Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (2) (2022), 455 – 465
DOI : 10.15672/hujms.816436

Research Article

On n-absorbing prime ideals of commutative rings

Mohammed Issoual[1], Najib Mahdou[2], Moutu Abdou Salam Moutui*[3]

[1] CRMEF Khemisset, Morocco
[2] Laboratory of Modeling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco
[3] Division of Science, Technology, and Mathematics American University of Afghanistan, Kabul, Afghanistan

Abstract

This paper investigates the class of rings in which every n-absorbing ideal is a prime ideal, called n-AB ring, where n is a positive integer. We give a characterization of an n-AB ring. Next, for a ring R, we study the concept of $\Omega(R) = \{\omega_R(I); I$ is a proper ideal of $R\}$, where $\omega_R(I) = \min\{n; I$ is an n-absorbing ideal of $R\}$. We show that if R is an Artinian ring or a Prüfer domain, then $\Omega(R) \cap \mathbb{N}$ does not have any gaps (i.e., whenever $n \in \Omega(R)$ is a positive integer, then every positive integer below n is also in $\Omega(R)$). Furthermore, we investigate rings which satisfy property (**) (i.e., rings R such that for each proper ideal I of R with $\omega_R(I) < \infty$, $\omega_R(I) =\mid Min_R(I) \mid$, where $Min_R(I)$ denotes the set of prime ideals of R minimal over I). We present several properties of rings that satisfy condition (**). We prove that some open conjectures which concern n-absorbing ideals are partially true for rings which satisfy condition (**). We apply the obtained results to trivial ring extensions.

Mathematics Subject Classification (2020). 13F05, 13A15, 13E05, 13F20, 13B99, 13G05, 13B21

Keywords. n-absorbing ideal, prime ideal, primary ideal, Prüfer ring, Noetherian ring, Artinian ring

1. Introduction

Throughout this work, all rings are assumed to be commutative with identity element and $1 \neq 0$. Recall from [3] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than a 2-absorbing ideal is the concept of n-absorbing ideal. Let $n \geq 1$ be a positive integer. Also, recall from [1] that a proper ideal I of R is called an *n-absorbing ideal* of R if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). For more details on n-absorbing ideals, we refer the reader to [11–13]. We investigate rings in which every n-absorbing

*Corresponding Author.
Email addresses: issoual2@yahoo.fr (M. Issoual), mahdou@hotmail.com (N. Mahdou), mmoutui@auaf.edu.af (M.A.S. Moutui)
Received: 26.09.2020; Accepted: 11.09.2021

ideal of R is a prime ideal, where $n \geq 2$ is an integer, called n-AB rings. Note that the authors in [6] studied rings where every 2-absorbing ideal of R is prime.

This paper aims at studying of rings in which every n-absorbing ideal is a prime ideal. We also study the concept of $\Omega(R) = \{\omega_R(I); I \text{ is a proper ideal of } R\}$, where $\omega_R(I) = \min\{n; I \text{ is an } n\text{-absorbing ideal of } R\}$. We establish results which give the possible values for $\Omega(R)$ in several classes of rings.

In section 2, we study the concept of n-AB ring and prove that for a ring R, the following assertions are equivalent:

(1) R is an n-AB ring.
(2) (a) The prime ideals of R are comparable. In particular, R is quasi-local with maximal ideal M.
  (b) If P is a minimal prime ideal over an n-absorbing ideal I, then $IM = P$.

Next, we use the notion of minimal n-absorbing ideal introduced in [17], to establish that for a ring R, the following statements are equivalent:

(1) R is an n-AB ring.
(2) (a) The prime ideals of R are comparable. In particular, R is quasi-local with maximal ideal M.
  (b) For every prime ideal P of R, $n - \text{Min}_R(P^n) = \{P\}$.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A. Suitable background on commutative trivial ring extensions is [2, 5, 10, 14, 15].

Let R be a ring and I be a proper ideal of R. If I is an n-absorbing ideal for some positive integer n, then it is easy to see that I is an m-absorbing ideal of R for every positive integer $m \geq n$. We define $\omega_R(I) = \min\{n; I \text{ is an } n\text{-absorbing ideal of } R\}$; Otherwise $\omega_R(I) = \infty$. It is convenient to define $\omega_R(R) = 0$. Then for any ideal of R, we have $\omega_R(I) \in \mathbb{N} \cup \{0, \infty\}$ with $\omega_R(I) = 1$ if and only if I is a prime ideal of R and $\omega_R(I) = 0$ if and only if $I = R$. We define $\Omega(R) = \{\omega_R(I); I \text{ is a proper ideal of } R\}$. Notice that $\{1\} \subseteq \Omega(R) \subseteq \mathbb{N} \cup \{\infty\}$. In [1] page 1668, Anderson-Badawi raised the following question:

• If $n \in \Omega(R)$ for some positive integers n, then $m \in \Omega(R)$ for every integer m with $1 \leq m \leq n$?

It is worth to mention that a positive answer (to the question of Anderson-Badawi) is given for Prüfer domains. In Section 3, we give a positive answer of Anderson-Badawi's question, and we establish another characterization of Artinian rings. If I is a proper ideal of R, $\text{Min}_R(I)$ denotes the set of prime ideals of R minimal over I. Recall that from [1, Theorem 2.5], $|\text{Min}_R(I)| \leq \omega_R(I)$.

In section 4, we study rings in which $|\text{Min}_R(I)| = \omega_R(I)$. We say that a ring R satisfies (**) if for every ideal of R with $\omega_R(I) < \infty$, we have $|\text{Min}_R(I)| = \omega_R(I)$. We prove in Theorem 4.9, that a Dedekind domain R satisfies (**) if and only if R is a field. Recall that from [1, Theorem 5.11(e)], Anderson-Badawi proved that $\Omega(R) \subseteq \Omega(R \propto E)$, where R is a commutative ring and E is an R-module. Notice that the inclusion may be strict. We end this paper by studying about when the equality between $\Omega(R)$ and $\Omega(R \propto E)$ is satisfied, where R is a ring and E an R-module. It is worth to mention that some of our proofs are easy, because we exploit earlier results. We are very grateful to [1, 7] for their results on n-absorbing ideals.

2. Main results on n-AB rings

We start this section by recalling the notion of n-AB ring defined in the introduction.

Definition 2.1. We say that a ring R is an n-AB ring for some positive integer n if every n-absorbing ideal of R is prime.

Now, we provide examples of rings which illustrate the notion of n-AB ring.

Example 2.2. Let R be a one-dimensional valuation domain with maximal ideal M which is not principal. Then R is an n-AB ring for any positive integer n.

Proof. Let I be a nonzero proper n-absorbing ideal of R. From [1, Theorem 5.5], $M^n \subseteq I$, as I is M-primary. On the other hand, we claim that $M^2 = M$. Assume not. Then, there exists $t \in M$ such that $t \notin M^2$. One can easily check that $M = tR$, making M, a principal ideal of R, which is a contradiction. So, $M^2 = M$. Therefore, $I = M$, making I a prime ideal. Hence, R is an n-AB ring, as desired. □

Example 2.3. Let R be a two-dimensional valuation domain with prime ideals $0 \subset P \subset M$ and value group $G = \mathbb{Q} \bigoplus \mathbb{Q}$ (all direct sums having lexicographic order). Then R is an n-AB ring.

Proof. We need to prove that $M^2 = M$ and $P^2 = P$. Indeed, let $(q, q') \in \mathbb{Q} \bigoplus \mathbb{Q}$ such that $(q, q') > (0, 0)$ if $q > 0$ or $q = 0$ and $q' > 0$. In the first case, $(q, q') = (q/2, q'/2) + (q/2, q'/2)$. In the second case, $(0, q') = (0, q'/2) + (0, q'/2)$. Hence, $M^2 = M$. With similar arguments as previously, we obtain $P^2 = P$. Next, let n be a positive integer and $I \neq M$ be a nonzero n-absorbing ideal of R. Then $\sqrt{I} = P$, and so $P^n = P \subseteq I \subseteq P$. Consequently, $I = P$ which is a prime ideal of R. Thus, R is an n-AB ring, as desired. □

Our next aim is to give a characterization of an n-AB ring. For this purpose, we establish the following lemma.

Lemma 2.4. *Let R be a quasi-local ring with maximal ideal M. Then the following statements hold:*

(1) *If I is an n-absorbing ideal of R, then IM is an $(n+1)$-absorbing ideal of R.*
(2) *If P is a prime ideal of R, then PM is an n-absorbing ideal of R for each $n \geq 2$; moreover, PM is a prime ideal of R if and only if $PM = P$.*

Proof. (1) Let $x_1, x_2, \cdots, x_{n+2} \in R$ be such that $x_1 \cdots x_{n+2} \in IM \subset I$. Since I is an n-absorbing ideal of R, then without loss of generality, we may assume that $x_1 \cdots x_n \in I$. Now, if $x_{n+2} \in M$, then we are done. Otherwise; we have $x_1 \cdots x_{n+1} \in IM$ since R is a quasi-local ring. Thus, IM is an $(n+1)$-absorbing ideal of R.

(2) Let P be a prime ideal of R. By assertion (1) above, PM is a 2-absorbing ideal of R and so an n-absorbing ideal for every positive integer $n \geq 2$. If $PM = P$, then PM is a prime ideal. Conversely, assume that PM is a prime ideal of R and let $x \in P$. Then $x^2 \in PM$, as R is a quasi-local ring. Thus, $x \in PM$ since PM is a prime ideal and so $PM = P$, as desired.

□

Now, we establish the following characterization of an n-AB ring.

Theorem 2.5. *A ring R is an n-AB ring if and only if the following two assertions hold:*

(1) *The prime ideals of R are comparable. (In particular, R is quasi-local with maximal ideal M.)*
(2) *If P is a minimal prime ideal over an n-absorbing ideal I, then $IM = P$.*

Proof. (⇒) (1) Let P_1 and P_2 be two prime ideals of R. By [1, Theorem 2.1(c)], $P_1 \cap P_2$ is a 2-absorbing ideal of R and so an n-absorbing ideal from [1, Theorem 2.1(b)](as $n \geq 2$). So, $P_1 \cap P_2$ is a prime ideal of R. Thus, P_1 and P_2 are comparable prime ideals. Now using the fact that the prime ideals of R are comparable, it follows that R is quasilocal

with maximal ideal M.
(2) Let I be an n-absorbing ideal of R and P be a minimal prime ideal over I. Then by assumption, I is a prime ideal of R. On the other hand, $\sqrt{I} = P$. Therefore, $I = P$ and so by Lemma 2.4, it follows $IM = I$.
($\Leftarrow$) Assume that the assertions (1) and (2) hold. Let I be an n-absorbing ideal of R. Since the prime ideals are comparable, then $\sqrt{I}$ is a prime ideal, say P which is the unique minimal prime ideal over I. By assertion (2) above, it follows that $P = IM \subseteq I$. Therefore, $I = P$ is a prime ideal of R. Hence, R is an n-AB ring, as desired. □

As a first application of Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let R be a ring. If R is an n-AB ring, then R is quasi-local with maximal ideal M satisfying $M^2 = M$.*

Proof. If R is an n-AB ring, then by Theorem 2.5, R is a quasi-local ring with maximal ideal M. On the other hand, M^n is an n-absorbing ideal of R [1, Lemma 2.8]. Consequently, M^n is a prime ideal of R. And so, $M^n = M \subseteq M^2 \subseteq M$. Finally, $M^2 = M$. □

It is worth to mention that the converse of Corollary 2.6 is not true, in general, as shown by the next example which exhibits a quasi-local ring R which is not 2-AB.

Example 2.7. Let R be a one-dimensional valuation domain with maximal ideal M which is not principal. Then $M^2 = M$. Now, let I be an ideal of R such that $0 \subset I \subset M$. Clearly, I is an M-primary ideal of R. We claim that I is not an n-absorbing ideal of R for every positive integer n. Deny. by [1, Theorem 5.5], $M = M^n \subset I$, which is a contradiction. Therefore, the only n-absorbing ideals of R are 0 and M. Next, let $A := R \propto R$ be the trivial ring extension of R by the $R-$module R. Clearly, A is a quasi-local ring with maximal ideal $m := M \propto R$. Consider a prime ideal P of R. Then by [1, Theorem 4.10], $0 \propto P$ is a 2-absorbing ideal of A which is not prime. Thus, A is not a 2-AB ring. □

The next corollary is another application of Theorem 2.5 which gives a characterization of n-AB rings in the special case of Noetherian setting.

Corollary 2.8. *A ring R is a Noetherian n-AB ring if and only if R is a field.*

Proof. Assume that R is a Noetherian n-AB ring. Then by Theorem 2.5, R is a quasi-local ring with maximal ideal M. Let P be a prime ideal of R. By Lemma 2.4, we have $MP = P$ and so $P = 0$ by Nakayama's lemma. Thus, R is a field. The converse is straightforward. □

Recall that a prime ideal P of a ring R is called a divided prime ideal if P is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided ring. Now, we give a necessary and sufficient condition for a divided domain to be an n-AB ring.

Theorem 2.9. *Let R be a divided domain. Then R is an n-AB ring if and only if $P^2 = P$ for every prime ideal P of R.*

Proof. Assume that R is an n-AB ring. Let P be a nonzero prime ideal. From [1, Theorem 3.3], P^n is an n-absorbing ideal of R and so a prime ideal of R. Therefore, $P^n = P$. It follows that $P^2 = P$, as $P^n \subseteq P^2 \subseteq P$. Conversely, assume that for every prime ideal P of R, $P^2 = P$. Let I be a nonzero n-absorbing proper ideal of R. Using the fact that R is a divided domain, then $\sqrt{I} = P$ is a nonzero divided prime ideal. From [7], it follows that $P^n \subseteq I \subseteq P$. Consequently, $I = P$ is a prime ideal of R. Hence, R is an n-AB ring, as desired. □

Theorem 2.9 covers the special case of valuation domains, as recorded below.

Corollary 2.10. *Let R be a valuation domain. Then R is an n-AB ring if and only if $P^2 = P$ for every prime ideal of R.*

Recall that in [8], Gilmer defined an ideal I of a commutative ring R to be semi-primary if its radical is a prime ideal of R. Also a ring R satisfies (*) if every semi-primary ideal is primary. These rings have been studied in [9]. The next theorem shows that for the class of n-AB rings which satisfy (*), every prime ideal is idempotent.

Theorem 2.11. *Let R be an n-AB ring which satisfies (*). Then every prime ideal of R is idempotent.*

Proof. Let R be an n-AB ring which satisfies (*). Consider a prime ideal P of R. Then from assumption, P^n is a P-primary ideal of R and so an n-absorbing ideal of R by [1, Theorem 3.1]. It follows that $P^n = P$ which is a prime ideal of R. Consequently, $P^2 = P$, as $P^n \subseteq P^2 \subseteq P$. □

Now, we establish another characterization of an n-AB ring using the notion of minimal n-absorbing ideal introduced by Moghimi and Naghani in [17], in the following way:

Definition 2.12 ([17])**.** Let I be an ideal of a ring R. An n-absorbing ideal P of R is said to be a minimal n-absorbing ideal over I, if there is no n-absorbing ideal Q of R such that $I \subseteq Q \subset P$. And the set of minimal n-absorbing ideals over I is denoted by $n - \mathrm{Min}_R(I)$.

Theorem 2.13. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is an n-AB ring.*
(2) (a) *The prime ideals of R are comparable. In particular, R is quasi-local with maximal ideal M.*
  (b) *If P is a minimal prime over an n-absorbing ideal I, then $IM = P$.*
(3) (a) *The prime ideals of R are comparable. In particular, R is quasi-local with maximal ideal M.*
  (b) *For every prime ideal P of R, $n - Min_R(P^n) = \{P\}$.*

Proof. (1) ⇔ (2) Follows from Theorem 2.5. (2) ⇒ (3) By Theorem 2.5, it remains to show that for every prime ideal P of R, $n - \mathrm{Min}_R(P^n) = \{P\}$. Let P be a prime ideal of R. By [17, Corollary 2.2], we have $n - \mathrm{Min}_R(P^n) \neq \emptyset$. Therefore, it is sufficient to show that $n - \mathrm{Min}_R(P^n) \subseteq \{P\}$. Let $J \in n - \mathrm{Min}_R(P^n)$. Then J is an n-absorbing ideal of R, and hence J is a prime ideal of R. Since $P^n \subseteq J$, it follows that $P \subseteq J$, which implies $P^n \subseteq P \subseteq J$. Since P is an n-absorbing ideal of R, we have $J = P$.
(3) ⇒ (1) Suppose that P is a minimal prime ideal over an n-absorbing ideal I. Then by [7], $P^n \subseteq I \subseteq P$. Since $n - \mathrm{Min}_R(P^n) = \{P\}$, then $I = P$ and so $IM = PM = P$ by Lemma 2.4. □

The next result is an immediate consequence of Theorem 2.13 with the well-known fact that the prime ideals of a divided ring are comparable.

Corollary 2.14. *Let R be a divided ring with unique maximal ideal M. Then the following statements are equivalent:*

(1) *For every minimal prime P over an n-absorbing ideal I of R, $IM = P$.*
(2) *For every prime ideal P of R, we have $n - Min_R(P^n) = \{P\}$.*

Moreover, if one of the above equivalent statements holds, then R is an n-AB ring.

In the following result, we show that for each prime ideal P of an n-AB ring, either P idempotent or P^j is not an n-absorbing ideal of R for every positive integer j with $2 \leq j \leq n$.

Corollary 2.15. *Let R be an n-AB ring. For each prime ideal P of R, either $P^2 = P$ or P^j is not an n-absorbing ideal of R for every positive integer j with $2 \leq j \leq n$.*

Proof. Assume that there exists a positive integer j with $2 \leq j \leq n$ such that P^j is an n-absorbing ideal. Then P^j is a prime ideal of R and so $P^j = P$. Hence $P^2 = P$. □

We end this section by studying the transfer of n-AB ring notion to trivial ring extension.

Theorem 2.16. *Let A be a ring, E be a finitely generated A-module and $R := A \propto E$. Then the following statements are equivalent:*

(1) *R is an n-AB ring,*
(2) *A is an n-AB ring and $E = 0$.*

Proof. (1) $\Rightarrow$ (2) Assume that R is an n-AB ring. Since $R/0 \propto E \simeq A$, it follows that A is an n-AB ring. Let M be the unique maximal ideal of A. By Corollary 2.6, we have $M^2 = M$ and so $(M \propto E)^2 = M \propto ME = M \propto E$ (since $M \propto E$ is the unique maximal ideal of $A \propto E$). Therefore, $ME = E$ and so by Nakayama's lemma, $E = 0$.
(2) $\Rightarrow$ (1) Straightforward since $A \simeq A \propto 0 = R$. □

Recall that from [2, Corollary 3.4], if A is an integral domain and E is a divisible A-module, then every ideal of $A \propto E$ has the form $I \propto E$ for some ideal I of A or $0 \propto N$ for some submodule N of E. The next theorem develops a result on the transfer of $n-$AB property for the special case of trivial extensions of integral domains by vector spaces over their quotient fields.

Theorem 2.17. *Let A be an integral domain with quotient field K and E be a K-vector space and $R := A \propto E$. Then the following statements are equivalent:*

(1) *R is an n-AB ring.*
(2) *A is an n-AB ring and $E = 0$.*

Proof. (1) $\Rightarrow$ (2) Assume that R is an n-AB ring. Then it is easy to see that A is an n-AB ring. Recall that from [11, Theorem 2.2], if F is an A-submodule of E, then $0 \propto F$ is a 2-absorbing ideal of R if and only if F is a K-subspace of E. Therefore, for $F = 0$, we obtain $0 \propto 0$ is a 2-absorbing ideal and so a prime ideal of R. We conclude that R is an integral domain, making $E = 0$. (2) $\Rightarrow$ (1) Clear since $A \simeq A \propto 0 = R$. □

The next result establishes the transfer of the n-AB property to trivial ring extension in the special case of Noetherian setting.

Corollary 2.18. *Let A be a Noetherian ring, E be a finitely generated A-module and $R := A \propto E$. Then R is an n-AB ring if and only if so is A and $E = 0$.*

Proof. Assume that R is an n-AB ring. From [2, Theorem 4.8], it follows that R is Noetherian. By Corollary 2.8, R is a field. Thus, A is a field and $E = 0$. The converse is trivial. □

3. On $\Omega(R)$ where R is a ring

Recall that $\omega_R(I) = \min\{n; I$ is an n-absorbing ideal of $R\}$; Otherwise $\omega_R(I) = \infty$. It is convenient to define $\omega_R(R) = 0$. Then for any ideal of R, we have $\omega_R(I) \in \mathbb{N} \cup \{0, \infty\}$ with $\omega_R(I) = 1$ if and only if I is a prime ideal of R and $\omega_R(I) = 0$ if and only if $I = R$. Also recall that $\Omega(R) = \{\omega_R(I); I$ is a proper ideal of $R\}$. Then $\{1\} \subseteq \Omega(R) \subseteq \mathbb{N} \cup \{\infty\}$. The first result of this section gives a characterization of Artinian rings.

Theorem 3.1. *Let R be a ring. Then R is an Artinian ring if and only if R is a Noetherian ring and $\Omega(R) = \{1, ..., n\}$ for some positive integer n.*

The proof of this theorem involves the following lemma.

Lemma 3.2. *Let M be a finitely generated maximal ideal of a ring R. If $\Omega(R) = \{1, ..., n\}$ for some positive integer n, then $ht(M) = 0$.*

Proof. From [1, Lemma 2.8], it follows that M^{n+1} is an $(n+1)$-absorbing ideal of R with $\omega_R(M^{n+1}) \leq n$ (since $\Omega(R) = \{1, ..., n\}$). We claim that $M^{n+1} = M^{n+2}$. Deny. $M^{n+2} \subset M^{n+1}$ and so by [1, Lemma 2.8], $n+1 \in \Omega(R) = \{1, \cdots, n\}$, which is a contradiction. Hence, $M^{n+1} = M^{n+2}$. Now the result follows from [1, Lemma 5.10]. □

Proof of Theorem 3.1. Assume that R is a Noetherian ring with $\Omega(R) = \{1, ..., n\}$ for some positive integer n. By Lemma 3.2, $ht(M) = 0$ for every maximal ideal M of R. Therefore, $\dim(R) = 0$, and so R is an Artinian ring. The converse is clear from [1, Theorem 5.11] and the fact that an Artinian ring is Noetherian. □

Recall that incomparable prime ideals in a Prüfer domain are comaximal since R is locally a valuation domain. In the case of a Prüfer domain, we give a positive answer to the following Anderson-Badawi's question: if $n \in \Omega(R)$ for some positive integer, is $m \in \Omega(R)$ for every positive integer m with $1 \leq m \leq n$?

Theorem 3.3. *Let R be a Prüfer domain and n be a positive integer in $\Omega(R)$. Then $m \in \Omega(R)$ for every positive integer $m \in \{1, ..., n\}$.*

Proof. Let n be a positive integer in $\Omega(R)$ and let I be an n-absorbing ideal of R with $\omega_R(I) = n$. By [1, Theorem 5.7], the ideal I is a product of a prime ideals of R. We may assume that $I = P_1^{n_1}...P_k^{n_k}$, where $P_1, \cdots, P_k$ are comaximal prime ideals of R since R is a Prüfer domain and $n_i's$ are positive integers with $n = n_1 + \cdots + n_k$. Now let m be a positive integer such that $1 \leq m \leq n$. We may set $m = m_1 + \cdots + m_k$ where $m_i's$ are non-negative integers such that $1 \leq m_i \leq n_i$ for every positive integer $1 \leq i \leq k$ and consider the ideal $J := P_1^{m_1}...P_k^{m_k}$ of R. From [1, Theorem 5.7], J is an m-absorbing ideal of R and so $\omega_R(J) = m$. Hence, $\{1, \cdots, n\} \subseteq \Omega(R)$, as desired. □

4. Rings satisfying $|\mathrm{Min}_R(I)| = \omega_R(I)$ if $\omega_R(I) < \infty$

Let I be an n-absorbing ideal of a ring R for some positive integer n. We denote by $\mathrm{Min}_R(I)$ the set of minimal prime ideals over I. Recall that from [1, Theorem 2.14], if I has exactly n minimal prime ideals, say $P_1, \cdots, P_n$. Then $P_1...P_n \subseteq I$ and so $\omega_R(I) = n$. In this section, we investigate rings in which every n-absorbing ideal has exactly n minimal prime ideals.

Remark 4.1. Let I be a proper ideal of a ring R. Notice that if I is an n-absorbing ideal of R for some positive integer n, then $\sqrt{I} = \cap_{P \in \mathrm{Min}_R(I)} P$ is also an n-absorbing ideal of R. Set $\mathrm{Min}_R(I) = \{P_1, \cdots, P_m\}$. Since $P_1, \cdots, P_m$ are incomparable prime ideals, then by [1, Remark 2.2], $m = \omega_R(\sqrt{I}) = \omega_R(\cap_{i=1}^m P_i) = |\mathrm{Min}_R(I)| \leq \omega_R(I)$.

Now, we introduce the following definition:

Definition 4.2. We say that a ring R satisfies the property (**) if for every proper ideal I such that $\omega_R(I) < \infty$, we have $\omega_R(I) = |\mathrm{Min}_R(I)|$.

As illustrative examples of Definition 4.2, we provide families of rings satisfying the property (**).

Example 4.3. If R is a field, then R satisfies (**).

Example 4.4. If R is a Von Neumann regular ring, then R satisfies (**), since every ideal of I is a radical ideal.

Example 4.5. Let R be a two-dimensional valuation domain with prime ideals $0 \subset P \subset M$ and value group $G = \mathbb{Q} \bigoplus \mathbb{Q}$. Notice that $M^2 = M$ and $P^2 = P$; so $0, P$ and M are the only n-absorbing ideals of R with $\omega_R(0) = \omega_R(P) = \omega_R(M) = 1$. Then R satisfies (**).

The next example exhibits a ring R satisfying the property (**) and having an ideal I that is not n-absorbing ideal.

Example 4.6. The ring $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ satisfies (**) since it is a Von Neumann regular ring. Let $I = \{(x_i) \in R; x_{2i+1} = 0, i \in \mathbb{N}\}$ be an ideal of R. One can easily check that I is not an n-absorbing ideal of R for every positive integer n. Therefore, $\omega_R(I) = \infty$.

Now, we give the following characterization of an n-absorbing ideal I of an integral domain R which satisfies (**) with $dim(R) \leq 1$ or a ring R satisfying (**) with $dim(R) = 0$ with $\omega_R(I) = n$ for some positive integer n.

Theorem 4.7. *Let R be a ring satisfying (**) and which is either an integral domain with $dim(R) \leq 1$ or $dim(R) = 0$. Let I be a proper ideal of R. Then I is an n-absorbing ideal of R with $\omega_R(I) = n$ if and only if I is a product of n incomparable prime ideals.*

Proof. Assume that R is an integral domain satisfying (**) with $\dim(R) \leq 1$ and pick an ideal I of R with $\omega_R(I) = n$. Then I has exactly n minimal prime ideals which are comaximal by assumption. From [1, Corollary 2.15], we obtain $I = P_1...P_n$ where P_i is a minimal prime ideal over I, for every $i = 1, 2, ..., n$. The converse is straightforward via [1, Remark 2.2].
Now, assume that R satisfies (**) with $\dim(R) = 0$. Let I be an n-absorbing ideal of R. Since $\omega_R(I) = n$, then the ideal I has exactly n minimal prime ideals, say $P_1, ..., P_n$ which are maximal, as $\dim(R) = 0$. From [1, Corollary 2.15], $I = P_1...P_n$. The converse follows from [1, Theorem 2.9]. □

Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. Recall that Anderson and Badawi in [1] proposed the following three conjectures.

(1) Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.
(2) Conjecture two: If I is an n-absorbing ideal of A, then $(\sqrt{I})^n \subseteq I$. Notice that an affirmative answer to this conjecture is given in [7].
(3) Conjecture three: If I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$.

The next theorem studies some properties of a ring R which satisfies (**).

Theorem 4.8. *Let R be a ring which satisfies (**), I be an ideal of R and P be a prime ideal of R. Then the following statements hold:*

(1) *If there exists a positive integer $n \geq 2$ such that P^n is P- primary, then P is idempotent. In particular, this holds if R is a valuation domain.*
(2) *Every maximal ideal of R is idempotent.*
(3) *Assume that P is a divided prime ideal of R such that $Nil(R) \subset P$. Then P is idempotent. Moreover, if I is an ideal of R such that $\sqrt{I} = P$, then I is an n-absorbing ideal of R for some positive integer if and only if $I = P$.*
(4) *If $P^2 \neq P$, then there is no n-absorbing ideal of R between P and P^2 for every positive integer n.*
(5) *The Conjecture three holds for every radical ideal of R.*
(6) *The Conjecture one holds for every radical ideal of R.*
(7) *Let n be a positive integer in $\Omega(R)$, then $\{1, \cdots, n\} \subseteq \Omega(R)$.*

Proof. (1) Let P be a prime ideal of R such that P^n is P-primary for some positive integer $n \geq 2$. From [1, Theorem 3.1], P^n is an n-absorbing ideal of R and so P^n is a prime ideal of R since $\omega_R(P^n) = |\text{Min}_R(P^n)| = 1$. Therefore, $P^n = P$. Hence, $P^2 = P$, as desired.
The "In particular" statement follows from [1, Theorem 5.5].
(2) Let M be a maximal ideal of R. Then M^2 is an M-primary ideal of R, and hence M is idempotent by assertion (1) above.

(3) Suppose $\text{Nil}(R) \subset P$. Let n be a positive integer. From [1, Theorem 3.3], P^n is a P-primary ideal of R which satisfies (**). Then $\omega_R(P^n) = 1$. Consequently, P^n is a prime ideal and so $P^2 = P$. Next, let I be an ideal of R. If $I = P$, then the claim is clear. Conversely, assume that I is an $n-$absorbing ideal of R such that $\sqrt{I} = P$. Then P is the unique minimal prime ideal over I. Suppose that $\omega_R(I) = n$ for some positive integer n. So, $P^n = P \subseteq I \subseteq P$. Hence, $I = P$.

(4) Assume by the way of contradiction that there exists an n-absorbing ideal I such that $P^2 \subset I \subset P$. One can easily check that $\sqrt{I} = P$. Therefore, by statement (3) above, it follows that $I = P$, which is a contradiction. Hence, there is no n-absorbing ideal between P and P^2 for every positive integer n.

(5) Let I be a radical ideal, which is an n-absorbing ideal of R and $\omega_R(I) = n$. Since R satisfies (**), the ideal I has exactly n minimal prime ideals, say $P_1, \cdots, P_n$. It is well known that $I[X]$ has exactly n-minimal prime ideals, say $P_1[X], \cdots, P_n[X]$. Then $\sqrt{I[X]} = P_1[X] \cap \cdots \cap P_n[X] = (P_1 \cap \cdots \cap P_n)[X] = \sqrt{I}[X] = I[X]$. From [1, Remark 2.2], we have $\omega_{R[X]}(I[X]) = n = \omega_R(I)$. Therefore, $I[X]$ is an n-absorbing ideal of $R[X]$. Hence, Conjecture three holds for I.

(6) Let I be a radical ideal of R which is n-absorbing. By assertion (5) above, we have $I[X]$ is an n-absorbing ideal of $R[X]$. By [16, Proposition 2.9(i)], it follows that I is a strongly n-absorbing ideal of R. The converse is trivial. Hence, Conjecture one holds for I.

(7) Let $n \in \Omega(R)$ be a positive integer. Then there exists an n-absorbing ideal I of R that $\omega_R(I) = n$. Since R satisfies (**), the ideal I has exactly n minimal prime ideals, say $P_1, P_2, ..., P_n$. Let $k \in \{1, ..., n\}$. Consider the ideal $J = P_1 \cap ... \cap P_k$ of R. We infer by [1, Remark 2.2] that $k = \omega_R(J)$. Therefore, $k \in \Omega(R)$, as desired. □

The next theorem gives a characterization of a Dedekind domain satisfying (**).

Theorem 4.9. *Let R be a ring. Then R is a Dedekind domain which satisfies (**) if and only if R is a field.*

The proof of this theorem requires the following lemma.

Lemma 4.10. *Let R be a Dedekind domain and I be a proper ideal of R. Then I is a radical ideal if and only if $\omega_R(I) = \omega_R(\sqrt{I})$.*

Proof. Assume that R is a Dedekind domain and I be a proper ideal of R such that $\omega_R(I) = \omega_R(\sqrt{I})$. Since $I \subseteq \sqrt{I}$, then the result follows readily from [17, Lemma 2.17]. □

Proof of Theorem 4.9. Assume that R is a Dedekind domain which satisfies (**) and let I be a proper ideal of R. Since I is a product of prime ideals, then $\omega_R(I) < \infty$, set $\omega_R(I) = n$, where n is a positive integer. So, $\sqrt{I} = \cap_{i=1}^n P_i$, where P_i is a minimal prime ideal over I, for every $1 \leq i \leq n$. Since $P_1, \cdots, P_n$ are incomparable prime ideals, we have $\cap_{i=1}^n P_i$ is an n-absorbing ideal of R. Moreover, $\omega_R(\cap_{i=1}^n P_i) = n$. Therefore, $\omega_R(\sqrt{I}) = \omega_R(I)$ and $I \subseteq \sqrt{I}$. Since R is a Dedekind domain, then by Lemma 4.10, $\sqrt{I} = I$. Hence, R is a field as it is a Von Neumann regular domain. □

Let R be a ring, E be an R-module and $R \propto E$ be the trivial ring extension of R by E. It is well known that $\Omega(R) \subseteq \Omega(R \propto E)$ [1, Theorem 5.11]. Notice that the inclusion may be strict. This allows us to investigate about when the equality between $\Omega(R) = \Omega(R \propto E)$ is satisfied. In the next theorem, we show that $\Omega(R \propto E) = \Omega(R) \cup \{2, \infty\}$ in the case R is an integral domain and E is a divisible R-module. Note that in this case, the ideals of $R \propto E$ are the form $I \propto E$ or $0 \propto N$ where I is a proper ideal of R and N a submodule of E such that $IE \subseteq N$ [2, Theorem 3.11].

Theorem 4.11. *Let R be an integral domain which is not a field with quotient field K and E be a K-vector space. Then $\Omega(R \propto E) = \Omega(R) \cup \{2, \infty\}$.*

Proof. Notice that $\Omega(R) \subseteq \Omega(R \propto E)$ from [1, Theorem 5.11(e)]. On the other hand, $0 \propto E$ is not an n-absorbing ideal for every positive integer n from [4, Corollary 3.3]. So, $\infty \in \Omega(R \propto E)$. Thus $\Omega(R) \cup \{\infty\} \subseteq \Omega(R \propto E)$. Now if N is a proper K-subspace of E and by [4, Theorem 3.2], $0 \propto N$ is a 2-absorbing ideal of $R \propto E$ which is not a prime ideal of $R \propto E$. Then $\omega_S(0 \propto N) = 2 \in \Omega(R \propto E)$. Hence, $\Omega(R) \cup \{2, \infty\} \subseteq \Omega(R \propto E)$. Now let $n \in \Omega(R \propto E)$ and let J be a proper ideal of $R \propto E$ such that $\omega_S(J) = n$. If $J = I \propto E$, then from [1], we have $\omega_R(I) = n \in \Omega(R)$. If $J = 0 \propto E$, then J is a prime ideal (as R is an integral domain) and so $\omega_S(J) = 1$. If N is a proper K-subspace of E, the ideal $0 \propto N$ is a 2-absorbing ideal of $R \propto E$ by [4, Theorem 3.3]. Therefore, $n = 2$. If N is not K-subspace of E, then the ideal $0 \propto N$ is not n-absorbing ideal of $R \propto E$ for every positive integer n [4, Corollary 3.3]. Thus, $\omega(J) = \infty$. Finally, we conclude that $\Omega(R \propto E) = \Omega(R) \cup \{2, \infty\}$. □

For the special case of trivial extensions of a Prüfer domain R or an integral domain R with $dim(R) = 0$ by vector spaces over their quotient fields, we obtain the following result.

Corollary 4.12. *Let R be an integral domain which is not a field with quotient field K and E be a K-vector space. Then the following assertions hold:*

(1) *If R is a Prüfer domain, then $\Omega(R \propto E) = \Omega(R) \cup \{\infty\}$.*
(2) *If $dim(R) = 0$, then $\Omega(R \propto E) = \Omega(R) \cup \{\infty\}$.*

Proof.
(1) By Theorem 2.15, we have $\Omega(R \propto E) = \Omega(R) \cup \{2, \infty\}$. Since R is a Prüfer domain, then there exists $n \geq 2$ such that $2 \in \Omega(R)$, thus $2 \in \{1, ..., n\} \subseteq \Omega(R)$ by Theorem 2.14. Hence, $2 \in \Omega(R)$ and so $\Omega(R \propto E) = \Omega(R) \cup \{\infty\}$, as desired.
(2) Let $P \neq Q$ be two prime ideals, which are incomparable since $\dim(R) = 0$. From [1, Theorem 2.1], we have $P \cap Q$ is a non-prime 2-absorbing ideal. Consequently, $2 \in \Omega(R)$. Hence, $\Omega(R \propto E) = \Omega(R) \cup \{\infty\}$.

□

The next example completes Theorem 4.11 by treating the case the ring R is a field. In this case, we show that $\Omega(R \propto R) = \Omega(R) \cup \{2\}$.

Example 4.13. Let K be a field and $S := K \propto K$ be the trivial ring extension of K by the K-vector space K. It is clear that the only proper ideals of S are $0 \propto K$ and $0 \propto 0$. Furthermore, $\omega_S(0 \propto K) = 1$ and $\omega_S(0 \propto 0) = 2$. So, $\Omega(S) = \{1, 2\}$ and $\Omega(K) = \{1\}$. Therefore, $\Omega(S) = \Omega(K) \cup \{2\}$.

Acknowledgment. The authors would like to express their sincere thanks to the anonymous referee for his/her helpful suggestions and comments.

References

[1] D.F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Commun. Algebra, **39** (5), 1646-1672, 2011.

[2] D.D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra, **1** (1), 3-56, 2009.

[3] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Aust. Math. Soc. **75**, 417-429, 2007.

[4] A. Badawi, M. Issoual and N. Mahdou, *On n-absorbing ideals and (m, n)-closed ideals in trivial extension rings*, J. Algebra Appl. **18** (7), 2019.

[5] C. Bakkari, S. Kabbaj and N. Mahdou, *Trivial extension definided by Prûfer conditions*, J. Pure App. Algebra **214**, 53-60, 2010.

[6] D. Bennis and B. Fahid, *Rings in which every 2-absorbing ideal is prime*, Beitr. Algebra Geom. **59** (2), 391-396, 2018.
[7] H.S. Choi and A. Walker, *The radical of an* n*-absorbing ideal*, J. Commut. Algebra, **12** (2), 171-177, 2020.
[8] R.W. Gilmer, *Ring in which every semi-primary ideals are primary*, Pacific J. Math. **12** (4), 1273-1276, 1962.
[9] R.W. Gilmer and J. Leonard Mott, *Multiplication rings as rings in which ideals with prime radical are primary*, Trans. Amer. Math. Soc. **114**, 40-52, 1965.
[10] S. Glaz, *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics, 13-71, 1989.
[11] M. Issoual and N. Mahdou, *Trivial extensions defined by 2-absorbing-like conditions*, J. Algebra Appl. **17** (11), 1850208, 2018.
[12] M. Issoual, N. Mahdou and M.A.S. Moutui, *On* n*-absorbing and strongly* n*-absorbing ideals of amalgamation*, J. Algebra Appl. **19** (10) 2050199, 2020.
[13] M. Issoual, N. Mahdou and M.A.S. Moutui, *On* (m,n)*-closed ideals in amalgamated algebra*, Int. Electron. J. Algebra, **29**, 134-147, 2021.
[14] S. Kabbaj, *Matlis' semi-regularity and semi-coherence in trivial ring extensions : a survey*, Moroccan J. Algebra Geometry Appl. In Press.
[15] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra, **32** (10), 3937-3953, 2004.
[16] A. Laradji, *On* n*-absorbing rings and ideals*, Colloq. Math. **147** (2) 265-273, 2017.
[17] H.F. Moghimi and S.R. Naghani, *On* n*-absorbing ideals and the* n*-Krull dimension of a commutative ring*, J. Korean Math. Soc. **53** (6), 1225-1236, 2016.