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On the Riemannian Curvature Invariants of Totally η-Umbilical Real Hypersurfaces of a Complex Space Form

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Abstract: Some relations involving the Ricci and scalar curvatures of totally η-umbilical real hypersurfaces of a complex space form are examined. With the help of these relations, some results on totally η-umbilical real hypersurfaces of a complex space form are given. Furthermore, these results are discussed on totally η-umbilical real hypersurfaces of the 6-dimensional complex space form. Some characterizations dealing totally η-umbilical real hypersurfaces of the 6-dimensional complex space form are obtained.

Keywords: Curvature, hypersurface, complex space form.

1. Introduction

Since Riemannian curvature invariants play a significant role in classifying Riemannian manifolds and their submanifolds, borrowing a term from biology, Chen called these invariants as Riemannian DNA for Riemannian manifolds in [7–9] and established some important relations between the intrinsic curvature invariants and extrinsic curvature invariants for submanifolds of a Riemannian manifolds in 1990s. cf. [4–6]. Recently, many authors investigated these kind of inequalities on submanifolds of various Riemannian manifolds such as Hermitian manifolds, contact metric manifolds and Riemannian product manifolds cf. [1, 2, 12, 13, 18, 20, 24] etc.

On the other hand, the study of real hypersurfaces in complex space forms has been an attractive topic in differential geometry since this kind of hypersurfaces admits a almost contact structure induced from the almost complex structure defined on a complex space form. These properties present us very rich geometric view point. Real hypersurface of complex space forms are examined by various geometers cf. [3, 10, 14, 15] etc. In [21], Tashiro and Tachibana proved that there do not exist any totally umbilical real hypersurface of non flat complex space and therefore the authors introduced the notion of totally η-umbilical real hypersurface as follows:

A real hypersurface of a complex space form is said to be η-umbilical if the shape operator

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A_N satisfies the following relation:

$$A_N X = aX + b\eta(X)\xi \tag{1}$$

for any tangent vector field X on M and some functions a and b. Here ξ is known as the structure vector field on tangent space of real hypersurface.

Later, totally η-umbilical real hypersurfaces of a complex projective space and a complex hyperbolic space are determined by Takagi [22] and Montiel [19]. Totally η-umbilical real hypersurfaces and ruled real hypersurfaces of a complex space form by the help of holomorphic distribution are investigated by Kon in [16].

Motivated by these facts, we study the Riemannian curvature invariants for totally η-umbilical real hypersurfaces of a complex space form and we obtain some relations for these hypersurfaces. With the help of these relations, we get some special characterizations for these hypersurfaces of 6-dimensional complex space forms.

2. Preliminaries

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and Π be a plane section spanned by linearly independent vector fields X and Y on $\widetilde{M}$. The sectional curvature of Π denoted by $\widetilde{K}(\Pi)$ and it is defined by [17]

$$\widetilde{K}(\Pi) \equiv \widetilde{K}(X,Y) = \frac{\widetilde{g}\left(\widetilde{R}(X,Y)Y,X\right)}{g(X,X)\,g(Y,Y) - g(X,Y)^2} \tag{2}$$

where $\widetilde{R}$ denotes the Riemannian curvature tensor of $\widetilde{M}$. The manifold $(\widetilde{M},\widetilde{g})$ is called as a space form if the value of $\widetilde{K}$ is constant for any tangent plane Π at every point $p \in \widetilde{M}$. A space form of constant curvature c is generally denoted by $\widetilde{M}(c)$ and the following equation holds

$$\widetilde{R}(X,Y)Z = \frac{c}{4}\left[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\right]. \tag{3}$$

We note that a space form $\widetilde{M}(c)$ becomes

- (i) The Euclidean space if $c = 0$.
- (ii) The sphere if $c > 0$.
- (iii) The hyperbolic space if $c < 0$.

Now let $(\widetilde{M},\widetilde{g})$ be a Riemannian manifold and $\{e_1,\ldots,e_m\}$ be an orthonormal basis for $T_p\widetilde{M}$ at a point $p \in \widetilde{M}$. The Ricci tensor $\widetilde{\mathrm{Ric}}$ is defined by

$$\widetilde{\mathrm{Ric}}(X,Y) = \sum_{j=1}^{m} \widetilde{g}\left(\widetilde{R}(e_j,X)Y,e_j\right) \tag{4}$$

for any $X, Y \in T_p\widetilde{M}$. For a fixed $i \in \{1, \ldots, m\}$, we have

$$\widetilde{\mathrm{Ric}}\,(e_i, e_i) \equiv \widetilde{\mathrm{Ric}}\,(e_i) = \sum_{j \neq i}^{m} \widetilde{K}(e_i, e_j). \tag{5}$$

Furthermore, the scalar curvature $\widetilde{\tau}$ at p is defined by

$$\widetilde{\tau}\,(p) = \sum_{i<j} \widetilde{K}(e_i, e_j). \tag{6}$$

Let Π_k be a k-plane subsection of $T_p\widetilde{M}$ and X be a unit vector in Π_k. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of Π_k such that $e_1 = X$. Then, the Ricci curvature Ric_{Π_k} of Π_k at X is defined by

$$\mathrm{Ric}_{\Pi_k}(X) = \widetilde{K}_{12} + \widetilde{K}_{13} + \cdots + \widetilde{K}_{1k}. \tag{7}$$

Here, $\mathrm{Ric}_{\Pi_k}(X)$ is called as k-Ricci curvature [6]. Thus for each fixed e_i, $i \in \{1, ..., k\}$ we get

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^{k} \widetilde{K}(e_i, e_j). \tag{8}$$

The scalar curvature $\widetilde{\tau}\,(\Pi_k)$ of the k-plane section Π_k is given by

$$\widetilde{\tau}\,(\Pi_k) = \sum_{1 \leq i < j \leq k} \widetilde{K}(e_i, e_j). \tag{9}$$

From (9), we have

$$\widetilde{\tau}\,(\Pi_k) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j \neq i}^{k} \widetilde{K}(e_i, e_j) = \frac{1}{2} \sum_{i=1}^{n} \mathrm{Ric}_{\Pi_k}(e_i). \tag{10}$$

Let (M, g) be an n-dimensional submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with the induced metric g from $\widetilde{g}$. The Gauss and Weingarten formulas are given respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h\,(X, Y) \text{ and } \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{11}$$

for all X, Y are any two tangent vector fields on the tangent bundle TM and N is the unit normal vector field on the normal bundle $T^{\perp}M$. Here, $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are, respectively, the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to the shape operator A by

$$\widetilde{g}\,(h\,(X, Y)\,, N) = g\,(A_N X, Y)\,. \tag{12}$$

Let R and $\widetilde{R}$ denotes the Riemannian curvature tensor fields of M and $\widetilde{M}$ respectively. The equation of Gauss is given by

$$g(R(X,Y)Z,W) = \widetilde{g}\big(\widetilde{R}(X,Y)Z,W\big) + \widetilde{g}(h(X,W),h(Y,Z)) - \widetilde{g}(h(X,Z),h(Y,W)) \tag{13}$$

for all $X,Y,Z,W \in TM$.

The mean curvature vector H is given by $H = \frac{1}{n}\,\mathrm{trace}(h)$. The submanifold M is called totally geodesic in $\widetilde{M}$ if $h = 0$, and minimal if $H = 0$. If $h(X,Y) = g(X,Y)H$ for all $X,Y \in TM$, then M is called totally umbilical [4].

Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r $(r = n+1,\ldots,m)$ belongs to an orthonormal basis $\{e_{n+1},\ldots,e_m\}$ of the normal space $T_p^{\perp}M$. Then we can write

$$h_{ij}^r = \widetilde{g}(h(e_i,e_j),e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n} \widetilde{g}(h(e_i,e_j),h(e_i,e_j)). \tag{14}$$

From (13), we have

$$K(e_i,e_j) = \widetilde{K}(e_i,e_j) + \sum_{r=n+1}^{m} \left(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\right) \tag{15}$$

where K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. Therefore, it follows from (15) that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2\|H\|^2 - \|h\|^2 \tag{16}$$

where

$$\widetilde{\tau}(T_pM) = \sum_{1\le i<j\le n} \widetilde{K}(e_i,e_j) \tag{17}$$

denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$.

In view of (16), we clearly have

$$\tau(p) \le \frac{1}{2}n^2\|H\|^2 + \widetilde{\tau}(T_pM). \tag{18}$$

The equality case of (18) satisfies if and only if M is totally geodesic [11].

An improved case of the inequality (18), the following theorem could be given:

Theorem 2.1 *[11, Theorem 4.2] For an n-dimensional submanifold M in a Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_pM) \tag{19}$$

with equality if and only if p is a totally umbilical point.

Now, we shall recall the Chen-Ricci inequality (20) in the following:

Theorem 2.2 *[11, Theorem 6.1] Let M be an n-dimensional submanifold of a Riemannian manifold. Then, the following statements are true.*

(a) *For any unit vector $X \in T_pM$, it follows that*

$$\operatorname{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_pM)}(X), \tag{20}$$

where $\widetilde{\operatorname{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of (20) is satisfied by a vector $X \in T_pM$ if and only if*

$$\begin{cases} h(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ 2h(X,X) = nH(p), \end{cases} \tag{21}$$

(c) *The equality case of (20) holds for all unit tangent vector $X \in T_pM$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

3. Real Hypersurfaces of Complex Space Forms

Let $\widetilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J, \widetilde{g})$ such that we have

$$J^2 = -I \tag{22}$$

and

$$\widetilde{g}(JX, JY) = \widetilde{g}(X,Y), \qquad X, Y \in T\widetilde{M}. \tag{23}$$

If J is integrable, that is, the Nijenhuis tensor $[J, J]$ of J vanishes then the almost Hermitian manifold is called a Hermitian manifold.

Let $(\widetilde{M}, J, \widetilde{g})$ be an almost Hermitian manifold and $\widetilde{\nabla}$ be the Riemannian connection of the Riemannian metric $\widetilde{g}$. The manifold is called a Kaehler manifold [23] if

$$\widetilde{\nabla} J = 0. \tag{24}$$

Similar to real space forms, in complex manifolds, we have the notion of complex space form. A Kaehler manifold $\widetilde{M}$ equipped with a Kaehler structure $(J, \widetilde{g}, \widetilde{\nabla})$, which has constant holomorphic sectional curvatures $4c$, is said to be a complex space form $\widetilde{M}(4c)$; and its Riemann curvature tensor $\widetilde{R}$ is given by [23]

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= c\{\widetilde{g}(X,W)\widetilde{g}(Y,Z) - \widetilde{g}(X,Z)\widetilde{g}(Y,W) \\ &\quad +\widetilde{g}(X,JZ)\widetilde{g}(JY,W) - \widetilde{g}(Y,JZ)\widetilde{g}(JX,W) \\ &\quad +2\widetilde{g}(X,JY)\widetilde{g}(JZ,W)\} \qquad (25)\end{aligned}$$

for any $X, Y, Z, W \in T\widetilde{M}$.

Let $\widetilde{M}(4c)$ be a $2n$-dimensional complex space form with constant holomorphic sectional curvature $4c$ and (M, g) be a real $(2n-1)$-dimensional hypersurface immersed in $\widetilde{M}(4c)$ with induced metric g. For a unit vector field $\xi \in TM$, we assume that $J\xi = N$, where N is the unit normal vector field. In this case, we write for any $X \in TM$ that

$$JX = \varphi X + \eta(X) N \quad \text{and} \quad JN = -\xi \qquad (26)$$

where φX is the tangential part of JX and η is $1-$ form on TM satisfying

$$\eta(X) = \widetilde{g}(JX, N) = g(X, \xi). \qquad (27)$$

For any real hypersurface M, there exist the following relations for any $X \in TM$:

$$\eta(\varphi X) = 0, \qquad (28)$$

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad (29)$$

$$\varphi\xi = 0. \qquad (30)$$

Furthermore, we have

$$g(\varphi X, Y) + g(X, \varphi Y) = 0$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \qquad (32)$$

From the above equalities, it is clear that the hypersurface M is an almost contact metric manifold with contact structure (φ, ξ, η, g). For more details, we refer to [16].

Let $\widetilde{\nabla}$ be the Riemannian connection of $\widetilde{M}(4c)$ and ∇ be the induced Riemannian connection on M. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(A_N X, Y) N, \qquad (33)$$

$$\widetilde{\nabla}_X N = -A_N X \qquad (34)$$

for any $X \in TM$ and $N \in T^{\perp}M$.

Using the fact that (φ, ξ, η, g) is the contact metric structure in the Gauss and Weingarten formulas, we have

$$\eta(\nabla_X \xi) = 0, \quad \nabla_X \xi = \varphi A_N X \tag{35}$$

and

$$(\nabla_X \varphi) Y = \eta(Y) A_N X - g(A_N X, Y) \xi. \tag{36}$$

Now let us denote the Riemannian curvature tensor field on M by R. From (13) and (25), we get

$$\begin{aligned} R(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X \\ &\quad - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} + g(A_N Y, Z)A_N X \\ &\quad - g(A_N X, Z)A_N Y \end{aligned} \tag{37}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\} \tag{38}$$

for any $X, Y, Z \in TM$ [16].

4. Main Results

Let $\widetilde{M}(4c)$ is an $2n$ dimensional complex space form and M be a real hypersurface of $\widetilde{M}(4c)$. Let us define a distribution T_0, so called holomorphic distribution on $\widetilde{M}(4c)$, given by

$$T_0 = \{X \in T_pM : \eta(X) = 0\}. \tag{39}$$

If T_0 is integrable and its integral manifold is a totally geodesic submanifold, then M is called as a ruled real hypersurface. A hypersurface M of $\widetilde{M}(4c)$ is said to be η-umbilical if the following relation holds:

$$AX = aX + b\eta(X)\xi \tag{40}$$

for any vector field $X \in TM$ and some functions a and b [21].

Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form. Let T_0 denotes the holomorphic distribution on M. Assume that we have $g(AX,Y) = ag(X,Y)$ for any $X, Y \in T_0$. Then we can consider an orthonormal basis $\{e_1, e_2, ..., e_{2n-2}, \xi\}$ such that the shape operator takes form as follows [16]:

$$A_N = \begin{pmatrix} a & \dots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a & h_{2n-2} \\ h_1 & \dots & h_{2n-2} & b \end{pmatrix} \tag{41}$$

where $h_i = g(A_N e_i, \xi)$ for $i \in \{1, \ldots, 2n\}$ and $b = g(A_N \xi, \xi)$.

Now we shall recall the following theorem:

Theorem 4.1 *[16, Theorem 3.1] Let M be a real hypersurface of a complex space form $\widetilde{M}(4c)$ and T_0 be the holomorphic distribution on M. If the following equation holds*

$$g(AX, Y) = ag(X, Y)$$

for any $X, Y \in T_0$, then M is either totally η-umbilical or it is a locally ruled real hypersurface.

Taking into consideration the above facts, we obtain followings:

Lemma 4.2 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$ and T_0 be the holomorphic distribution on M. Then we have the following equalities:*

(i) For any unit vector X in T_0, we have

$$\widetilde{Ric}_{T_pM}(X) = c\{2n+3\}. \tag{42}$$

(ii) For the structure vector field ξ of M, we have

$$\widetilde{Ric}_{T_pM}(\xi) = 2nc.$$

Proof Under the assumption, let us choose an orthonormal basis $\{e_1, e_2, ..., e_{2n-2}, \xi\}$ on TM. Putting $X = e_i$ and $Y = e_j$ in (37), we have

$$\widetilde{R}(e_i, e_j, e_j, e_i) = c\left\{1 + 3g(Je_j, e_i)^2\right\}. \tag{43}$$

Furthermore, if write ξ instead of e_j, then we get

$$\begin{aligned} \widetilde{R}(e_i, \xi, \xi, e_i) &= c\left\{1 + 3g(J\xi, e_i)^2\right\} \\ &= c\left\{1 + 3g(N, e_i)^2\right\} \\ &= c. \end{aligned} \tag{44}$$

Using the fact that

$$\widetilde{Ric}_{T_pM}(e_j) = \left[\sum_{i=1}^{2n-1} \widetilde{R}(e_i, e_j, e_j, e_i)\right] + \widetilde{R}(\xi, e_i, e_i, \xi) \tag{45}$$

and considering the equation (43) and (44), we obtain

$$\widetilde{Ric}_{T_pM}(e_j) = c\{2n+3\}. \tag{46}$$

Putting $X = e_j$ in (46), the proof of (i) statement is completed.

The proof of statement (ii) is straightforward by using the fact

$$\widetilde{Ric}_{T_pM}(\xi) = \left[\sum_{i=1}^{2n-1} \widetilde{R}(e_i, \xi, \xi e_i)\right]. \tag{47}$$

□

Taking into account of the Gauss equation and Lemma 4.2, we obtain the following lemma:

Lemma 4.3 *Let M be an $(2n+1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$ and T_0 be the holomorphic distribution on M. Then we have the following equalities:*

(i) For any unit vector X in T_0, we have

$$Ric(X) = (2n+1)c + (2n-3)a^2 + ab. \tag{48}$$

(ii) For the structure vector field ξ of M, we have

$$Ric(\xi) = (2n-2)c + (2n-2)ab. \tag{49}$$

Lemma 4.4 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$. Then we have*

$$H(p) = \frac{1}{2n-1}\left[\left(\sum_{i=1}^{2n-2} aN\right) + bN\right]. \tag{50}$$

Proof From the definition of mean curvature vector field, we write

$$H(p) = \frac{1}{2n-1}\left[\left(\sum_{i=1}^{2n-2} h(e_i, e_i)\right) + h(\xi, \xi)\right]. \tag{51}$$

On the other hand, we have

$$\begin{aligned} h(e_i, e_i) &= g(Ae_i, e_i)N \\ &= ag(e_i, e_i)N \\ &= aN \end{aligned} \tag{52}$$

and

$$\begin{aligned} h(\xi, \xi) &= g(A\xi, \xi)N \\ &= bN. \end{aligned} \tag{53}$$

If we put (52) and (53) in (51) we obtain the equation (50). □

From (9) and Lemma 4.2, we get the following lemma:

Lemma 4.5 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$. Then we have*

$$\widetilde{\tau}\left(T_pM\right) = \left(2n^2 + 3n - \frac{3}{2}\right)c. \tag{54}$$

From (16), Lemma 4.4 and Lemma 4.6, we obtain the following lemma:

Lemma 4.6 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$. Then we have*

$$\tau\left(p\right) = \left(n-1\right)\left(2n+2\right)c + \left(n-1\right)\left(2n-3\right)a^2 + 2\left(n-1\right)ab. \tag{55}$$

Proposition 4.7 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$. Then the following inequality holds:*

$$\left[\left(2n-4\right)a + b\right]^2 \geq -8c. \tag{56}$$

Proof Considering Lemma 4.2, Lemma 4.3 and Lemma 4.4 in (20), the proof is straightforward. □

For the special case $n = 3$, we have the following corollaries:

Corollary 4.8 *Let M be a real hypersurface of a 6-dimensional complex space form $\widetilde{M}$. Then we have*

$$\left(2a + b\right)^2 \geq -8c. \tag{57}$$

The equality case of (57) holds for all $p \in M$ if and only if M is totally geodesic and $\widetilde{M}$ is the complex Euclidean space.

Corollary 4.9 *Let M be a real hypersurface of a 6-dimensional complex space form $\widetilde{M}$. If $a = -\frac{b}{2}$ then $c \geq 0$.*

Proposition 4.10 *Let M be an $(2n-1)$-dimensional real hypersurface of a complex space form $\widetilde{M}(4c)$. Then the following inequality holds:*

$$\left(-2n+2\right)a^2 - b^2 \leq \left(6n+1\right)c. \tag{58}$$

Proof Using Lemma 4.4, Lemma 4.5, Lemma 4.6 in (20), the proof is straightforward. □

For the special case $n = 3$, we have the following corollaries:

Corollary 4.11 *Let M be a real hypersurface of a 6-dimensional complex space form $\widetilde{M}$. Then we have the following inequality:*

$$4a^2 + b^2 \geq -19c. \tag{59}$$

The equality case of this inequality holds if and only if M is a totally geodesic hypersurface of complex Euclidean space.

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