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ON THE (s, t)-PELL AND (s, t)-PELL-LUCAS MATRİX SEQUENCES

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Abstract

Number sequence matrices is a widely studied subject in matrix analysis. Especially number sequence matrices whose entries are well-known number sequences have become a very interesting research subject in recent years. We have seen many studies on the different number sequences in the last years. Fibonacci and Lucas number sequences are the best of these number sequences. In this sequences each term is the sum of two previous terms, with initial values $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ respectively. In Pell and Pell-Lucas number sequences, n*th* term of the sequence is equal to the sum of (n-2)*th* term and two times (n-1)*th* term. In literature, many proporties belong to number and matrix sequences constructed by recursion relations like these sequences. In this study, we present some important relationships between (s, t)-Pell and (s, t)-Pell-Lucas matrix sequences. Some identities for (s, t)-Pell and (s, t)-Pell-Lucas sequences are obtained by using these matrix sequences. Furthermore, we give the Binet Formulas for nth (s, t)-Pell and (s, t)-Pell-Lucas sequences. And in this formulas we will determine some relations between (s, t)-Pell and (s, t)-Pell-Lucas sequences.

Key Words: *(s, t)-Pell sequence, (s, t)-Pell-Lucas sequence, (s, t)-Pell matrix sequence, (s, t)-Pell-Lucas matrix sequence.*

(s, t)-PELL VE (s, t)-PELL-LUCAS MATRİS DİZİLERİ ÜZERİNE

Özet

Sayı dizili matrisler, matris analizinde yaygın bir çalışma alanına sahiptir. Özellikle elemanları iyi bilinen sayı dizilerinden oluşan sayı dizili matrisler, günümüzde çok ilgi çekici olmuşlardır. Son yıllarda farklı sayı dizileri üzerine pek çok çalışma görmekteyiz. Fibonacci ve Lucas sayı dizileri, bu sayı dizilerinin en önemlilerindendir. Bu dizinin her bir terimi $F_0 = 0$, $F_1 = 1$ ve $L_0 = 2$, $L_1 = 1$ olmak üzere, kendinden önceki iki terimin toplamıdır. Benzer şekilde Pell ve Pell-Lucas sayı dizileri de kendinden önceki terimin iki katı ile iki önceki terimin toplamıdır. Literatürde buna benzer sayı dizi ve matrislerinin bazı özellikleri verilmiştir. Biz de bu çalışmada, (s, t)-Pell ve (s, t)-Pell-Lucas matris dizileri arasındaki bazı önemli ilişkileri sunacağız. Bu matris dizilerini kullanarak (s, t)-Pell ve (s, t)-Pell-Lucas sayı dizileri için bazı eşitlikleri ele alacağız. Ayrıca, (s, t)-Pell ve (s, t)-Pell-Lucas sayı dizileri için Binet Formülünü vereceğiz ve bu formüllerle (s, t)-Pell ve (s, t)-Pell-Lucas matrisleri arasındaki bazı ilişkileri göstereceğiz.

Anahtar Kelimeler: *(s, t)-Pell sayıları, (s, t)-Pell-Lucas sayıları, (s, t)-Pell matrisi, (s, t)-Pell-Lucas matrisi.*

1. INTRODUCTION

Fibonacci and Lucas numbers are the terms of the sequences 0, 1, 1, 2, 3, 5, … and 2, 1, 3, 4, 7, 11, … wherein each term is the sum of the previous terms, beginning with the values $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ respectively. In the literature, in [6, 7], there are the some generalizations of the Fibonacci, Pell and Jacobsthal families. For instance, in [2], Falcon and Plaza introduce *k*-Fibonacci sequence by using Fibonacci and Pell sequences. İn [4, 5], İpek defined (s,t)-Fibonacci and (s,t)-Lucas matrix sequences by using (s,t)-Fibonacci and (s,t)-Lucas sequences. He also gave some properties related to these matrix sequences. İn [1], Catarino and Vasco present some basic proporties involving the *k*-Pell numbers. İn [8], Uygun establish (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas matrix sequences and present some important relationships between these matrix sequences.

In this study, we present some important relationships between (s, t) Pell and (s, t) Pell-Lucas matrix sequences. Some identities for (s, t)-Pell and (s, t)-Pell-Lucas sequences are obtained by using these matrix sequences. Furthermore, we give the Binet Formulas for nth (s, t)-Pell and (s, t)-Pell-Lucas sequences. And in this formulas we will determine some relations between (s, t)-Pell and (s, t)-Pell-Lucas sequences.

Definition: For any real numbers s, t; the (s,t)-Pell and the (s,t)-Pell-Lucas sequences are defined recurrently by

$$P_{n+1}(s,t) = 2sP_n(s,t) + tP_{n-1}(s,t), \quad P_0(s,t) = 0, P_1(s,t) = 1, \; n \geq 1$$

and

$$Q_{n+1}(s,t) = 2sQ_n(s,t) + tQ_{n-1}(s,t), \quad Q_0(s,t) = 2, Q_1(s,t) = 2s, \; n \geq 1$$

respectively, where $s > 0, t \neq 0$ *and* $s^2 + t > 0$. If s=1, t=1 the classic Pell and Pell-Lucas sequences are obtained.

Definition: For any real numbers s, t; the (s,t)-Pell matrix and the (s,t)-Pell-Lucas matrix sequences are defined recurrently by

$$\mathbf{P}_{n+1}(s,t) = 2s\mathbf{P}_n(s,t) + t\mathbf{P}_{n-1}(s,t), \quad \mathbf{P}_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \mathbf{P}_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}, \; n \geq 1$$

and

$$\mathbf{Q}_{n+1}(s,t) = 2s\mathbf{Q}_n(s,t) + t\mathbf{Q}_{n-1}(s,t), \quad \mathbf{Q}_0(s,t) = \begin{pmatrix} 2s & 2 \\ 2t & -2s \end{pmatrix}, \ \mathbf{Q}_1(s,t) = \begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}, \ n \geq 1$$

respectively, where $s > 0, t \neq 0 \ and \ s^2 + t > 0.$

Lemma: For any integer $m, n \geq 0$

$$\mathbf{P}_{m+n}(s,t) = \mathbf{P}_m(s,t)\mathbf{P}_n(s,t) \text{ [3].}$$

Lemma: For any integer $n \geq 0$

$$\mathbf{Q}_{n+1}(s,t) = \mathbf{Q}_1(s,t)\mathbf{P}_n(s,t) \text{ and } \mathbf{Q}_{n+1}(s,t) = \mathbf{Q}_n(s,t)\mathbf{P}_1(s,t) \text{ [3].}$$

2. MAIN RESULTS

Theorem: For any integer $n \geq 0$ we have

$$\mathbf{P}_n(s,t) = \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix}$$

Proof: By considering induction steps,

For n=0, $\mathbf{P}_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{-1}(s,t) = \frac{1}{t}, \ P_0(s,t) = 0, \ P_1(s,t) = 1$

For n=1, $\mathbf{P}_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$

Let us suppose that the equality holds for all $n = N \in$ □$^{+}$. To end up the prof, we have to show that the case also holds for n=N+1. Therefore we can write,

$$\mathbf{P}_{n+1}(s,t) = 2s\mathbf{P}_n(s,t) + t\mathbf{P}_{n-1}(s,t)$$

$$\begin{pmatrix} P_{n+2}(s,t) & P_{n+1}(s,t) \\ tP_{n+1}(s,t) & tP_n(s,t) \end{pmatrix} = 2s\begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix} + t\begin{pmatrix} P_n(s,t) & P_{n-1}(s,t) \\ tP_{n-1}(s,t) & tP_{n-2}(s,t) \end{pmatrix}$$

$$= \begin{pmatrix} 2sP_{n+1}(s,t) + tP_n(s,t) & 2sP_n(s,t) + tP_{n-1}(s,t) \\ t\left(2sP_n(s,t) + tP_{n-1}(s,t)\right) & t\left(2sP_{n-1}(s,t) + tP_{n-2}(s,t)\right) \end{pmatrix}$$

Hence the result.

Theorem: For any integer $m,\ n \geq 0$ we get

$$P_{m+n+1}(s,t) = P_{m+1}(s,t)P_{n+1}(s,t) + tP_m(s,t)P_n(s,t)$$

$$P_{m+n}(s,t) = P_m(s,t)P_{n+1}(s,t) + tP_{m-1}(s,t)P_n(s,t)$$

$$P_{m+n}(s,t) = P_{m+1}(s,t)P_n(s,t) + tP_m(s,t)P_{n-1}(s,t)$$

$$P_{m+n-1}(s,t) = P_m(s,t)P_n(s,t) + tP_{m-1}(s,t)P_{n-1}(s,t)$$

Proof:

$$\mathbf{P}_{m+n}(s,t) = \begin{pmatrix} P_{m+n+1}(s,t) & P_{m+n}(s,t) \\ tP_{m+n}(s,t) & tP_{m+n-1}(s,t) \end{pmatrix} = \begin{pmatrix} P_{m+1}(s,t) & P_m(s,t) \\ tP_m(s,t) & tP_{m-1}(s,t) \end{pmatrix} \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix} = \mathbf{P}_m(s,t)\mathbf{P}_n(s,t)$$

$$\begin{pmatrix} P_{m+n+1}(s,t) & P_{m+n}(s,t) \\ tP_{m+n}(s,t) & tP_{m+n-1}(s,t) \end{pmatrix} = \begin{pmatrix} P_{m+1}(s,t)P_{n+1}(s,t) + tP_m(s,t)P_n(s,t) & P_{m+1}(s,t)P_n(s,t) + tP_m(s,t)P_{n-1}(s,t) \\ t\left(P_m(s,t)P_{n+1}(s,t) + tP_{m-1}(s,t)P_n(s,t)\right) & t\left(P_m(s,t)P_n(s,t) + tP_{m-1}(s,t)P_{n-1}(s,t)\right) \end{pmatrix}$$

Theorem: For any integer $n \geq 0$ we have

$$\mathbf{P}_n(s,t) = \mathbf{P}_1^n(s,t)$$

Proof: By considering induction steps,

For n=1, $\mathbf{P}_1(s,t) = \mathbf{P}_1^1(s,t)$

Let us suppose that the equality holds for all $n = N \in \mathbb{N}^{+}$. To end up the prof, we have to show that the case also holds for n=N+1. Therefore we can write,

$$\mathbf{P}_1^{N+1}(s,t) = \mathbf{P}_1^N(s,t)\mathbf{P}_1(s,t) = \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix} \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2sP_{n+1}(s,t) + tP_n(s,t) & P_{n+1}(s,t) \\ t\left(2sP_n(s,t) + tP_{n-1}(s,t)\right) & tP_n(s,t) \end{pmatrix} = \begin{pmatrix} P_{n+2}(s,t) & P_{n+1}(s,t) \\ tP_{n+1}(s,t) & tP_n(s,t) \end{pmatrix}$$

$$= \mathbf{P}_{N+1}(s,t)$$

Hence the result.

Theorem: For any integer $n \geq 0$ we have

$$det\mathbf{P}_n(s,t) = (t)^n$$

Proof:

$\mathbf{P}_1(s,t)=\begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$ and $det\mathbf{P}_1=t$. From $\mathbf{P}_n(s,t)=\mathbf{P}_1^n(s,t)$

$$det\mathbf{P}_n(s,t)=\det\left(\mathbf{P}_1^n(s,t)\right)=\left(det\mathbf{P}_1(s,t)\right)^n=\left(t\right)^n$$

Theorem: For any integer $n\geq 0$ we have

$$\mathbf{Q}_n(s,t)=\begin{pmatrix} Q_{n+1}(s,t) & Q_n(s,t) \\ tQ_n(s,t) & tQ_{n-1}(s,t) \end{pmatrix}$$

Proof: By considering induction steps,

For n=0, $\mathbf{Q}_0(s,t)=\begin{pmatrix} 2s & 2 \\ 2t & -2s \end{pmatrix}$, $Q_{-1}(s,t)=-\dfrac{2s}{t}$, $Q_0(s,t)=2$, $Q_1(s,t)=2s$

For n=1, $\mathbf{Q}_1(s,t)=\begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}$

Let us suppose that the equality holds for all $n=N\in\mathbb{N}^{+}$. To end up the prof, we have to show that the case also holds for n=N+1. Therefore we can write,

$$\mathbf{Q}_{n+1}\left(s,t\right)=2s\mathbf{Q}_n\left(s,t\right)+t\mathbf{Q}_{n-1}\left(s,t\right)$$

$$\begin{pmatrix} Q_{n+2}(s,t) & Q_{n+1}(s,t) \\ tQ_{n+1}(s,t) & tQ_n(s,t) \end{pmatrix}=2s\begin{pmatrix} Q_{n+1}(s,t) & Q_n(s,t) \\ tQ_n(s,t) & tQ_{n-1}(s,t) \end{pmatrix}+t\begin{pmatrix} Q_n(s,t) & Q_{n-1}(s,t) \\ tQ_{n-1}(s,t) & tQ_{n-2}(s,t) \end{pmatrix}$$

$$=\begin{pmatrix} 2sQ_{n+1}(s,t)+tQ_n(s,t) & 2sQ_n(s,t)+tQ_{n-1}(s,t) \\ t\left(2sQ_n(s,t)+tQ_{n-1}(s,t)\right) & t\left(2sQ_{n-1}(s,t)+tQ_{n-2}(s,t)\right) \end{pmatrix}$$

Hence the result.

Theorem: For any integer $n\geq 0$ we have

$$det\mathbf{Q}_{n+1}(s,t)=\left(4s^2t+4t^2\right)\left(t\right)^n$$

Proof:

$\mathbf{Q}_1(s,t)=\begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}$ and $det\mathbf{Q}_1(s,t)=4s^2t+4t^2$. From $\mathbf{P}_n(s,t)=\mathbf{P}_1^n(s,t)$

$$det\mathbf{Q}_{n+1}(s,t)=\det\left(\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\right)=det\left(\mathbf{Q}_1(s,t)\mathbf{P}_1^n(s,t)\right)$$

$$=\left(det\mathbf{Q}_1(s,t)\right)\left(det\mathbf{P}_1(s,t)\right)^n=\left(4s^2t+4t^2\right)(t)^n$$

Theorem: For any integer $n\geq 0$ we have

$$\mathbf{Q}_{n+1}(s,t)=2s\mathbf{P}_{n+1}(s,t)+2t\mathbf{P}_n(s,t)$$

Proof: By considering induction steps,

For *n*=0, $\mathbf{Q}_1(s,t)=2s\mathbf{P}_1(s,t)+2t\mathbf{P}_0(s,t)=\begin{pmatrix}4s^2+2t & 2s\\ 2st & 2t\end{pmatrix}=2s\begin{pmatrix}2s & 1\\ t & 0\end{pmatrix}+2t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Let us suppose that the equality holds for all $n=N\in\square^{+}$. To end up the prof, we have to show that the case also holds for *n*=*N*+1. Therefore we can write,

$$\mathbf{Q}_{N+1}(s,t)=\mathbf{Q}_1(s,t)\mathbf{P}_N(s,t)=\left(2s\mathbf{P}_1(s,t)+2t\mathbf{P}_0(s,t)\right)\mathbf{P}_N(s,t)$$

$$=2s\mathbf{P}_{N+1}(s,t)+2t\mathbf{P}_N(s,t)$$

Hence the result.

Theorem: For any integer $m,\ n\geq 0$ we get

$$\mathbf{P}_m(s,t)\mathbf{Q}_{n+1}(s,t)=\mathbf{Q}_{n+1}(s,t)\mathbf{P}_m(s,t)$$

Proof:

$$\begin{aligned}\mathbf{P}_m(s,t)\mathbf{Q}_{n+1}(s,t)&=\mathbf{P}_m(s,t)\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\\&=\mathbf{P}_m(s,t)\left(2s\mathbf{P}_1(s,t)+2t\mathbf{P}_0(s,t)\right)\mathbf{P}_n(s,t)\\&=2s\mathbf{P}_{n+m+1}(s,t)+2t\mathbf{P}_{n+m}(s,t)\\&=\left(2s\mathbf{P}_1(s,t)+2t\mathbf{P}_0(s,t)\right)\mathbf{P}_{n+m}(s,t)\\&=\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\mathbf{P}_m(s,t)\\&=\mathbf{Q}_{n+1}(s,t)\mathbf{P}_m(s,t)\end{aligned}$$

Theorem: For any integer $n\geq 0$ we have

$$\mathbf{P}_n(s,t)=\mathbf{Q}_{n+2}(s,t)\mathbf{Q}_2^{-1}(s,t)$$

Proof: By considering induction steps,

For *n*=0,

$$\begin{aligned}\mathbf{P}_0(s,t) &= \mathbf{Q}_2(s,t)\mathbf{Q}_2^{-1}(s,t)\\ &= \mathbf{Q}_1(s,t)\mathbf{P}_1(s,t)\left(\mathbf{Q}_1(s,t)\mathbf{P}_1(s,t)\right)^{-1}\\ &= \mathbf{Q}_1(s,t)\mathbf{P}_1(s,t)\mathbf{P}_1^{-1}(s,t)\mathbf{Q}_1^{-1}(s,t)\\ &= I_2\end{aligned}$$

Let us suppose that the equality holds for all $n = N \in \mathbb{N}^{+}$. To end up the prof, we have to show that the case also holds for n=N+1. Therefore we can write,

$$\mathbf{Q}_{N+3}(s,t)\mathbf{Q}_2^{-1}(s,t) = \mathbf{Q}_1(s,t)\mathbf{P}_{N+2}(s,t)\mathbf{P}_1^{-1}(s,t)\mathbf{Q}_1^{-1}(s,t)$$

$$= \mathbf{P}_{N+2}(s,t)\mathbf{P}_1^{-1}(s,t) = \mathbf{P}_{N+1}(s,t)\mathbf{P}_1(s,t)\mathbf{P}_1^{-1}(s,t) = \mathbf{P}_{N+1}(s,t)$$

Hence the result.

Theorem: For any integer $n \geq 0$ we get

a) $\mathbf{Q}_{n+1}^2(s,t) = \mathbf{Q}_1^2(s,t)\mathbf{P}_{2n}(s,t)$

b) $\mathbf{Q}_{n+1}^2(s,t) = \mathbf{Q}_1(s,t)\mathbf{Q}_{2n+1}(s,t)$

c) $\mathbf{Q}_{2n+1}(s,t) = \mathbf{P}_n(s,t)\mathbf{Q}_{n+1}(s,t)$

Proof:

a) $\mathbf{Q}_{n+1}^2(s,t) = \mathbf{Q}_{n+1}(s,t)\mathbf{Q}_{n+1}(s,t) = \mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)$

$$= \mathbf{Q}_1(s,t)\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\mathbf{P}_n(s,t) = \mathbf{Q}_1^2(s,t)\mathbf{P}_{2n}(s,t)$$

b) $\mathbf{Q}_{n+1}^2(s,t) = \mathbf{Q}_1^2(s,t)\mathbf{P}_{2n}(s,t) = \mathbf{Q}_1(s,t)\mathbf{Q}_1(s,t)\mathbf{P}_{2n}(s,t) = \mathbf{Q}_1(s,t)\mathbf{Q}_{2n+1}(s,t)$

c) $\mathbf{Q}_{2n+1}(s,t) = \mathbf{Q}_1(s,t)\mathbf{P}_{2n}(s,t) = \mathbf{Q}_1(s,t)\mathbf{P}_n(s,t)\mathbf{P}_n(s,t) = \mathbf{P}_n(s,t)\mathbf{Q}_1(s,t)\mathbf{P}_n(s,t) = \mathbf{P}_n(s,t)\mathbf{Q}_{n+1}(s,t)$

Theorem: For any integer $m,\ n \geq 0$ we get

$$2P_{m+n}(s,t) = P_n(s,t)Q_m(s,t) + P_m(s,t)Q_n(s,t)$$

Proof:

$$P_{m+n}(s,t) = P_m(s,t)P_{n+1}(s,t) + tP_{m-1}(s,t)P_n(s,t)$$

$$P_{m+n}(s,t) = P_{m+1}(s,t)P_n(s,t) + tP_m(s,t)P_{n-1}(s,t)$$

When the equations are gathered to the side,

$$2P_{m+n}(s,t) = P_m(s,t)\left(P_{n+1}(s,t) + tP_{n-1}\right)(s,t) + P_n(s,t)\left(P_{m+1}(s,t) + tP_{m-1}(s,t)\right)$$

$$= P_m(s,t)\left(2sP_n(s,t) + 2tP_{n-1}(s,t)\right) + P_n(s,t)\left(2sP_m(s,t) + 2tP_{m-1}(s,t)\right)$$

$$= P_m(s,t)Q_n(s,t) + P_n(s,t)Q_m(s,t)$$

Theorem: For any integer $m,\ n \geq 0$ we get

$$Q_{m+n+1}(s,t) = P_{m+1}(s,t)Q_{n+1}(s,t) + tP_m(s,t)Q_n(s,t)$$

Proof:

$$2sP_{m+n+1}(s,t) = 2sP_{m+1}(s,t)P_{n+1}(s,t) + 2stP_m(s,t)P_n(s,t)$$

$$2tP_{m+n}(s,t) = 2tP_{m+1}(s,t)P_n(s,t) + 2t^2P_m(s,t)P_{n-1}(s,t)$$

When the equations are gathered to the side,

$$2sP_{m+n+1}(s,t) + 2tP_{m+n}(s,t) = P_{m+1}(s,t)\left(2sP_{n+1}(s,t) + 2tP_n(s,t)\right) + tP_m(s,t)\left(2sP_n(s,t) + 2tP_{n-1}(s,t)\right)$$

$$Q_{m+n+1}(s,t) = P_{m+1}(s,t)Q_{n+1}(s,t) + tP_m(s,t)Q_n(s,t)$$

Theorem: For any integer $n \geq 0$ we have

$$Q_{n+2}^2(s,t) + tQ_{n+1}^2(s,t) = (4s^2 + 4t)P_{2n+3}(s,t)$$

$$Q_{n+2}^2(s,t) + tQ_{n+1}^2(s,t) = Q_{2n+4}(s,t) + tQ_{2n+2}(s,t)$$

$$Q_{2n}(s,t) = P_n(s,t)Q_{n+1}(s,t) + tP_{n-1}(s,t)Q_n(s,t)$$

Proof:

From $\mathbf{Q}_{n+1}^2(s,t) = \mathbf{Q}_1^2(s,t)\mathbf{P}_{2n}(s,t)$

$$\begin{pmatrix} Q_{n+2}(s,t) & Q_{n+1}(s,t) \\ tQ_{n+1}(s,t) & tQ_n(s,t) \end{pmatrix}^2 = \begin{pmatrix} 4s^2 + 2t & 2s \\ 2st & 2t \end{pmatrix}^2 \begin{pmatrix} P_{2n+1}(s,t) & P_{2n}(s,t) \\ tP_{2n}(s,t) & tP_{2n-1}(s,t) \end{pmatrix}$$

we get $Q_{n+2}^2(s,t) + tQ_{n+1}^2(s,t) = (4s^2 + 4t)P_{2n+3}(s,t).$

From $\mathbf{Q}^2_{n+1}(s,t) = \mathbf{Q}_1(s,t)\mathbf{Q}_{2n+1}(s,t)$

$$\begin{pmatrix} Q_{n+2}(s,t) & Q_{n+1}(s,t) \\ tQ_{n+1}(s,t) & tQ_n(s,t) \end{pmatrix}^2 = \begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}\begin{pmatrix} Q_{2n+2}(s,t) & Q_{2n+1}(s,t) \\ tQ_{2n+1}(s,t) & tQ_{2n}(s,t) \end{pmatrix}$$

we get $Q^2_{n+2}(s,t) + tQ^2_{n+1}(s,t) = Q_{2n+4}(s,t) + tQ_{2n+2}(s,t).$

From $\mathbf{Q}_{2n+1}(s,t) = \mathbf{P}_n(s,t)\mathbf{Q}_{n+1}(s,t)$

$$\begin{pmatrix} Q_{2n+1}(s,t) & Q_{2n}(s,t) \\ tQ_{2n}(s,t) & tQ_{2n-1}(s,t) \end{pmatrix} = \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix}\begin{pmatrix} Q_{n+2}(s,t) & Q_{n+1}(s,t) \\ tQ_{n+1}(s,t) & tQ_n(s,t) \end{pmatrix}$$

We get $Q_{2n}(s,t) = P_n(s,t)Q_{n+1}(s,t) + tP_{n-1}(s,t)Q_n(s,t).$

Theorem: For any integer $n \geq 0$ the Binet Formulas for nth (s,t)-Pell number and (s,t)-Pell-Lucas number are given by

$$P_n(s,t) = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

and

$$Q_n(s,t) = r_1^n + r_2^n$$

Proof:

The proof of first equality is obvious from the principle of induction n. Let us prove second equality.

From $Q_n(s,t) = 2sP_n(s,t) + 2tP_{n-1}(s,t)$

$$Q_n(s,t) = 2s\frac{r_1^n - r_2^n}{r_1 - r_2} + 2t\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}$$

$$= 2\left(\frac{r_1^n\left(\frac{sr_1 + t}{r_1}\right) - r_2^n\left(\frac{sr_2 + t}{r_2}\right)}{r_1 - r_2} \right) = 2\left(\frac{r_1^n\left(\frac{r_1^2 - r_1r_2}{2r_1}\right) - r_2^n\left(\frac{r_2^2 - r_1r_2}{2r_2}\right)}{r_1 - r_2} \right)$$

$$= 2\left(\frac{r_1^n\left(\frac{r_1 - r_2}{2}\right) - r_2^n\left(\frac{r_2 - r_1}{r_2}\right)}{r_1 - r_2} \right) = r_1^n + r_2^n$$

Theorem: For any integer $n \geq 0$ we have

$$\mathbf{P}_n(s,t) = \left(\frac{\mathbf{P}_1(s,t) - r_2\mathbf{P}_0(s,t)}{r_1 - r_2}\right) r_1^n - \left(\frac{\mathbf{P}_1(s,t) - r_1\mathbf{P}_0(s,t)}{r_1 - r_2}\right) r_2^n$$

and

$$\mathbf{Q}_{n+1}(s,t) = \left(\frac{\mathbf{Q}_2(s,t) - r_2\mathbf{Q}_1(s,t)}{r_1 - r_2}\right) r_1^n - \left(\frac{\mathbf{Q}_2(s,t) - r_1\mathbf{Q}_1(s,t)}{r_1 - r_2}\right) r_2^n$$

Proof:

$$\mathbf{P}_n(s,t) = \frac{r_1^n}{r_1 - r_2}\left[\begin{pmatrix} 2s - r_2 & 1 \\ t & -r_2 \end{pmatrix}\right] - \frac{r_2^n}{r_1 - r_2}\left[\begin{pmatrix} 2s - r_1 & 1 \\ t & -r_1 \end{pmatrix}\right]$$

$$= \frac{1}{r_1 - r_2}\left[\begin{pmatrix} 2sr_1^n - 2sr_2^n - r_2 r_1^n + r_1 r_2^n & r_1^n - r_2^n \\ tr_1^n - tr_2^n & -r_2 r_1^n + r_1 r_2^n \end{pmatrix}\right]$$

$$= \begin{pmatrix} P_{n+1}(s,t) & P_n(s,t) \\ tP_n(s,t) & tP_{n-1}(s,t) \end{pmatrix}$$

By using the above method,

$$\mathbf{Q}_{n+1}(s,t) = \left(\frac{\mathbf{Q}_2(s,t) - r_2\mathbf{Q}_1(s,t)}{r_1 - r_2}\right) r_1^n - \left(\frac{\mathbf{Q}_2(s,t) - r_1\mathbf{Q}_1(s,t)}{r_1 - r_2}\right) r_2^n$$

can be clearly seen.

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