{\left[1+\beta(\alpha-1)+\beta t\left(\frac{t}{\beta}\right)^{\alpha}\right](\alpha+1)^{-1}-t}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,t \le \beta \\ \frac{1-t}{\alpha+1} & ,t>\beta \end{cases}$$

Together with the hazard plot, MRL plot is a useful and good indication to investigate the behaviour of lifetime data [15]. The MRL plot which are given in Fig. 5 shows that the MRL of a lifetime data under the STSP distribution brings with convex curve to concave curve with increasing shape parameter α. Similar to results which are obtained with hazard plot, for $\alpha < 1$, MRL is rapidly increasing in early life as parallel to rapidly decreasing failure. Then, MRL is rapidly decreasing in wear out stage after a stationary process in useful lifetime on peak. Examples can be increased for all possible conditions of the parameters α and β.

Further, when a component has already reached given age t, life expectancy at age t is named as mean life expectancy at age-t and denoted by $E(X|X>t) = t + r(t)$. If a component has a lifetime under the STSP distribution, the mean life expectancy at age t it is obtained as in the following

$$E(X|X>t) = t + r(t) = \begin{cases} \frac{\left[1+\beta(\alpha-1)-\alpha\beta t\left(\frac{t}{\beta}\right)^{\alpha}\right](\alpha+1)^{-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & , t \leq \beta \\ \frac{\alpha t+1}{\alpha+1} & , t > \beta \end{cases}$$

Similar to MRL plot, the plots of the mean life expectancy at age-t are given in Fig. 5. The behaviour of the mean life expectancy shows consistents results with the hazard (Fig. 4) and MRL (Fig. 5) plots.

There is an other index for replacement policies is computation of the probability of that an A-year-old component reaches age-B. Under the STSP distribution, it can be obtained easily as in the following

$$e^{-\int_A^B \lambda(x)dx} = \begin{cases} \frac{\beta^{\alpha-1}-B^{\alpha}}{\beta^{\alpha-1}-A^{\alpha}} & , A < B \leq \beta \\ \frac{\beta^{\alpha-1}(1-B)^{\alpha}}{(\beta^{\alpha-1}-A^{\alpha})(1-\beta)^{\alpha-1}} & , A \leq \beta < B \\ \left(\frac{1-B}{1-A}\right)^{\alpha} & , \beta \leq A < B \end{cases}$$

Additionally, the expected service life (ESL) of a component under a replacement policy [3] whereby the component is replaced when it reaches age t is defined as the expected value of the mixture random variable, namely $Z = min\{X, t\}$ and $ESL(t)$ is given as in the following [18].

$$ESL(t) = \int_0^t xf(x)dx + \int_t^1 tf(x)dx$$

For the STSP distribution the expected service life of a component is considered for two cases as given below

If $t \leq \beta$,
$ESL(t) = \int_0^t xf_1(x)dx + \int_t^{\beta} tf_2(x)dx + \int_{\beta}^1 tf_2(x)dx = \int_0^t xf_1(x)dx + tR_1(t)$

If $t > \beta$,
$ESL(t) = \int_0^{\beta} xf_1(y)dx + \int_{\beta}^t xf_2(x)dx + \int_t^1 tf_2(x)dx$
$ESL(t) = \int_0^{\beta} xf_1(x)dx + \int_{\beta}^t xf_2(x)dx + tR_2(t)$

where $f_1(x)$ and $f_2(x)$ are the two sides of the pdf (1) of the STSP distribution. Thus, ESL(t) under the STSP distribution is obtained as in the following

$$ESL(t) = \begin{cases} t - \frac{\beta t\left(\frac{t}{\beta}\right)^{\alpha}}{\alpha+1} & , t \leq \beta \\ \frac{\alpha\beta+(1-\beta)\left[1-(1-t)\left(\frac{1-t}{1-\beta}\right)^{\alpha}\right]}{\alpha+1} & , t > \beta \end{cases}$$

The plots of EST(t) for different cases of the parameters are given in Fig. 6 and Fig. 7. In symmetrical case, that is if $\beta = 0.5$, is changing to concave curve with increasing α. For fixed $\alpha > 1$, ESL(t) has larger values and similar concavity with increasing β. On the contrary, for $\alpha < 1$, ESL(t) has smaller values and similar

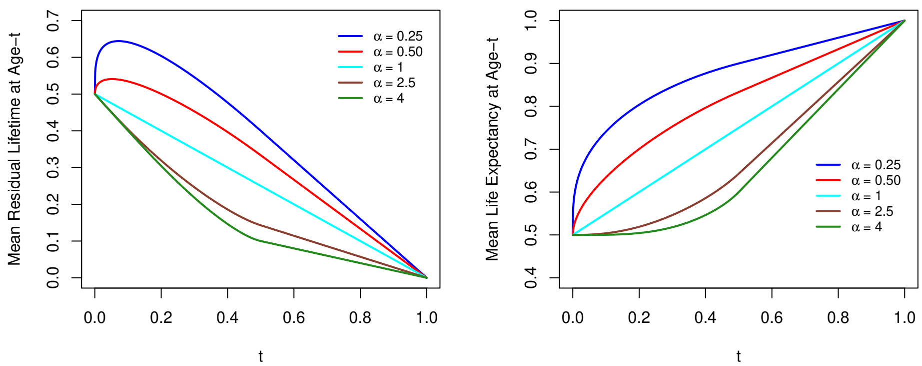

FIGURE 5. Plots of mean residual lifetime (left) and mean life expectancy (right) at age-t for the symmetrical STSP distribution

concavity with increasing β.

All these indices which are given and interpreted above is quite useful to evaluate the behaviour of a lifetime data. In engineering. maintenance and replacement policies of components and systems have been considered, seriously.

2.1. **Classifiying the distribution based on notions of aging.** Many lifetime distributions are considered under particular replacement policies. The maintanence policies are useful to reduce the deficit of the system failures and provide operational sustainability. In this purpose, the STSP distribution has been evaluated based on its aging. Firstly, the behaviour of the hazard function is considered and life characteristics for a lifetime data from the STSP distribution is determined as in the following and summarized in Table 1.

Theorem 1. *In the case of $x \leq \beta$, $\lambda(x)$ is increasing namely it has increasing failure rate (IFR) for $\alpha > 1$ and either decreasing on $x \leq min\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta\right)$ and increasing on $\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \leq x \leq \beta$ for $\alpha < 1$.*

Proof. If $x \leq \beta$, then

$$\lambda^{'}(x) = \alpha\left[\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta} - 1\right]^{-2}\frac{1}{x^2}\left[\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta}(\alpha-1)+1\right]$$

Note that; $\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta} > 1$. So,

the sign of $\lambda^{'}(x)$ depends on the sign of $\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta}(\alpha-1)+1$.

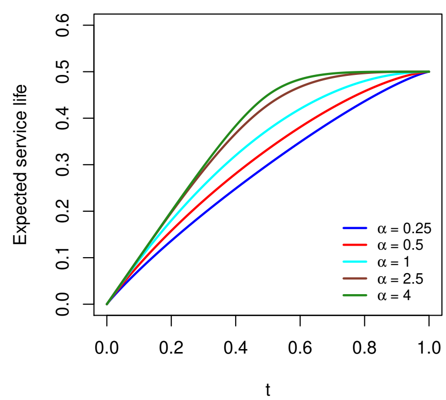

FIGURE 6. Plots of expected service life (ESL) for the symmetrical STSP distribution for various α values

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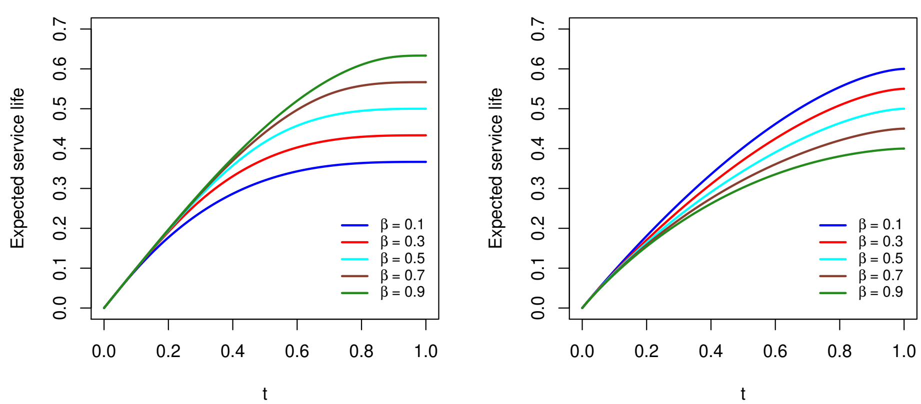

FIGURE 7. Plots of expected service life (ESL) for various β values in the case of $\alpha > 1(\alpha = 2)$ on the left and $\alpha < 1(\alpha = 0.5)$ on the right)

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Table 1. Life characteristics for a lifetime data from the STSP distribution.

Parameters	**Domain**	**Failure Type**
$\alpha < 1$	$x \leq min\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$	Decreasing Hazard
$\alpha < 1$	$\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \leq x \leq \beta$	Increasing Hazard
$\alpha \geq 1$	$x \leq \beta$	Increasing Hazard
$\alpha \geq 0$	$x \geq \beta$	Increasing Hazard

For $\alpha > 1$, $\lambda(x)$ is increasing on $(0, \beta)$
$\left(\frac{\beta}{x}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 > 0 \Longleftrightarrow x > (1-\alpha)\beta^{1-1/\alpha}$
For $\alpha < 1$, $\lambda(x)$ is either increasing or decreasing on $(0, \beta)$
$\left(\frac{\beta}{x}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 > 0 \Longleftrightarrow x > \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}$

Thus, $\lambda(x)$ is increasing on $\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$, if $1 < \alpha + \beta$.

So if $1 - \beta < \alpha < 1$ then $\lambda(x)$ is increasing on $\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$

$\left(\frac{\beta}{t}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 < 0 \Longleftrightarrow t < \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}$

So $\lambda(x)$ is decreasing on $\left(0, \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \right)$, if $\alpha + \beta < 1$

So if $\alpha < 1 - \beta < 1$ then $\lambda(x)$ is decreasing on $\left(0, \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \right)$. □

Theorem 2. *In the case of $x > \beta$, $\lambda(x)$ is an increasing function and namely it has IFR on $(\beta, 1)$ for both $\alpha > 1$ and $\alpha < 1$.*

Proof. $\lambda^{'}(x) = \frac{\alpha}{(1-x)^2}$. In this way, $\lambda^{'}(x) > 0$ for all $\alpha > 0$ values. □

In the hazard function of the STSP distribution for $\alpha > 1$ values of shape parameter $\lambda_1^{'}(\beta) \neq \lambda_2^{'}(\beta)$ and it is not differentiable in the $x = \beta$ point so there is a cusp as seen in Fig. 4 (Here, $\lambda_1^{'}(.)$ and $\lambda_2^{'}(.)$ denotes to two side of the hazard function (3)).

If a lifetime distribution, has a hazard function with non-decreasing avarage, it is increasing failure rate avarage (IFRA) class of lifetime distribution. This class could be alternately defined by a condition intuitively related to wear out for each $x \geq 0$ [4]. An IFR limetime distribution is also IFRA. The both proporties of a lifetime distribution are notions of aging. The IFR, the IFRA or the NBU class of distributions have a number of benefits. For instance, the distribution or reliability functions of these distributions can be bounded from lower and upper in terms of their mean or quantiles. Many other useful properties of these class of distributions are elaborated by Barlow and Proschan [1] such as relating to the reliability of a simple system, a coherent system, a system subject to cumulative shocks and etc. [19].

An IFRA component Tends to more survive any shorter period and on the contrary, less surviving any longer period. The IFRA class contains the exponential survival probabilities. It contains all IFR survival probabilities. Birnbaum et al. [4] mentioned that the IFRA class is closed under the formation of coherent systems and that it is essentially the smallest class containing the exponentials which is so closed.

Remark 1. *A distribution has IFRA (Increasing failure rate avarage) if* $-(1/x)\ln R(x)$ *is increasing in* $x \geq 0$*. Similarly a distribution has DFRA (Decreasing failure rate avarage) if* $-(1/x)\ln R(x)$ *is decreasing in* $x \geq 0$ *[1].*

Theorem 3. *The STSP distribution is an IFRA class of distribution for* $\alpha > 1$ *in the both* $x \leq \beta$ *and* $x \geq \beta$ *cases.*

Proof.

$$\psi_1(x) = -\frac{\ln R_1(x)}{x} = -\frac{\ln\left[1-\beta(\frac{x}{\beta})^\alpha\right]}{x} = \frac{\ln\left[\frac{1}{1-\beta(\frac{x}{\beta})^\alpha}\right]}{x}$$

Using the expansion of $\ln\left[\frac{1}{1-\beta(\frac{x}{\beta})^\alpha}\right]$ as

$$\psi_1(x) = \frac{\beta(\frac{x}{\beta})^\alpha + \frac{\left[\beta(\frac{x}{\beta})^\alpha\right]^2}{2} + \frac{\left[\beta(\frac{x}{\beta})^\alpha\right]^3}{3} + \cdots}{x}$$

Then,

$$\psi_1^{'}(x) = (\alpha-1)\beta^{1-\alpha}x^{\alpha-2} + \frac{(2\alpha-1)\beta^{2-2\alpha}x^{2\alpha-1}}{2} + \frac{(3\alpha-1)\beta^{3-3\alpha}x^{3\alpha-2}}{3} + \cdots$$

It is clearly seen that for $\alpha > 1$, $\psi_1^{'}(x) > 0$. Thus, the STSP distribution is IFRA in the case of $x \leq \beta$ if and only if $\alpha > 1$.

On the other hand

$$\begin{aligned}\psi_2(x) &= -\frac{\ln R_2(x)}{x} = -\frac{\ln\left[\frac{(1-x)^\alpha}{(1-\beta)^{\alpha-1}}\right]}{x} = \frac{\ln\left[\frac{(1-\beta)^{\alpha-1}}{(1-x)^\alpha}\right]}{x} \\ &= \frac{\ln(1-\beta)^{\alpha-1} - \ln(1-x)^\alpha}{x} = \frac{\ln(1-\beta)^{\alpha-1} + \alpha\ln\left(\frac{1}{1-x}\right)}{x}\end{aligned}$$

Using the expansion of $\ln\left(\frac{1}{1-y}\right)$ as

$$\psi_2(x) = \frac{(\alpha-1)\ln(1-\beta) + \alpha\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)}{x}$$

Then

$$\psi_2^{'}(x) = \frac{(1-\alpha)\ln(1-\beta)}{x^2} + \frac{\alpha}{2} + \frac{2x\alpha}{3} + \frac{3x^2\alpha}{4}\cdots$$

In this equation $\ln(1-\beta) > 0$ for $\alpha > 1$ values. Thus, it makes $\frac{(1-\alpha)\ln(1-\beta)}{x^2} \geq 0$ and $\psi_2^{'}(x) > 0$ □

2.2. **Hazard plot.** A hazard plot is a simple plot of the points $\left(a_j, x_j\right)$, where $a_j = \sum_{i=1}^{j} \frac{1}{n-i+1}$ are called the hazard plot scores [18]. For using a hazard plot to determine if a data comes from the STSP distribution, note that, cumulative hazard function of the STSP distribution

$$H(x) = -\ln\{R(x)\}\begin{cases} \ln\left[1-\beta(\frac{x}{\beta})^{\alpha}\right]^{-1} & ,0 < x \leq \beta \\ \ln\left[\frac{(1-\beta)^{\alpha-1}}{(1-x)^{\alpha}}\right] & ,\beta \leq x < 1 \end{cases}$$

Therefore, if a data comes from the STSP distribution the relationship between $\ln(a_j)$ and $\ln(x_j)$ should be a 45^0 line similarly to hazard plot for the Weibull distribution. Many engineers regard hazard plot as a simpler diagnostic test than a probability plot [18].

3. Classical estimation

In this section, we have obtained the maximum likelihood estimation (MLE) of the reliability and hazard functions of the STSP distribution. Let us suppose that $x_1, x_1, \ldots, x_n$ is the independent and identical (IID) random samples from $STSP(\alpha, \beta)$. Then the likelihood function is given by

$$L(\alpha,\beta) = \alpha^n \left\{\frac{\prod_{i=1}^{r} x_{(i)} \prod_{i=r+1}^{n}(1-x_{(i)})}{\beta^r(1-\beta)^{n-r}}\right\}^{\alpha-1}$$

where $x_{(r)} \leq \beta < x_{(r+1)}$ with $x_{(0)} \equiv 0$ and $x_{(n+1)} \equiv 1$.

The maximum likelihood estimators of the parameters are obtained by van Dorp and Kotz [21], and they are given by

$$\hat{\beta} = X_{(\hat{r})}$$

$$\hat{\alpha} = -\frac{n}{\log M(\hat{r})}$$

where $\hat{r} = arg\ max_{\{r \in 1,2,\cdots,n\}} M(r)$ and

$$M(r) = \prod_{i=1}^{r-1} \frac{X_{(i)}}{X_{(r)}} \prod_{i=r+1}^{n} \frac{1 - X_{(i)}}{1 - X_{(r)}}$$

Thus, by using the invariance property of the MLEs, the maximum likelihood estimators of the reliability function and hazard function can be obtained by replacing the parameters in Eq.(2) and Eq.(3) with their estimates and denoted by $\hat{R}_{ML}$ and $\hat{\lambda}_{ML}$.

4. Bayesian estimation

In this section, we provide Bayes estimates of reliability function $R(x)$ and hazard function $\lambda(x)$. Under considering different loss functions, these estimates are obtained and compared with respect to their expected risks (ER). In Bayesian estimation, squared error loss function (SELF) is the most commonly used loss function due to it is symmetrical and it provides equal distance to the losses through overestimation and underestimation. However, in some situations such as reliability and hazard estimates overestimation is more considerable than underestimation or vice-vera [16]. In this purpose, Linex loss function (LLF) defined by Varian [22] and general entropy loss function (GELF) defined by Calabria and Pulcini [5] are considered as asymmetric loss functions which are defined as, respectively,

SELF $\Longrightarrow L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$
LLF $\Longrightarrow L_2(\hat{\theta}, \theta) = e^{p(\hat{\theta}-\theta)} - p(\hat{\theta} - \theta) - 1, \quad p \neq 0$
GELF $\Longrightarrow L_3(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta}\right)^c - c \log\left(\frac{\hat{\theta}}{\theta}\right) - 1$

where p and c reflects the departure from the symmetry, $\hat{\theta}$ represents an estimate for parameter θ. Thus, Bayes estimates of the parameters under these loss functions can be obtained from their posterior distributions as in the following;

SELF $\Longrightarrow \hat{\theta}_{B1} = E(\theta|data)$
LLF $\Longrightarrow \hat{\theta}_{B2} = -\frac{1}{p} \log\{E(e^{-p\theta}|data)\}$
GELF $\Longrightarrow \hat{\theta}_{B3} = \{E(\theta^{-c}|data)\}^{-1/c}$

Under these loss functions, the Bayes estimators of reliability $R(x|\alpha, \beta)$ and hazard $\lambda(x|\alpha, \beta)$ functions which are given in Eq.(2) and Eq.(3), respectively, are expressed as in the following,

$$\hat{R}_{B1} = \int_0^\infty \int_0^1 R(data|\alpha, \beta)\ \pi(\alpha, \beta|data) d\beta d\alpha \tag{4}$$

$$\hat{R}_{B2} = -\frac{1}{p} \log\left\{ \int_0^\infty \int_0^1 e^{-pR(data|\alpha,\beta)} \pi(\alpha, \beta|data) d\beta d\alpha \right\} \tag{5}$$

$$\hat{R}_{B3} = \left\{ \int_0^\infty \int_0^1 R(data|\alpha,\beta)^{-c} \pi(\alpha,\beta|data) d\beta d\alpha \right\}^{-1/c} \tag{6}$$

where $\pi(\alpha,\beta|data)$ is posterior distribution of the parameters. Estimators of the $\lambda(t)$, denoted by $\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$ and $\hat{\lambda}_{B3}$, can be obtained by changing $R(data|\alpha,\beta)$ with Eq.(3), similarly.

However, the form of the $STSP$ distribution given in (1) is not proper for developing Bayesian models. Since the its support depends on the reflection parameter, posterior distributions of α and β, namely $\pi(\alpha,\beta|data)$ can not be obtained. Also, estimators given in (4),(5) and (6) can not be expressed in closed form and hence it can not be evaluated analytically. This fact was previously pointed out for the triangular distribution which is special form of the $STSP$ distribution ($\alpha = 2$ case) by Ho et al. [11]. To overcome this adversity and obtain a Bayesian inference for the STSP distribution, Çetinkaya and Genç [9] proposed a hierarchical model construction. This model provides conditional distributions of parameters to build a Markov Chain Monte Carlo (MCMC) algorithm using a Gibbs sampler as given in the following.

Çetinkaya and Genç [9] developed marginal densities by introducing an auxiliary or talent variable as in the following.

Let V be a random variable with parameter $\alpha > 1$. Suppose that V has the pdf

$$f_V(v;\alpha) = \alpha\big[1-(1-v)^{1/(\alpha-1)}\big], \quad 0 < v < 1.$$

Further, let the conditional distribution of X given $V = v$ be the uniform distribution represented by

$$U\big[\beta(1-v)^{1/(\alpha-1)}, 1-(1-\beta)(1-v)^{1/(\alpha-1)}\big].$$

Then the marginal distribution of X has the STSP distribution with pdf given in (1). Thus, this hierarchical model will simplify the computational procedures for Bayesian calculations. In order to implement a Gibbs sampler, Çetinkaya and Genç [9] are obtained the conditional distributions of α, β and v as in the following

$$\begin{aligned} f(v|\alpha,\beta,x) &\propto f(v|\alpha) f(x|\alpha,\beta,v) \\ &\propto I\left(max\left\{ 1-\left(\frac{x}{\beta}\right)^{\alpha-1}, 1-\left(\frac{1-x}{1-\beta}\right)^{\alpha-1} \right\} < v < 1 \right) \end{aligned}$$

$$\begin{aligned} f(\beta|\alpha,v,x) &\propto \pi(\beta) f(x|\beta,v,\alpha) \\ &\propto \pi(\beta) I\left(1-\frac{1-x}{(1-v)^{1/(\alpha-1)}} < \beta < \frac{x}{(1-v)^{1/(\alpha-1)}} \right) \end{aligned}$$

$$\begin{aligned} f(\alpha|v,\beta,x) &\propto \pi(\alpha) f(v|\alpha) f(x|\beta,v,\alpha) \\ &\propto \pi(\alpha) I\left(1 < \alpha < min\left\{ \frac{\ln(1-v)}{\ln(\frac{x^{<}}{\beta})} + 1, \frac{\ln(1-v)}{\ln(\frac{1-x^{>}}{1-\beta})} + 1 \right\} \right) \end{aligned}$$

where $I(.)$ denotes indicator function, $x^{<}$ denotes observations below β and $x^{>}$ observations above β, $\pi(\alpha)$ and $\pi(\beta)$ denotes prior distributions for the parameters. Thus, MCMC samples using Gibbs algorithm can be obtained by using the following steps;

Step 1: Assign initial $\alpha^{(0)}$ and $\beta^{(0)}$ values for α and β.
Step 2: Set t=1.
Step 3: Given $\alpha^{(t-1)}$ and $\beta^{(t-1)}$ and $\{x_1, x_2, \cdots, x_n\}$ generate $\{v_1, v_2, \cdots, v_n\}$ using Eq.(4).
Step 4: Considering uniform prior on $[0, 1]$ for β, given $\alpha^{(t-1)}$, $\{x_1, x_2, \cdots, x_n\}$ and $\{v_1, v_2, \cdots, v_n\}$, generate $\beta^{(t)}$ using

$$I\left(max\left\{1 - \frac{1 - x_i}{(1 - v_i)^{1/(\alpha^{(t-1)}-1)}}, 0\right\} < \beta < min\left\{\frac{x_i}{(1 - v_i)^{1/(\alpha^{(t-1)}-1)}}, 1\right\}\right)$$

Step 5: Considering uniform prior on $[1, c]$ for α and choosing $c = 100$ generate α^t from the pdf $\left[(n + 1)/(b^{n+1} - 1)\right]\alpha^n$ using inverse transformation method, where

$$b = min\left\{1 + \frac{\ln(1 - v_i)}{\ln(\frac{x_i^{<}}{\beta^{(t)}})}, 1 + \frac{\ln(1 - v_i)}{\ln(\frac{1-x_i^{>}}{1-\beta^{(t)}})}, c\right\}$$

Step 6: Using Eq.(2) and Eq.(3), compute $R_B^{(t)}$ and $\lambda_B^{(t)}$ at $(\alpha^{(t)}, \beta^{(t)})$.
Step 7: Set $t = t + 1$.
Step 8: Repeat steps $2 - 7$, M times and obtain posterior samples $(R_B^{(t)} : t = 1, 2, \cdots, M)$ and $(\lambda_B^{(t)} : t = 1, 2, \cdots, M)$.

Finally, the posterior mean under mean sqaured error, linex loss and general entropy loss functions, say $\hat{R}_{B1}$, $\hat{R}_{B2}$, $\hat{R}_{B3}$ and $\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$, $\hat{\lambda}_{B3}$, can be obtained as follows;

$$\hat{R}_{B1} = \frac{1}{M}\sum_{t=1}^{M} R_B^{(t)} \quad , \quad \hat{R}_{B2} = -\frac{1}{p}\ln\left\{\frac{1}{M}\sum_{t=1}^{M} e^{-pR_B^{(t)}}\right\}$$
$$\hat{R}_{B3} = \left\{\frac{1}{M}\sum_{t=1}^{M}(R_B^{(t)})^{-c}\right\}^{-1/c} \tag{7}$$

$\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$, $\hat{\lambda}_{B3}$ are obtained similarly.

5. Simulation Studies

In this section, performances of the maximum likelihood and Bayes estimators under different loss functions are compared. According to various fixed point (t) and sample sizes, avarage estimates and corresponding expected risks (ER) of $R(t)$

are obtained and reported in Tables 2 and 3. Similar results are also obtained for $\lambda(t)$ and reported in Table 4 and 5.

The expected risks of estimates under all considered loss functions (SELF, LLF and GELF), when θ is estimated by $\hat{\theta}$, can be obtained by using the following equation,

$$\mathrm{ER}(\hat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \theta)^2,$$

where

$$\hat{\theta} = E(\theta|data), \qquad \text{for SELF}$$
$$\hat{\theta} = -\frac{1}{p}\log\{E(e^{-p\theta}|data), \qquad \text{for LLF}$$
$$\hat{\theta} = \{E(\theta^{-c}|data)\}^{-1/c}, \qquad \text{for GELF}$$

respectively. Choosen arbitrary values of the parameters (α, β) are taken as $(2.8, 0.8)$ and $(1.5, 0.5)$, respectively. The Bayes point estimates are obtained under SELF, LLF$(p = -0.5, 0.5, 1)$ and GELF$(c = -0.5, 0.5, 1)$ loss functions. We generate 2000 samples of size n (small sample size $n = 10$, moderate sample sizes $n = 20, 30$ and large sample sizes $n = 50, 100$). For Bayesian estimation, we run the Gibbs sampler to generate a Markov chain with 3500 observations using the given algorithm in Section 4. As burn-in period, we discard the first 500 values and take every third variate as a independent and identically distributed observation in thinning procedure. Thus, a sample of 1000 resulted which is used to calculate the posterior estimates. Then, the simulation is performed via MCMC for 2000 replicates. We report all the results of this simulation scheme in Table 2, 3 for reliability estimates. We observed that all the estimates are close to the actual values of $R(t)$. As expected, the ERs of all estimators decrease as sample size increases in all considered cases. In all cases $(t \leq \beta, t > \beta)$, maximum likelihood estimates tend to give overestimates. Being underestimating or overestimating is not only depend on loss parameters, it is also related to relation between t and β. Bayes estimates under squared error $\hat{R}_{B1}$ and Linex loss functions $\hat{R}_{B2}$ gives under estimates for $t \leq \beta$ and over estimates for $t > \beta$. Bayes estimates under general entropy loss function $\hat{R}_{B3}$ gives under estimates for $t \leq \beta$. On the other hand, for $t > \beta$ it gives under estimate for $c = 0.5$ and $c = 1$, overestimates for $c = -0.5$. Expected risks show that MLE and Bayes estimates under SELF have larger risks. Bayesian estimates under LLF and GELF gives better results in terms of expected risks. Especially, estimates give smallest risks for loss parameters $c = 0.5$ and $p = 0.5$. While loss parameter values converges to 1, risks are getting larger.

Furthermore, similar simulation scenario are applied for $\lambda(t)$ and reported in Table 4, 5. However, Linex loss function is not considered for hazard estimates, only SELF and GELF are used in Bayesian estimates in addition to MLE. Since the

TABLE 2. Avarage estimates and corresponding mean squared errors/risks of $R(t)$ for different choise of n and t when $\alpha = 2.8$ and $\beta = 0.8$ where actual $R(0.2) = 0.984$, $R(0.5) = 0.785$ and $R(0.9) = 0.029$.

t	n	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}(Linex)$			$\hat{R}_{B3}(GELF)$		
				$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	10	0.983448	0.970658	0.970911	0.970400	0.970138	0.970381	0.969809	0.969514
		0.000381	0.000818	0.000801	0.000836	0.000853	0.000837	0.000877	0.000898
	20	0.982662	0.975747	0.975863	0.975631	0.975512	0.975623	0.975369	0.975239
		0.000239	0.000437	0.000431	0.000443	0.000449	0.000444	0.000458	0.000465
	30	0.983106	0.978792	0.978854	0.978730	0.978668	0.978728	0.978596	0.978529
		0.000138	0.000209	0.000207	0.000212	0.000214	0.000212	0.000216	0.000219
	50	0.983434	0.980887	0.980916	0.980858	0.980828	0.980857	0.980796	0.980766
		0.000083	0.000111	0.000110	0.000111	0.000112	0.000111	0.000113	0.000113
	100	0.983348	0.982076	0.982088	0.982065	0.982053	0.982064	0.982040	0.982028
		0.000041	0.000050	0.000050	0.000050	0.000050	0.000050	0.000050	0.000050
0.5	10	0.805895	0.769507	0.772007	0.766961	0.764367	0.765790	0.757843	0.753595
		0.011199	0.011945	0.011693	0.012211	0.012491	0.012449	0.013632	0.014321
	20	0.795528	0.778710	0.779972	0.777433	0.776138	0.776950	0.773281	0.771367
		0.005247	0.005728	0.005655	0.005804	0.005886	0.005864	0.006173	0.006348
	30	0.793182	0.783086	0.783890	0.782275	0.781457	0.782004	0.779787	0.778650
		0.003232	0.003476	0.003450	0.003503	0.003532	0.003523	0.003631	0.003691
	50	0.790239	0.785236	0.785692	0.784777	0.784316	0.784638	0.783426	0.782812
		0.001903	0.001997	0.001990	0.002004	0.002012	0.002010	0.002038	0.002054
	100	0.787610	0.785392	0.785609	0.785173	0.784955	0.785111	0.784545	0.784261
		0.000964	0.001019	0.001017	0.001021	0.001023	0.001022	0.001029	0.001033
0.9	10	0.036569	0.029611	0.029755	0.029468	0.029327	0.025943	0.018368	0.014756
		0.002212	0.000556	0.000565	0.000547	0.000538	0.000495	0.000457	0.000478
	20	0.033779	0.034082	0.034217	0.033949	0.033817	0.031104	0.025323	0.022586
		0.001096	0.000477	0.000484	0.000470	0.000463	0.000408	0.000330	0.000318
	30	0.031665	0.034080	0.034189	0.033973	0.033866	0.031723	0.027279	0.025209
		0.000628	0.000401	0.000406	0.000396	0.000391	0.000347	0.000278	0.000262
	50	0.030256	0.033022	0.033088	0.032956	0.032891	0.031505	0.028678	0.027359
		0.000242	0.000236	0.000239	0.000234	0.000232	0.000209	0.000175	0.000166
	100	0.028860	0.030302	0.030325	0.030278	0.030255	0.029645	0.028380	0.027770
		0.000082	0.000090	0.000090	0.000090	0.000089	0.000085	0.000079	0.000077

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

TABLE 3. Avarage estimates and corresponding mean squared errors/risks of $R(t)$ for different choise of n and t when $\alpha = 1.5$ and $\beta = 0.5$ where actual $R(0.2) = 0.874$, $R(0.5) = 0.500$ and $R(0.9) = 0.045$.

t	n	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}(Linex)$			$\hat{R}_{B3}(GELF)$		
				$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	10	0.874192	0.880441	0.881192	0.879683	0.878918	0.879536	0.877680	0.876729
		0.008142	0.003009	0.002987	0.003033	0.003059	0.003050	0.003142	0.003195
	20	0.873982	0.872569	0.873088	0.872047	0.871521	0.871956	0.870711	0.870078
		0.004265	0.001863	0.001853	0.001873	0.001885	0.001878	0.001912	0.001930
	30	0.874393	0.869598	0.869997	0.869197	0.868793	0.869128	0.868176	0.867693
		0.002691	0.001442	0.001434	0.001450	0.001459	0.001453	0.001478	0.001491
	50	0.875334	0.868961	0.869237	0.868683	0.868404	0.868636	0.867980	0.867649
		0.001292	0.000944	0.000939	0.000950	0.000955	0.000951	0.000966	0.000975
	100	0.874437	0.869591	0.869733	0.869449	0.869306	0.869425	0.869091	0.868923
		0.000609	0.000606	0.000604	0.000609	0.000612	0.000610	0.000617	0.000621
0.5	10	0.501461	0.499542	0.502393	0.496691	0.493841	0.493212	0.479664	0.472407
		0.024020	0.011413	0.011422	0.011418	0.011438	0.011813	0.013025	0.013872
	20	0.498163	0.497941	0.499603	0.496279	0.494617	0.494423	0.487152	0.483395
		0.011601	0.005787	0.005783	0.005797	0.005812	0.005910	0.006251	0.006474
	30	0.502838	0.502116	0.503334	0.500899	0.499681	0.499604	0.494469	0.491846
		0.007896	0.00430	0.004307	0.004296	0.004295	0.004341	0.004465	0.004550
	50	0.497907	0.498000	0.498795	0.497205	0.496411	0.496369	0.493065	0.491393
		0.004915	0.003033	0.003030	0.003036	0.003041	0.003061	0.003133	0.003178
	100	0.499269	0.499646	0.500082	0.499209	0.498772	0.498762	0.496985	0.496091
		0.002365	0.001661	0.001661	0.001662	0.001662	0.001667	0.001684	0.001694
0.9	10	0.042168	0.039468	0.039622	0.039315	0.039163	0.035352	0.026263	0.021649
		0.001781	0.000528	0.000530	0.000527	0.000525	0.000587	0.000807	0.000959
	20	0.045160	0.046053	0.046180	0.045926	0.045800	0.043146	0.036971	0.033781
		0.001053	0.000394	0.000395	0.000392	0.000390	0.000400	0.000459	0.000513
	30	0.044577	0.047299	0.047403	0.047196	0.047092	0.045040	0.040359	0.037972
		0.000727	0.000357	0.000359	0.000356	0.000354	0.000354	0.000373	0.000396
	50	0.044447	0.048333	0.048408	0.048257	0.048182	0.046789	0.043676	0.042119
		0.000379	0.000279	0.000280	0.000278	0.000277	0.000270	0.000264	0.000268
	100	0.043858	0.046646	0.046685	0.046607	0.046568	0.045844	0.044252	0.043462
		0.000155	0.000165	0.000166	0.000165	0.000164	0.000161	0.000156	0.000156

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

TABLE 4. Avarage estimates and corresponding mean squared errors/risks of $\lambda(t)$ for different choise of n and t when $\alpha = 2.8$ and $\beta = 0.8$ where actual $\lambda(0.2) = 0.235$, $\lambda(0.5) = 1.530$ and $\lambda(0.9) = 28$.

t	n	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$		
				$c=-0.5$	$c=0.5$	$c=1$
0.2	10	0.212659	0.313104	0.268433	0.179495	0.139538
		0.040227	0.059365	0.050593	0.042289	0.041897
	20	0.227699	0.279875	0.255613	0.206944	0.183191
		0.024096	0.032751	0.029739	0.026911	0.026982
	30	0.228798	0.260926	0.244609	0.211838	0.195616
		0.015070	0.017770	0.016607	0.015801	0.016135
	50	0.228658	0.247889	0.238138	0.218507	0.208703
		0.009430	0.010604	0.010247	0.010088	0.010284
	100	0.232933	0.242591	0.237783	0.228127	0.223305
		0.004708	0.005225	0.005131	0.005085	0.005132
0.5	10	1.527713	1.759105	1.692216	1.562116	1.497889
		0.645253	0.692297	0.626148	0.542050	0.522746
	20	1.523361	1.615436	1.587562	1.533140	1.506383
		0.227766	0.237356	0.224401	0.206981	0.202233
	30	1.508165	1.558570	1.542440	1.510611	1.494848
		0.103195	0.108809	0.105687	0.102085	0.101525
	50	1.515442	1.535820	1.527515	1.510964	1.502708
		0.056820	0.059281	0.058701	0.058129	0.058130
	100	1.526108	1.534501	1.530836	1.523501	1.519830
		0.024845	0.025726	0.025636	0.025553	0.025559
0.9	10	33.641810	34.949716	34.095302	32.380401	31.522558
		216.803883	232.935939	212.975812	177.496690	161.996142
	20	30.473946	30.805958	30.405633	29.597499	29.190319
		71.428341	71.478819	68.002931	62.007851	59.498323
	30	29.760384	29.848291	29.587420	29.059841	28.793272
		38.963410	40.020913	38.694668	36.473211	35.583057
	50	28.950512	28.947054	28.798466	28.498208	28.346406
		20.345468	21.397870	20.993493	20.333355	20.080180
	100	28.537238	28.599188	28.531815	28.396683	28.328937
		8.511153	9.343395	9.232614	9.041654	8.961717

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

TABLE 5. Avarage estimates and corresponding mean squared errors/risks of $\lambda(t)$ for different choise of n and t when $\alpha = 1.5$ and $\beta = 0.5$ where actual $\lambda(0.2) = 1.086$, $\lambda(0.5) = 3$ and $\lambda(0.9) = 15$.

t	n	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$		
				$c = -0.5$	$c = 0.5$	$c = 1$
0.2	10	1.169160	1.138610	1.083326	0.969943	0.911189
		0.578881	0.282840	0.264464	0.256365	0.267456
	20	1.126438	1.130396	1.101100	1.042949	1.013914
		0.241296	0.119590	0.114397	0.110948	0.112758
	30	1.115528	1.136663	1.116351	1.076537	1.056966
		0.149128	0.083364	0.079938	0.076227	0.075905
	50	1.100535	1.127605	1.114830	1.089977	1.077876
		0.072287	0.047207	0.045266	0.042646	0.041937
	100	1.095927	1.119173	1.112864	1.100583	1.094604
		0.033968	0.029776	0.028847	0.027331	0.026735
0.5	10	3.163512	3.274372	3.189939	3.029371	2.953750
		2.061529	1.592072	1.445463	1.206419	1.113028
	20	2.955649	2.951840	2.909191	2.826440	2.786523
		0.805802	0.538925	0.521216	0.495661	0.487587
	30	2.891219	2.847796	2.817406	2.757840	2.728774
		0.535141	0.375209	0.373863	0.375581	0.378551
	50	2.873531	2.806055	2.786332	2.747164	2.727771
		0.324230	0.256955	0.261111	0.271222	0.277145
	100	2.866385	2.799225	2.787858	2.765075	2.753677
		0.175262	0.165486	0.169514	0.178249	0.182950
0.9	10	19.682280	20.553253	20.138921	19.332664	18.942773
		74.221823	77.820319	70.489912	57.178282	51.195827
	20	17.047429	17.258969	17.069952	16.697927	16.515474
		20.687151	19.057091	17.735260	15.325558	14.236774
	30	16.403576	16.389651	16.262977	16.011816	15.887572
		12.517064	11.323384	10.766275	9.747012	9.284522
	50	15.779786	15.640292	15.564618	15.413565	15.338271
		5.990385	5.627023	5.470833	5.190874	5.067049
	100	15.459294	15.358812	15.321466	15.246624	15.209147
		2.569578	2.791105	2.754111	2.688752	2.660413

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

second case ($t > \beta$) of the hazard function which is given in Eq. (3) is depend on only shape (α) parameter and dividing it to $(1 - t)$ bring along large deviations even if small changes on α, the ERs under LLF do not provide consistent results. Also, many authors implied that LLF is not as appropriate for estimation of scale parameter as it is for location parameter and GELF is proposed as a suitable alternative to the modified LINEX loss function [2], [17]. Table 4, 5 show that the Bayes estimates under GELF has smaller expected risks and loss parameter $c = 0.5$ gives smallest risks for actual $\lambda > 1$ values of hazard function. On the contrary, MLE estimates has smaller risks while actual values converges to 0. In this case, ML gives better results than Bayes estimates in terms of ER. Similar to reliability estimates, the ERs of all hazard estimators decrease as sample size increases as expected.

6. Real Data Studies

In this section, a real data analysis is used to illustrate the proposed methods. In this purpose, breaking strengths of $1mm$ length single carbon fibers data, from Crowder [7], is used. We scaled the data by subtracting 2 and multiplying 5, respectively. Thus, the data lie in the interval $(0, 1)$. The sample size of the data is 58. The scaled data is given in Table 6.

Table 6. Re-scaled breaking strengths of $1mm$ length single carbon fibers data,($n = 58$).

0.0494	0.3570	0.5356	0.3362	0.5110	0.2656
0.2710	0.4222	0.6718	0.3824	0.5664	0.3456
0.3566	0.4914	0.1816	0.4432	0.7368	0.4126
0.4164	0.5272	0.3162	0.5084	0.2490	0.4652
0.4804	0.6268	0.3792	0.5476	0.3454	0.5264
0.5268	0.1684	0.4282	0.7142	0.4100	0.6086
0.6198	0.3144	0.5038	0.2252	0.4524	0.7996
0.8120	0,3572	0.5396	0.3452	0.5228	0.8120
0.1280	0.4236	0.6946	0.3928	0.5848	
0.2766	0.4932	0.2198	0.4502	0.7442	

We fit the STSP distribution to this dataset and we used maximum likelihood and Bayesian estimation methods. Estimations of the parameters α and β are reported in Table 7. Then, we applied to data Kolmogorov-Simirnov test to evaluate goodness of fit and test statistics are reported in Table 8, respectively. For sample size $n = 58$ and significance level $0, 05$, the critical Kolmogorov-Simirnov test value is $D_{58,0.05} = 0, 1783$. Thus, the null hypothesis that the data come from the STSP distribution cannot reject. Also, the QQ-plot and hazard plot, Fig. 8, support this observation.

Table 7. ML and Bayes estimates of the parameters for the real data set.

	MLE	SELF	LLF			GELF		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
α	2.6704	2.6734	2.7060	2.6417	2.6111	2.6614	2.6373	2.6253
β	0.4164	0.4092	0.4097	0.4087	0.4083	0.4080	0.4056	0.4044

Estimates of reliability $R(t)$ and failure rate $\lambda(t)$ under maximum likelihood and Bayes method are obtained for different choise of t, say $t = 0.2, 0.4, 0.6, 0.8$, and reported in Table 9 and Table 10, respectively. We perform the algorithm which is given above for Bayes estimations with 100 000 iteration. We start the iteration with the maximum likelihood estimates of parameters and with these good starting values we prefer not to use burn-in operation. Also, we take every tenth variate as a independent and identically distributed observation in thinning procedure. Thus, a sample of 10 000 resulted which is used to calculate the posterior estimates. We used R program [20] to obtain the simulation results. Convergence of the simulated Markov chains is assessed by graphical methods.

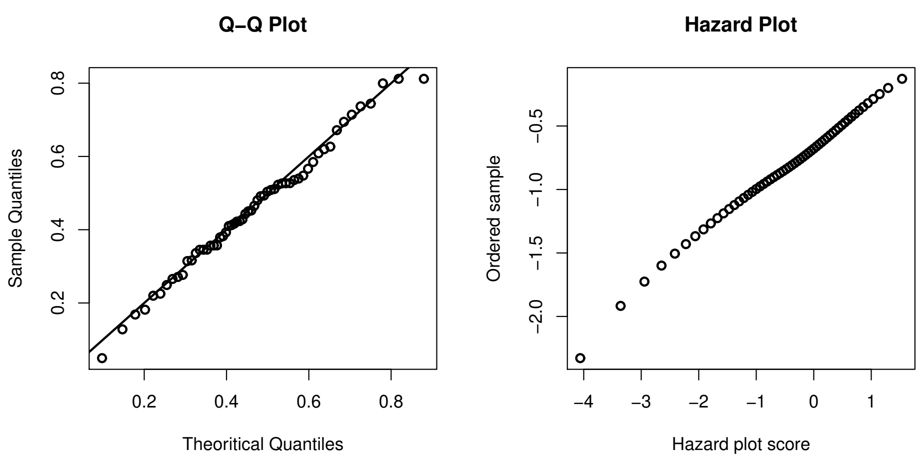

Figure 8. Q-Q and the hazard plots of the real dataset.

In this purpose, trace plots (Fig. 9, Fig. 10) which is a plot of the iteration number, t, against the value of the $R_B^{(t)}$ and $\lambda_B^{(t)}$ at each iteration. Also, density plots of the posterior distribution of the R and λ are drawn at the same time. It is observed that Markov chains fluctuates around their center with similar variation.

TABLE 8. Kolmogorov-Simirnov test statistics for the real data set. Kolmogorov-Simirnov critical test value $D_{58,0.05} = 0,1783$.

MLE	SELF	LLF			GELF		
		$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.1207	0.0862	0.1552	0.1034	0.0690	0.1379	0.1207	0.1379

TABLE 9. Reliability estimates of the real data set under various t values.

t	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}$			$\hat{R}_{B3}$		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	0.9412	0.9363	0.9364	0.9362	0.9361	0.9362	0.9360	0.9359
0.4	0.6260	0.6137	0.6146	0.6129	0.6120	0.6123	0.6093	0.6078
0.6	0.2128	0.2142	0.2146	0.2139	0.2135	0.2125	0.2091	0.2074
0.8	0.0334	0.0354	0.0355	0.0354	0.0353	0.0341	0.0314	0.0300

The density plots seems in a symmetrical and unimodal shape. Morever, autocorrelation of the chains are evaluated and their plots are given in Fig. 11. The ACF plots show that thinning is succesful. Also, we computed the sample lag-t autocorrelation function by *autocorr* command in library *coda* [6] in R. For reliability estimates, the lag-10 autocorrelation is 0.02165095 and the lag-50 autocorrelation is -0.01679917. In addition to this, the lag-10 autocorrelation is 0.09367374 and the lag-50 autocorrelation is -0.02822016 for hazard estimates. Thus, we can say that convergence of the Markov chain is satisfactory.

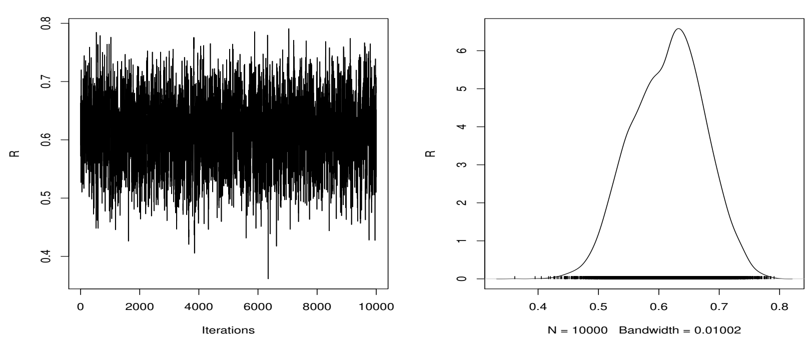

FIGURE 9. Trace plot of reliability estimates on the left and the density plot of the posterior distribution of reliability on the right.

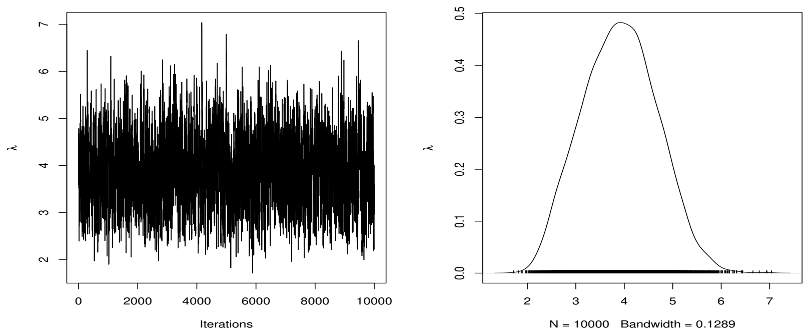

FIGURE 10. Trace plot of hazard estimates on the left and the density plot of the posterior distribution of hazard on the right.

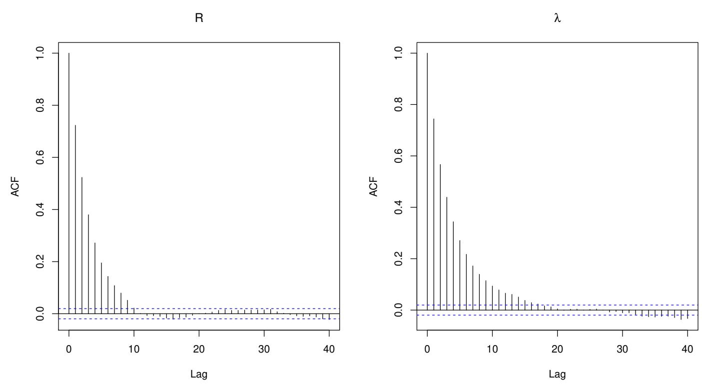

FIGURE 11. Autocorrelation plot for reliability estimates on the left and for hazard estimates on the right.

7. Bayesian Prediction

In this section, we studied Bayesian prediction of future ordered sample based on informative of current observed data. Let $y_{1:m}, y_{2:m}, \cdots, y_{m:m}$ be a future ordered observation independent of the given informative sample data $x_{1:n}, x_{2:n}, \cdots, x_{n:n}$. Then, Bayesian predictive density of the $s^{th}\{s = 1, 2, \cdots, m\}$ ordered future sample can be obtained by using

$$g_{s:m}(y|x) = \int_0^\infty \int_0^1 f_{s:m}(y|\alpha, \beta)\pi(\alpha, \beta|x)d\beta d\alpha$$

TABLE 10. Failure rate estimates of the real data set under various t values.

t	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$			$\hat{\lambda}_{B3}$		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	0.8334	0.8852	0.8964	0.8745	0.8644	0.8732	0.8493	0.8374
0.4	3.9891	3.9164	4.0665	3.7712	3.6333	3.8779	3.7989	3.7584
0.6	6.6760	6.6874	6.8823	6.5091	6.3441	6.6597	6.6044	6.5767
0.8	13.3521	13.3790	14.2480	12.6505	12.0244	13.3201	13.2022	13.1433

where $\pi(\alpha,\beta|x)$ denotes the posterior density of the parameters and $f_{s:m}(y|\alpha,\beta)$ denotes the pdf of the s^{th} order statistic in the future sample as given in the following

$$f_{s:m}(y|\alpha,\beta) = \frac{m!}{(s-1)!(m-s)!}\left[F(y|\alpha,\beta)\right]^{s-1}\left[1-F(y|\alpha,\beta)\right]^{m-s} f(y|\alpha,\beta)$$

here $f(.|\alpha,\beta)$ denotes the pdf which is given in Eq. (1) and $F(.|\alpha,\beta)$ denotes the distribution function of the STSP distribution. Çetinkaya and Genç [8] studied the STSP distribution in detailed in terms of its order statistics. The density of the s^{th} order statistics is given as

$$f_{s:m}(y) = \alpha C_{m,s}\begin{cases} \beta^{(1-\alpha)s}\sum_{i=0}^{m-s}(-1)^i\binom{m-s}{i}\beta^{i(1-\alpha)}x^{\alpha(s+i)-1} & ,0<y\le\beta \\ (1-\beta)^{\varphi_1}\sum_{i=0}^{s-1}(-1)^i\binom{r-1}{i}(1-\beta)^{i(1-\alpha)}(1-x)^{\varphi_2} & ,\beta\le y<1 \end{cases}$$

where $C_{m,s} = \frac{m!}{(s-1)!(m-s)!}$, $\varphi_1 = (1-\alpha)(m-s+1)$ and $\varphi_2 = \alpha(i+m-s+1)-1$. If we denote the predictive density of $y_{s:m}$ as $\hat{g}_{s:m}(y|x)$, it can be obtained by using

$$\hat{g}_{s:m}(y|x) = \int_0^\infty\int_0^1 f_{s:m}(y|\alpha,\beta)\pi(\alpha,\beta|x)d\beta d\alpha \tag{8}$$

However, it is be noted that Eq. (8) cannot be expressed in closed form and hence it cannot be evaluated analytically. Thus, we propose a simulation consistent estimator of $\hat{g}_{s:m}(y|x)$, which can be obtained by using Gibbs sampling MCMC method described in Section 4. Let suppose that MCMC sample $\{(\alpha_i,\beta_i); i = 1,2,\cdots,M\}$ obtained from $\pi(\alpha,\beta|x)$ using the algorithm given in Section 4, then a simulation consistent estimator of $\hat{g}_{s:m}(y|x)$ can be obtained as

$$\hat{g}_{s:m}(y|x) = \frac{1}{M}\sum_{i=1}^{M} f_{s:m}(y|\alpha_i,\beta_i)$$

Further, a simulation consistent estimator of predictive distribution of s^{th} order statistics, say $\hat{G}_{s:m}(y|x)$, can be obtained as

$$\hat{G}_{s:m}(y|x) = \frac{1}{M}\sum_{i=1}^{M} F_{s:m}(y|\alpha_i,\beta_i)$$

where $F_{s:m}(y|\alpha,\beta)$ denotes the distribution function of the s^{th} order statistics, i.e. $F_{s:m}(y|\alpha,\beta) = C_{m,s}\int_0^y [F(z|\alpha,\beta)]^{s-1}[1-F(z|\alpha,\beta)]^{m-s} f(z|\alpha,\beta)dz$

$$= C_{m,s}\begin{cases} B(\beta(\frac{y}{\beta})^{\alpha}; s, m-s+1) & ,0<y\leq\beta \\ B(s,m-s+1) - B\left((1-\beta)(\frac{1-y}{1-\beta})^{\alpha}, m-s+1, s\right) & ,\beta\leq y<1 \end{cases}$$

here $B(a,b)$ denotes the beta function and $B(\cdot,a,b)$ denotes the incomplete beta function. It should be note that, $\hat{g}_{s:m}(y|x)$ is not a point prediction, it is a predictive density. The point prediction for the future observations under squared error loss function can be obtained as in the following

$$\hat{Y}_S = \int_0^1 y\hat{g}_{s:m}(y|x) = \frac{1}{M}\sum_{i=1}^{M}\int_0^1 y f_{s:m}(y|\alpha_i,\beta_i)dy = \frac{1}{M}\sum_{i=1}^{M}\mu_{s:m}$$

where $\mu_{s:m}$ is the first moment of s^{th} order statistics of the STSP distribution and it was given by Çetinkaya and Genç [8] as in the following

$$\mu_{s:m} = C_{m,s}\Big[\beta^{1-1/\alpha}B(\beta;1/\alpha+s,m-s+1) + B(1-\beta;m-s+1,s) \\ -(1-\beta)^{1-1/\alpha}B(1-\beta;1/\alpha+m-s+1,s)\Big]$$

Finally, point estimation under SELF, denoted by $\hat{Y}_S$, can be obtained as in the following;

$$\hat{Y}_S = \frac{1}{M}\sum_{i=1}^{M} C_{m,s}\Big[\beta_i^{1-1/\alpha_i}B(\beta_i;1/\alpha_i+s,m-s+1) + B(1-\beta_i;m-s+1,s) \\ -(1-\beta_i)^{1-1/\alpha_i}B(1-\beta_i;1/\alpha_i+m-s+1,s)\Big] \tag{9}$$

Further, point prediction under general entropy loss function, denoted by $\hat{Y}_G$ can be obtained as

$$\hat{Y}_G = \left[\int_0^1 y^{-c}\hat{g}_{s:m}(y|x)\right]^{-1/c} = \left[\frac{1}{M}\sum_{i=1}^{M}\int_0^1 y^{-c} f_{s:m}(y|\alpha_i,\beta_i)dy\right]^{-1/c}$$

Then, solution of this integral is obtained as

$$\int_0^1 y^{-c} f_{s:m}(y|\alpha,\beta)dy = \int_0^{\beta} y^{-c}\left[\beta\left(\frac{y}{\beta}\right)^{\alpha}\right]^{s-1}\left[1-\beta\left(\frac{y}{\beta}\right)^{\alpha}\right]^{m-s}\alpha\left(\frac{y}{\beta}\right)^{\alpha-1}dy \\ +\int_{\beta}^{1} y^{-c}\left[1-(1-\beta)\left(\frac{1-y}{1-\beta}\right)^{\alpha}\right]^{s-1}\left[(1-\beta)\left(\frac{1-y}{1-\beta}\right)^{\alpha}\right]^{m-s}\alpha\left(\frac{1-y}{1-\beta}\right)^{\alpha-1}dy$$

In the first integral, by change of variable $U = \beta\left(\frac{y}{\beta}\right)^{\alpha}$ and binomial expansion for $\left[1-(1-\beta)\left(\frac{1-y}{1-\beta}\right)^{\alpha}\right]^{s-1}$ and $1-y=v$ transformation in the second integral, the solution can be obtained as

$$\int_0^1 y^{-c} f_{s:m}(y|\alpha,\beta)dy = \beta^{c(1/\alpha-1)}B(\beta; s-c/\alpha, m-s+1) + \alpha\sum_{j=0}^{s-1}\binom{s-1}{j}$$
$$(-1)^j(1-\beta)^{(1-\alpha)(m-s+j+1)}B(1-\beta;\alpha(m-s+j+1,1-c))$$

where $c<1$. Thus; $\hat{Y}_G$ can be obtained as

$$\hat{Y}_G = \left[\frac{1}{M}\sum_{i=1}^{M} C_{m,s}\left(\beta_i^{c(1/\alpha_i-1)}B(\beta_i; s-c/\alpha_i, m-s+1) + \alpha_i\sum_{j=0}^{s-1}\binom{s-1}{j}\right.\right.$$
$$\left.\left.(-1)^j(1-\beta_i)^{(1-\alpha_i)(m-s+j+1)}B(1-\beta_i;\alpha_i(m-s+j+1),1-c)\right)\right]^{-1/c} \tag{10}$$

Moreover, we can construct a $100\gamma\%$ predictive interval for $y_{s:m}$. A symmetrical predictive interval for future sample can be obtained by solving the following non-linear equations for the lower bound L and upper bound U,

$$\begin{aligned}\frac{1+\gamma}{2} &= P(Y_{s:m} > L|data) = 1 - F^*_{s:m}(L|data) \Longrightarrow F^*_{s:m}(L|data) = \frac{1-\gamma}{2}\\ \frac{1-\gamma}{2} &= P(Y_{s:m} > U|data) = 1 - F^*_{s:m}(U|data) \Longrightarrow F^*_{s:m}(U|data) = \frac{1+\gamma}{2}\end{aligned} \tag{11}$$

It is not possible to obtain the solutions analytically and we need to apply suitable numerical techniques for solving these nonlinear equations.

Example. Under the future prediction framework, the prediction values of the first two and last two observations of future sample,$y_{1:m}$,$y_{2:m}$,$y_{m-1:m}$ and $y_{m:m}$, of size $m = 5, 10, 15, 20$ based on real data given in Sec. 6 are obtained with their constructed 95% symmetric predictive interval and reported in Table 11. We performed similar algorithm process which in given in Sec. 4 with the iteration number $M = 10000$ and we used Eq. (9) and Eq. (10) to obtain prediction and Eq. (11) for their predictive intervals. We take the first 500 values as burn-in period and take every third variable as a thinng procedure. Estimations are obtained under symmetric (SELF) and asymmetric (GELF) loss functions and they are represented with their expected risks. Under GELF, three different loss parameters ($c=-0.5, c=0.5, c=0.75$) are considered. For example, based on given real data set, prediction of the first observation of a future sample with size $m=5$ is obtained as 0.263357 with ER 0.000419 under SELF and 0.252411 with ER 0.000964 under SELF ($c=-0.5$). Table 11 shows that predictions are closer to each other for the last observations. For all sample sizes and orders, GELF with $c=-0.5$ loss parameter has smallest expected risks. Prediction intervals are getting shorter by increasing sample size m for each order.

8. Conclusions

Mance et al. [13] first considered the TSP distribution under reliability properties. Recently, moments of order statistics and stress-strength reliability estimation under the STSP distribution were studied by Çetinkaya and Genç [8], [9]. In this study, the STSP distribution is considered as a further research in statistical reliability analysis. In this purpose, we introduced the importance of the distribution as defined on a finite range and two-sided distribution in reliability context. Particular reliability indices with their plots are presented. It has both convex and concave reliability curves according to various cases of its parameters. Also, it has bathtube failure rate for $\alpha < 1$ so it is useful for modelling early life, useful life and wear out proccesses of a component with only single model. By considering the behaviour of the hazard function, the STSP distribution is IFR class of distribution for $\alpha > 1$ and has better chance of surviving any shorter period and the worse chance of surviving any larger period. For the various cases of its parameters, it has both increasing and decreasing failure rate. We showed that the hazard plot is usable to determine if a data comes from the STSP distribution or not. Estimation of the reliability and hazard rate of the STSP distribution are obtained with maximum likelihood method and Bayesian estimation method under different loss functions. Loss functions are considered as symmetrical (SELF) and asymmetrical (LLF and GELF). Based on reliability and hazard estimation studies, our conclusions can be listed as follows;

- In all cases ($t \leq \beta, t > \beta$), maximum likelihood estimates tend to give overestimates.
- Being underestimating or overestimating is not only depend on loss parameters, it is also related to relation between t and β.
- Bayes estimates under squared error $\hat{R}_{B1}$ and Linex loss functions $\hat{R}_{B2}$ gives under estimates for $t \leq \beta$ and over estimates for $t > \beta$.
- Bayes estimates under general entropy loss function $\hat{R}_{B3}$ gives under estimates for $t \leq \beta$. On the other hand, for $t > \beta$ it gives under estimate for $c = 0.5$ and $c = 1$.
- Linex loss function is not proposed to obtain consistent estimations for hazard rate since the second case of the hazard function in Eq. 3 brings along large deviations even if small changes on α.
- For $\lambda > 1$ actual values of hazard function, Bayes estimates under GELF has smaller expected risks and loss parameter $c = 0.5$ gives smallest risks.
- MLEs of hazard rate have smaller risks while actual values converges to zero.
- While actual values of hazard rate λ converges to zero, ML gives better results than Bayes estimates in terms of expected risks.

Table 11. Bayesian point future predictions under SELF and GELF, their expected risks and corresponding predictive bounds for various sample size (m) and the first and last two ordered samples (r) based on given real dataset.

		Bayes Point Predictors				
			GELF			
m	r	SELF	$c=-0.5$	$c=0.5$	$c=0.75$	Prediction Interval
5	1	0.263357	0.252411	0.225168	0.216594	0.078675 0.445661
		(0.000419)	(0.000964)	(0.003864)	(0.012944)	(0.366985)
	2	0.368356	0.362191	0.348675	0.344976	0.191825 0.504161
		(0.000397)	(0.000400)	(0.000741)	(0.002049)	(0.312337)
	4	0.542520	0.537029	0.525876	0.523051	0.364299 0.763211
		(0.000536)	(0.000237)	(0.000346)	(0.000886)	(0.398912)
	5	0.663595	0.657488	0.644880	0.641650	0.432850 0.897675
		(0.000640)	(0.000190)	(0.000282)	(0.000726)	(0.464825)
10	1	0.211261	0.202036	0.179387	0.172369	0.062818 0.370456
		(0.000411)	(0.001357)	(0.004510)	(0.014746)	(0.307638)
	2	0.285261	0.279898	0.267940	0.264628	0.142756 0.419541
		(0.000430)	(0.000701)	(0.001144)	(0.003090)	(0.276785)
	9	0.638138	0.634510	0.627164	0.625311	0.468758 0.817628
		(0.000981)	(0.000300)	(0.000334)	(0.000786)	(0.348871)
	10	0.744783	0.740974	0.733117	0.731103	0.543045 0.928041
		(0.000817)	(0.000185)	(0.000213)	(0.000508)	(0.384996)
15	1	0.175294	0.166855	0.146081	0.139594	0.049536 0.317008
		(0.000378)	(0.001819)	(0.005730)	(0.018808)	(0.267472)
	2	0.243967	0.238845	0.227423	0.224256	0.120202 0.366792
		(0.000559)	(0.001252)	(0.001789)	(0.004687)	(0.246590)
	14	0.690972	0.688235	0.682689	0.681288	0.534454 0.848207
		(0.000893)	(0.000234)	(0.000250)	(0.000580)	(0.313753)
	15	0.774991	0.772149	0.766314	0.764824	0.595999 0.935209
		(0.000720)	(0.000151)	(0.000165)	(0.000385)	(0.339210)
20	1	0.166227	0.158357	0.139102	0.133144	0.048739 0.298049
		(0.000447)	(0.002244)	(0.005961)	(0.019004)	(0.249311)
	2	0.225341	0.220638	0.210224	0.207360	0.111514 0.340080
		(0.000470)	(0.001219)	(0.001751)	(0.004582)	(0.228566)
	19	0.721969	0.719759	0.715280	0.714149	0.578303 0.863639
		(0.000883)	(0.000211)	(0.000221)	(0.000507)	(0.285337)
	20	0.793014	0.790657	0.785832	0.784603	0.629258 0.939188
		(0.000756)	(0.000151)	(0.000160)	(0.000369)	(0.309930)

*First rows in each coloumn represents the point estimation values and prediction interval (last coloumn), the second rows in brackets represents the expected risks of the estimates and length of prediction interval (last coloumn).

All obtained results are illustrated with a real data example. The reliability and hazard rate estimates for various fixed point are obtained. Convergency of the obtained Markov chain is checked and consistent estimations are reported. Finally, we obtained the prediction of the future observations based on given datasets. For various sample size, the first two and last two observations are predicted with their prediction interval.

There are still some other problems concerning the STSP distribution. For example, censored or truncated sampling schemes may be considered in the frame of reliability estimation and prediction.

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Declaration of Competing Interests The authors declare that they have no competing interest.

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