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A Closed Formula of Hausdorff Series in a Semigroup Ring

Osman KELEKCİ [1]*

*[1] Niğde Ömer Halisdemir Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümü, NİĞDE

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Abstract: Let K be a fileld of characteristic zero, u and v be algebraically independent variables and $K\langle u,v\rangle$ be the free associative noncommutative algebra of rank 2 over K. Each element in $K\langle u,v\rangle$ can be written as a noncommutative polynomial of u and v. The expression of the polynomial $w = w(u,v) = \log(e^u e^v)$ is a formal power series and a solution to this equation for w is given by the Hausdorff series expressed as nested commutators of u and v. However this series is not in its closed form in $K\langle u,v\rangle$. Obtaining a closed form of this series, one may consider another algebraic structure other than $K\langle u,v\rangle$ and evolute the series in it. We consider the right zero semigroup with finite elements, and the semigroup ring A spanned on this semigroup over the field of real numbers. In this paper, we provide a closed form of this formula in the semigroup ring A.

Yarıgrup Halkasında Hausdorff Serisinin Kapalı bir Formülü

Anahtar Kelimeler
Hausdorff serisi,
Sağ sıfır yarıgrup,
Yarıgrup halkası

Öz: K karakteristiği sıfır olan bir cisim, u ve v cebirsel bağımsız değişkenler ve $K\langle u,v\rangle$, K üzerinde rankı 2 olan serbest birleşmeli değişmeli olmayan bir cebir olsun. $K\langle u,v\rangle$'deki her eleman, u ve v'nin değişmeli olmayan bir polinomu olarak yazılabilir. $w = w(u,v) = \log(e^u e^v)$ polinomunun açılımı düzgün bir kuvvet serisidir ve bu denklemin w için çözümü u ve v'nin iç içe geçen komütatörleri olarak ifade edilebilen Hausdorff serisi tarafından verilir. Ancak bu seri $K\langle u,v\rangle$'de kapalı formda değildir. Bu serinin kapalı bir formunu elde ederken, $K\langle u,v\rangle$'den başka cebirsel yapılar düşünülebilir ve bu cebirsel yapılar içinde, seri geliştirilebilir. Sonlu elemanlı sağ sıfır yarıgubunu ve reel sayılar cismi üzerinde bu yarıgrup tarafından gerilen A yarıgrup halkasını ele alalım. Bu çalışmada, A yarıgrup halkasında bu formülün kapalı bir formu verilmiştir.

*İlgili Yazar, email: okelekci@ohu.edu.tr

1. Introduction

The Campbell-Baker–Hausdorff (CBH) series is of fundamental importance in the theory of Lie groups, their applications, physics, and physical chemistry. The CBH formula is a general result for the quantity $w = w(u,v) = \log(e^u e^v)$, where u and v are not necessarily commuting. Standard methods for the explicit construction of the CBH terms yield polynomial representations, which must be translated into the usually required commutator representation.

Evolutions of the Haussdorff series in various algebras and rings has been considered in obtaining a closed form of this formula after Campbell [1, 2], Baker [3] and Hausdorff [4] published their works on CBH formula. The infinite series for $\log(e^u e^v)$ for noncommuting u and v is expressible in terms of iterated commutators of u and v except for the linear term $u + v$. Finally, Dynkin [5] derived an explicit expression for the terms as a sum of iterated commutators over a certain set of sequences.

$$w(u,v) = u + v + \frac{1}{2}[u,v] + \frac{1}{12}\big[u,[u,v]\big] + \frac{1}{12}\big[v,[v,u]\big] - \frac{1}{24}\Big[u,\big[v,[u,v]\big]\Big] \ldots$$

Although the expression is known in the recursive form, it has been studied to simplify the formula in order to express the CBH formula in a closed form. For the free metabelian Lie algebra, Gerritzen [6] and Kurlin [7] suggested a closed formula in their works, which was used by Drensky and Fındık [8] in their work for describing the multiplication rule in the inner automorphism group of the free matebalian Lie algebra. Baker [9] computed the coefficients according to a fixed basis set. Recently, Fındık and the author [10] studied the CBH formula in semigroup rings spanned on right and right zero semigroups.

A semigroup S is said to be a right zero semigroup if $s_1s_2 = s_2$ for each $s_1, s_2 \in S$. For unexplained terms in semigroup theory see [11]. Now let $\mathbb{R}$ be the field of real numbers, $RZ_n = \{a_1, \dots, a_n\}$ be a right zero semigroup, and A be the semigroup ring spanned on RZ_n together with 1 over the base field $\mathbb{R}$. In the present paper, we suggest a closed formula for coefficients of $w = \log(e^u e^v)$, with respect to the ordered basis $\{1, \alpha_1, \alpha_2, \dots, \alpha_n\}$ in this semigroup.

2. Material and Method

Let $u, v, w \in A$ such that $e^w = e^u e^v$, where $u = x_0 + x_1a_1 + \cdots + x_na_n$, $v = y_0 + y_1a_1 + \cdots + y_na_n$ and $w = z_0 + z_1a_1 + \cdots + z_na_n$. If $x = x_1 + \cdots + x_n$, $y = y_1 + \cdots + y_n$ and $z = z_1 + \cdots + z_n$. Then by Theorem 2.3 in [10], we see that the followings hold.

$$z_0 = x_0 + y_0, z = x + y,$$

$$z_j = \frac{x+y}{e^{x+y}-1}\left(\frac{e^x-1}{x}x_j + \frac{e^x(e^y-1)}{y}y_j\right)$$

where $j = 1, \dots, n$. Let $\bar{u} = u - x_0$ and $\bar{v} = v - y_0$. Then we have also the followings.

$$uv = x_0v + y_0u + x\bar{v}$$

$$vu = y_0u + x_0v + y\bar{u}$$

$$[u, v] = uv - vu = x\bar{v} - y\bar{u}.$$

3. Results

Initially we state the next technical lemma which provides the operations on commutators.

Lemma 3.1. There exists a commutative polynomial $f(x, y) \in \mathbb{R}[x, y]$ such that $w = u + v + [u, v]f(x, y)$.

Proof. We start by showing that $[u, [u, v]] = x[u, v]$ and $[v, [u, v]] = y[u, v]$. Because x_0 is in the center of the ring A, and the fact that $[\bar{u}, \bar{u}] = 0$, we have that

$$[u, [u, v]] = \mathrm{ad}u[u, v] = [x_0 + \bar{u}, x\bar{v} - y\bar{u}] = [\bar{u}, x\bar{v} - y\bar{u}] = [\bar{u}, x\bar{v}] = x[\bar{u}, \bar{v}] = x[u, v]$$

and that similarly $[v, [u, v]] = y[u, v]$. It is direct consequence from the computation above that

$$(\mathrm{ad}u)^a(\mathrm{ad}v)^b[u, v] = x^a y^b (x\bar{v} - y\bar{u}) = x^a y^b [u, v], a, b \geq 0$$

and hence by Dynkin [5] formula,

$$w = u + v + f(\mathrm{ad}u, \mathrm{ad}v)[u, v] = u + v + f(x, y)[u, v]$$

for some function f. ■

Theorem 3.2. Let $u, v \in A$. Then

$$\log(e^u e^v) = u + v + \frac{xe^x(e^y-1) - y(e^x-1)}{xy(e^{x+y}-1)}[u, v]$$

Proof. We know from Lemma 3.1. that

$$\log(e^{u}\, e^{v}\,) = w(u,v) = u + v + f(x,y)[u,v]$$

which gives that

$$z_0 + z_1 a_1 + \cdots + z_n a_n = (x_0 + x_1 a_1 + \cdots + x_n a_n) + (y_0 + y_1 a_1 + \cdots + y_n a_n) + f(x,y)(x\bar{v} - y\bar{u}).$$

Now by Theorem 2.3 in [10], compairing the coefficients of a_1 from both side,

$$z_1 = \frac{x+y}{e^{x+y}-1}\left(\frac{e^x - 1}{x} x_1 + \frac{e^x(e^y - 1)}{y} y_1\right) = x_1 + y_1 + f(x,y)(xy_1 - yx_1)$$

Now comparing coefficients of algeraically independent variables x_1 and y_1, respectively, we have that

$$\frac{x+y}{e^{x+y}-1}\frac{e^x - 1}{x} = 1 - yf(x,y)$$

$$\frac{x+y}{e^{x+y}-1}\frac{e^x(e^y - 1)}{y} = 1 + xf(x,y)$$

which leads to obtain the desired form of the formula for f by simple computations. ∎

4. Discussion and Conclusion

Let S be a right zero semigroup and $\mathbb{R}\langle S\rangle$ be the semigroup ring with basis $S \cup \{1\}$. Consider the equation $w = \log(e^{u}e^{v})$ where $u, v \in \mathbb{R}\langle S\rangle$. Clearly, $w = u + v$ when S is commutative. However it is not easy to compute w in the case of noncommutativity. In the present paper, we suggest a closed formula of Hausdorff series in the semigroup ring $\mathbb{R}\langle S\rangle$ and show that in the Dynkin formula, all elements excluding the first two terms are a polynomial of x and y, as $f(x,y)$. It would be interesting to held the problem for some other noncommutative semigroups.

References

[1] Campbell, J.E. 1897. On a law of combination of operators bearing on the theory of continuous transformation groups. Proc. London Math. Soc., 28(1897), 381-390.

[2] Campbell, J.E. 1897. On a law of combination of operators (second paper). Proc. London Math. Soc., 29(1897), 14-32.

[3] Baker, H. F. 1905. Alternants and continuous groups. Proc. London Math. Soc. II, 3(1905), 24-47.

[4] Hausdorff, F. 1906. Die symbolische Exponential Formel in der Gruppentheorie. Berichte über die Verhandlungen Sachsischen Akademie der Wisssenchaften zu Leipzig, 58(1906), 19-48.

[5] Dynkin, E.B. 1947. Evaluation of the coefficients of the Campbell-Hausdorff formula. Dokl. Akad. Nauk. SSSR., 57(1947), 323-326.

[6] Gerritzen, L. 2003. Taylor expansion of noncommutative power series with an application to the Hausdorff series. J. Reine Angew. Math., 556(2003), 113-125.

[7] Kurlin, V. 2007. The Baker-Campbell-Hausdorff formula in the free metabelian Lie algebra. J. Lie Theory. 17(2007), 525–538.

[8] Drensky, V. and Fındık, Ş. 2012. Inner and Outer Automorphisms of Free Metabelian Nilpotent Lie Algebras. Commun. Alg., 40(2012), 4389-4403.

[9] Baker, H. F. 1901. On the exponential theorem for a simply transitive continuous group, and the calculation of the finite equations from the constants of structure. Proc. London Math. Soc., 34(1901), 91-127.

[10] Fındık, Ş. and Kelekci, O. 2020. Hausdorff series in a semigroup ring. Int. J. Alg. Comp., 30(2020), 853-859.

[11] Howie, J.M. 1995. Fundamentals of Semigroup Theory. Clarendon Press, Oxford, 351s.