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On Integral Transforms of Some Special Functions

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Abstract: In this study, known integral transforms such as Fourier and Hartley are studied and these integral transforms are studied in detail for bicomplex numbers. In addition, the properties of the bicomplex Hartley transform have been investigated. Also, the relation between Hartley and Fourier transform for bicomplex numbers is given.

Keywords: Bicomplex functions, Fourier transformations, Integral transformations.

1 Introduction

The concept of bicomplex number was first defined by Corrada Segre $(1863-1924)$ in 1892 . This number system has been considered as a generalization of complex numbers and is defined as $\mathbb{BC}=\{z_1+z_2j|z_1,\ \ z_2\in\mathbb{C},\ \ j^2=-1\}$ where j plays a role as an imaginary unit [10]. Any bicomplex number Z can be written as a linear combination of bases e_1 and e_2: $Z=(z_1-z_2i)e_1+(z_1+z_2i)e_2$, where the relationships between bases e_1 and e_2 are $e_1+e_2=1, e_1e_2=e_2e_1=0$. The idempotent coefficients are z_1-iz_2 and z_1+iz_2.
The aim of this study is to first recall the Fourier and Hartley transformations and then examine these transformations in a set of bicomplex numbers. As well known that the Fourier transform plays an important role in solving differential equations and integral equations. Especially in mathematical statistics, statistical mechanics problems, problems related to free vibration, diffusion, in geophysical engineering; measuring the resistance and strength of fault lines, the two-dimensional wave equation or Cauchy problems, solution of unknown $f(x)$ functions in integral equations, Fourier transforms are used effectively.
The Fourier transform of a function $f(x)$ with $k\in\mathbb{R}$ is represented by the symbol $\mathfrak{F}\{f(x)\}$, and it is defined as

$$\mathfrak{F}\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx. \tag{1}$$

This integral is called complex Fourier transform or exponential Fourier transform. The only condition for obtaining the Fourier transform of a function $f(x)$ is its absolute integration.
The inverse Fourier transform is represented by the symbol $\mathfrak{F}^{-1}\{F(k)\}=f(x)$ and is given as

$$\mathfrak{F}^{-1}\{F(k)\}=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}F(k)dk \tag{2}$$

where, $\mathfrak{F}^{-1}$ is called the inverse Fourier transform operator. Let a and b be any real constants. If the Fourier transforms of functions $f(x)$ and $g(x)$ are denoted by $\mathfrak{F}\{f(x)\}=F(k)$ and $\mathfrak{F}\{g(x)\}=G(k)$, respectively, the properties of the Fourier transform associated with these functions can be given in the following table.

$\mathfrak{F}\{af(x)+bg(x)\}=a\mathfrak{F}\{f(x)\}+b\mathfrak{F}\{g(x)\}=aF(k)+bG(k)$	Linearity property
$\mathfrak{F}\{e^{iax}f(x)\}=F(k-a)$	Translation Property
$\mathfrak{F}\{f(x-a)\}=e^{-ika}F(k)$	Shifting Property
$\mathfrak{F}\{f(ax)\}=\frac{1}{\lvert a\rvert}F(\frac{k}{a})$	Scaling property
$\mathfrak{F}\{x^nf(x)\}=(i)^n\frac{d^n}{dx^n}\{F(k)\}$	The property of multiplication by x^n
$If\ it's\ f^n(x)\to 0\ for\ \lvert x\rvert\to\infty,\ then\ \mathfrak{F}\{f^n(x)\}=(ik)^nF(k)$	Fourier Transform of Derivative

Table 1 Properties of the Fourier transform

Example. If $f(x)=e^{-a|x|}$ with $a\in\mathbb{R}$ and $w=u+iv\in\mathbb{C}$, then we find the Fourier transform of the function $f(x)$.

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{iwx} f(x)dx = \int_{-\infty}^{\infty} e^{iwx} e^{-a|x|} dx$$
$$\hat{f}(w) = \int_{-\infty}^{0} e^{iwx} e^{ax} dx + \int_{0}^{\infty} e^{iwx} e^{-ax} dx$$
$$\hat{f}(w) = \int_{-\infty}^{0} e^{(iw+a)x} dx + \int_{0}^{\infty} e^{(iw-a)x} dx.$$

If we write $u + iv$ instead of w, we get the following equality.

$$\hat{f}(w) = \int_{-\infty}^{0} e^{(iu-v+a)x} dx + \int_{0}^{\infty} e^{(iu-v-a)x} dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}(w) = \frac{2v}{a^2 + u^2 - v^2 + 2iuv}. \tag{3}$$

Now, let us mention from bicomplex Fourier transform. In this section, our aim is to extend the Fourier transform $\mathfrak{F} : \mathbb{D} \subset \mathbb{R} \to \mathbb{BC}$ in bicomlex variables from its complex version and examine its fundamental properties.

2 Bicomplex Fourier Transform

There are some studies on bicomplex Fourier transform in the literature. One of them belongs to Banerjee. In this section, we will first discuss and remind you of these transformations. Let $f(t)$ be a real valued continuous function for the values t, $-\infty < t < \infty$. Accordingly,

$$|f(t)| \leq c_1 e^{-\alpha t}, \ t \geq 0, \ \alpha > 0 \tag{4}$$

$$|f(t)| \leq c_2 e^{\beta t}, \ t \leq 0, \ \beta > 0. \tag{5}$$

The above equations show that f is absolutely integrable in the whole real plane [1].
The complex Fourier transform of $f(t)$ satisfying the condition $|\hat{f}_1(w_1)| < \infty$ where w_1 is a complex frequency, is defined as

$$\hat{f}_1(w_1) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iw_1 t} f(t)dt, \quad w_1 = x + iy.$$

Then,

$$|\hat{f}_1(w_1)| = |\int_{-\infty}^{\infty} e^{iw_1 t} f(t)dt| \leq \int_{-\infty}^{\infty} |e^{-yt} f(t)|dt = \int_{-\infty}^{0} e^{-yt}|f(t)|dt + \int_{0}^{\infty} e^{-yt}|f(t)|dt.$$

Thus, we write

$$|\hat{f}_1(w_1)| \leq c_2 \int_{-\infty}^{0} e^{(\beta-y)t} dt + c_1 \int_{0}^{\infty} e^{-(\alpha+y)t} dt = \frac{c_2}{\beta - y} + \frac{c_1}{\alpha + y}.$$

Note that in order to be $|\hat{f}_1(w_1)| < \infty$ it must be $\breve{}\alpha < y < \beta$.
That is, $|\hat{f}_1(w_1)|$ is holomorphic in the Ω_1 region below:

$$\Omega_1 = \{w_1 \in \mathbb{C} : -\infty < Re(w_1) < \infty, \ -\alpha < Im(w_1) < \beta\}. \tag{6}$$

Similarly, the complex Fourier transform of $f(t)$ associated with the other complex frequency w_2 is as follows:

$$\hat{f}_2(w_2) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iw_2 t} f(t)dt, \quad w_2 \in \mathbb{C}.$$

$\hat{f}_2(w_2)$ is holomorphic in the Ω_2 region below:

$$\Omega_2 = \{w_2 \in \mathbb{C} : -\infty < Re(w_2) < \infty, \ -\alpha < Im(w_2) < \beta\}. \tag{7}$$

The complex functions $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ can be written as follows with the help of idempotent bases e_1 and e_2:

$$\hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2 = \int_{-\infty}^{\infty} e^{iw_1 t} f(t)dte_1 + \int_{-\infty}^{\infty} e^{iw_2 t} f(t)dte_2$$

$$\hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2 = \int_{-\infty}^{\infty} e^{i(w_1 e_1 + w_2 e_2)t} f(t)dt = \int_{-\infty}^{\infty} e^{iwt} f(t)dt = \hat{f}(w).$$

Since $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ are complex holomorphic functions in Ω_1 and Ω_2, respectively, then the bicomplex function $\hat{f}(w)$ will be holomorphic in the region Ω:

$$\Omega = \{w \in \mathbb{BC} : w = w_1 e_1 + w_2 e_2, \ w_1 \in \Omega_1 \ and \ w_2 \in \Omega_2\}. \tag{8}$$

The complex-valued holomorphic functions $\hat{f}_1(w_1)$ and $\hat{f}_2(w_2)$ are absolutely convergent in the regions Ω_1 and Ω_2, respectively. Then the region of absolute convergence of $\hat{f}(w)$ is region Ω.

Let $w_1 = x_1 + ix_2,\ \ w_2 = y_1 + iy_2,\ \ x_1, x_2, y_1, y_2 \in \mathbb{R}$ be with $w_1, w_2 \in \mathbb{C}$. For $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$

$$-\infty < x_1, y_1 < \infty,\ \ -\alpha < x_2 < \beta,\ \ -\alpha < y_2 < \beta.$$

Using the last inequalities and the idempotent elements e_1, e_2, w transforms to $4-$ component form as follows:

$$w = \frac{x_1 + y_1}{2} + i\frac{x_2 + y_2}{2} + j\frac{y_2 - x_2}{2} + ij\frac{x_1 - y_1}{2} = a_0 + ia_1 + ja_2 + ija_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For the elements x_2, y_2 we have three possible states with respect to these. That is

1) If $x_2 = y_2$, then

$$-\alpha < a_1 < \beta,\ \ a_2 = 0.$$

2) If $x_2 > y_2$, then

$$-\alpha - a_2 < a_1 < \beta + a_2,\ \ -\frac{\alpha + \beta}{2} < a_2 < 0.$$

3) If $x_2 < y_2$, then

$$-\alpha + a_2 < a_1 < \beta - a_2,\ \ 0 < a_2 < \frac{\alpha + \beta}{2}.$$

In the above all three cases we have $-\infty < a_0, a_3 < \infty$. Thus, considering these inequalities we get the following results:

$$-\infty < a_0, a_3 < \infty$$

$$-\alpha + |a_2| < a_1 < \beta - |a_2| \quad 0 \leq |a_2| < \frac{\alpha + \beta}{2}.$$

And so, the convergence region Ω of $\hat{f}(w)$ is as follows:

$$\Omega = \{w \in \mathbb{BC} : -\infty < a_0, a_3 < \infty,\ \alpha + |a_2| < a_1 < \beta - |a_2|,\ \ 0 \leq |a_2| < \frac{\alpha + \beta}{2}\}. \tag{9}$$

Now, let us investigate the existence of the bicomplex Fourier transform $\hat{f}(w)$:

If $w = a_0 + ia_1 + ja_2 + ija_3 \in \Omega$, then we have

$$-\infty < a_0, a_3 < \infty,\ \alpha + |a_2| < a_1 < \beta - |a_2| \quad 0 \leq |a_2| < \frac{\alpha + \beta}{2}.$$

If we can write the number w using idempotent bases, that is

$$w = \{(a_0 + a_3) + i(a_1 - a_2)\}e_1 + \{(a_0 - a_3) + i(a_1 + a_2)\}e_2 = w_1e_1 + w_2e_2,$$

then we get the following inequalities:

1) If $a_2 = 0, -\alpha < a_1 < \beta$, then

$$-\alpha < a_1 - a_2 < \beta,\ \ -\alpha < a_1 + a_2 < \beta.$$

2) If $a_2 < 0$, using by the first side of inequality $-\alpha - a_2 < a_1 < \beta + a_2$, we get

$$-\alpha < a_1 + a_2,\ \ a_1 - a_2 < \beta.$$

Since $a_2 < 0$, if we combine these two results, we get

$$-\alpha < a_1 + a_2 < a_1 - a_2 < \beta.$$

From here, this state can be interpreted as

$$-\alpha < a_1 + a_2 < a_1 - a_2,\ \ a_1 + a_2 < a_1 - a_2 < \beta.$$

3) If $a_2 > 0$, from the first side of inequality we get

$$-\alpha + a_2 < a_1 < \beta - a_2,\ \ -\alpha < a_1 - a_2.$$

And $a_1 + a_2 < \beta$ is obtained from the end side. Since $a_2 > 0$, if we combine the two result, the we have

$$-\alpha < a_1 - a_2 < a_1 + a_2 < \beta.$$

So, we can write as

$$-\alpha < a_1 - a_2 < a_1 + a_2,\ \ a_1 - a_2 < a_1 + a_2 < \beta.$$

Let $f(t)$ be a real valued continuous function in the interval $(-\infty, \infty)$ that satisfies (4) and (5). Then the Fourier transform of $f(t)$ is as follows:

$$\hat{f}(w) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{iwt} f(t)dt, \quad w \in \mathbb{BC}. \tag{10}$$

We know that the Fourier transform $\hat{f}(w)$ exists and holomorphic for all $w \in \Omega$, where Ω is the convergence region of $\hat{f}(w)$ [1].
That is if the function $f(t)$ satisfies (4) and (5) is a continuous and real-valued function for $-\infty < t < \infty$, then the Fourier transform $\hat{f}(w)$

exist in the region (9). Let's give this theorem.

Theorem 1. *[1] Let the Fourier transforms of $f(t)$ and $g(t)$ functions be $\hat{f}(w)$ and $\hat{g}(w)$, respectively. If $\hat{f}(w) = \hat{g}(w)$, then*

$$f(t) = g(t). \tag{11}$$

Now, let's give some of basic Properties of the Fourier transform.

1. **Linearity Property**. Let the Fourier transforms of $f(t)$ and $g(t)$ be $\hat{f}(w)$ and $\hat{g}(w)$, respectively. Then linearity property can be given as follows:

$$\mathfrak{F}\{af(t) + bg(t)\} = a\hat{f}(w) + b\hat{g}(w)$$

where a and b are constants in the region Ω [1].

2. **Shifting Property**. Let $\hat{f}(w)$ be the Fourier transform of the continuous function $f(t)$. Then, $\mathfrak{F}\{f(t-a)\} = e^{iwa}\hat{f}(w)$ with $t - a \in \Omega$ [1].

3. **Scaling Property**. Let $\hat{f}(w)$ be the Fourier transform of the continuous function $f(t)$. Then, with $a \neq 0$,

$$\mathfrak{F}\{f(at)\} = \frac{1}{|a|}\hat{f}(\frac{w}{a}).$$

Before giving the convolution theorem for the Fourier transform, let us give the convolution property.

4. **Convolution Property.** If $f(t)$ and $g(t)$ functions are piecewise continuous in the interval $[0, \infty)$, then convolution of f and g is as follows and denoted by $f * g$:

$$f(t) * g(t) = \int_0^t f(x)g(t-x)dx.$$

Theorem 2. *(Convolution Theorem) [1] If the Fourier transforms of $f(t)$ and $g(t)$ functions be $\hat{f}(w)$ and $\hat{g}(w)$ respectively, then*

$$\mathfrak{F}f(t) * g(t) = \mathfrak{F}\{\int_{-\infty}^{\infty} f(u)g(t-a)du\} = \hat{f}(w)\hat{g}(w).$$

Theorem 3. *[1] If $f(t)$ and $t^r f(t)$ functions are integrable in the interval $-\infty < t < \infty$ with $r = 1, 2, \ldots, n$, then*

$$\mathfrak{F}\{t^n f(t)\} = i^n \frac{d^n}{dw^n}\hat{f}(w). \tag{12}$$

Where, $\hat{f}(w)$ is the Fourier transform of the function $f(t)$.

Proof: For $n = 1$, let's examine the right side of the equation.

$$\frac{d}{dw}\hat{f}(w) = \frac{\partial}{w_1}\hat{f}_1(w_1)e_1 + \frac{\partial}{w_2}\hat{f}_2(w_2)e_2$$

$$= \frac{\partial}{w_1}\int_{-\infty}^{\infty} e^{iw_1 t} f(t)dte_1 + \frac{\partial}{w_2}\int_{-\infty}^{\infty} e^{iw_2 t} f(t)dte_2$$

When the necessary operations are done, we obtain the following equality:

$$\frac{d}{dw}\hat{f}(w) = i\mathfrak{F}tf(t).$$

Thus, $\mathfrak{F}\{tf(t)\} = -i\frac{d}{dw}\hat{f}(w)$. Similarly, let's see for $n = 2$.

$$\frac{d^2}{dw^2}\hat{f}(w) = \frac{d}{dw}[\frac{d}{dw}\hat{f}(w)] = i\frac{d}{dw}[\int_{-\infty}^{\infty} e^{iwt} f(t)dt]$$

$$= i\frac{\partial}{w_1}\int_{-\infty}^{\infty} e^{iw_1 t} tf(t)dte_1 + i\frac{\partial}{w_2}\int_{-\infty}^{\infty} e^{iw_2 t} tf(t)dte_2$$

When the necessary operations are done, we obtain the following equality:

$$\frac{d^2}{dw^2}\hat{f}(w) = -\mathfrak{F}t^2 f(t).$$

Thus, $\mathfrak{F}\{t^2 f(t)\} = (-i)^2\frac{d^2}{dw^2}$. If this is continued, the following result will be obtained:

$$\mathfrak{F}\{t^n f(t)\} = i^n \frac{d^n}{dw^n}\hat{f}(w).$$

□

Theorem 4. *[1] If $f(t)$ and $f^{(r)}(t)$, $r = 1, 2, ..., n$, are piecewise smooth and tend to 0 as $|t| \to \infty$ and f with its derivatives of order up to n are integrable in $-\infty < t < \infty$, then*

$$\mathfrak{F}\{f^n(t)\} = (-iw)^n \hat{f}(w), \tag{13}$$

where $\hat{f}(w)$ is the Fourier transform of the function $f(t)$ and $f^r(t) = \frac{d^r}{dt^r} f(t)$.

Example. If $f(t) = \begin{cases} e^{-2t}, & t > 0 \\ 0, & x \leq 0 \end{cases}$, where $\alpha = 2$, β any positive number,
then we get $\hat{f}(w) = \frac{1}{2-iw}$. Actually,

$$\hat{f}(w) = \int_0^\infty e^{-2t} e^{iwt} dt = \int_0^\infty e^{-t(2-iw)} dt = [\frac{e^{-t(2-iw)}}{2-iw}]_0^\infty = \frac{1}{2-iw}.$$

Its the convergent region is

$$\Omega = \{w = a_0 + ia_1 + ja_2 + ija_3 \in \mathbb{BC} : 0 \leq |a_2| < \frac{2+\beta}{2},\ a_1 > -2\}$$

where $\beta \in \mathbb{Z}^+$.

Example. If $f(x) = e^{-a|x|}$ with $a \in \mathbb{R}$ and $w = w_1 + jw_2 \in \mathbb{BC}$, then we find the Fourier transform of the function $f(x)$. Where w_1 is $F_n + iF_{n+1}$, w_2 is $F_{n+2} + iF_{n+3}$ and F_n is nth Fibonacci number.
The Fourier transform of $\hat{f}(w)$ could be written as follows with the help of idempotent bases:

$$\hat{f}(w) = \hat{f}_1(w_1)e_1 + \hat{f}_2(w_2)e_2.$$

Let's first find the $\hat{f}_1(w_1)$ Fourier transform.

$$\hat{f}_1(w_1) = \int_{-\infty}^\infty e^{iw_1x} e^{-a|x|} dx$$

$$= \int_{-\infty}^0 e^{iw_1x} e^{ax} dx + \int_0^\infty e^{iw_1x} e^{-ax} dx$$

$$= \int_{-\infty}^0 e^{(iw_1+a)x} dx + \int_0^\infty e^{(iw_1-a)x} dx.$$

If we write $F_n + iF_{n+1}$ instead of w_1, we get the following equality:

$$\hat{f}_1(w_1) = \int_{-\infty}^0 e^{(iF_n - F_{n+1}+a)x} dx + \int_0^\infty e^{(iF_n - F_{n+1}-a)x} dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}_1(w_1) = \frac{2a}{a^2 + F_n^2 - F_{n+1}^2 + 2iF_nF_{n+1}}.$$

Now, let's find the $\hat{f}_2(w_2)$ Fourier transform.

$$\hat{f}_2(w_2) = \int_{-\infty}^\infty e^{iw_2x} e^{-a|x|} dx$$

$$= \int_{-\infty}^0 e^{iw_2x} e^{ax} dx + \int_0^\infty e^{iw_2x} e^{-ax} dx$$

$$= \int_{-\infty}^0 e^{(iw_2+a)x} dx + \int_0^\infty e^{(iw_2-a)x} dx.$$

If we write $F_{n+2} + iF_{n+3}$ instead of w_2, we get the following equality:

$$\hat{f}_2(w_2) = \int_{-\infty}^0 e^{(iF_{n+2} - F_{n+3}+a)x} dx + \int_0^\infty e^{(iF_{n+2} - F_{n+3}-a)x} dx.$$

When the necessary operations are done, we get the following equality:

$$\hat{f}_2(w_2) = \frac{2a}{a^2 + F_{n+2}^2 - F_{n+3}^2 + 2iF_{n+2}F_{n+3}}.$$

Thus,

$$\hat{f}(w) = 2a(\frac{1}{a^2 + F_n^2 - F_{n+1}^2 + 2iF_nF_{n+1}} e_1 + \frac{1}{a^2 + F_{n+2}^2 - F_{n+3}^2 + 2iF_{n+2}F_{n+3}} e_2).$$

3 Bicomplex Hartley Transform

Let us briefly mention about the Hartley transformation. The Hartley transform is an integral transformation that maps to a real valued frequency function through the kernel $casvx = cosvx + sinvx$. This new version of the Fourier transform was discovered by Ralph Vinton Lyon Hartley in 1942. Sinusoids are waves that acting between certain frequencies and certain amplitudes that repeat. The Fourier transform involves the complex sum of real and imaginary numbers and sinusoidal functions. The Hartley transform includes the real sum of real numbers and sinusoidal functions. Both transformations were effective in solving fluctuating events.Hartley transform is a transform obtained by taking the function cas instead of the exponential kernel in the Fourier transform. The Hartley transformation is as

$$H(f) = \int_{-\infty}^{\infty} caswtV(t)dt = \int_{-\infty}^{\infty} cas2\pi ftV(t)dt. \tag{14}$$

Where t is time, $V(t)$ is a absolute integrable function in the $(-\infty, \infty)$ depends on t, f is frequency, w is angular frequency and $cast = cost + sint$. This integral transform is called the Hartley transform of the $V(t)$ function. Additionally, $w = 2\pi f$. $V(t)$ function is found as

$$V(t) = \int_{-\infty}^{\infty} caswtH(f)dt = \int_{-\infty}^{\infty} cas2\pi ftH(f)dt. \tag{15}$$

Now, let's mention about the bicomplex Hartley transformation. So, let us first give the complex Hartley transform.
Let's define the complex Hartley transform $\mathbb{H} : \mathbb{D} \subset \mathbb{R} \to \mathbb{C}$. We can define the complex Hartley transformation of $V(t)$ satisfying the condition $|H_1(w_1)| < \infty$, where w_1 is a complex frequency as

$$H_1(w_1) = \int_{-\infty}^{\infty} cas2\pi w_1 tV(t)dt.$$

The transform $H_1(w_1)$ is holomorphic in the region

$$\Omega_1 = w_1 \in \mathbb{C} : -\infty < Re(w_1) < \infty.$$

Similarly, the complex Hartley transform of $V(t)$ associated with the other complex frequency w_2 is as follows:

$$H_2(w_2) = \int_{-\infty}^{\infty} cas2\pi w_2 tV(t)dt.$$

The transform $H_2(w_2)$ is holomorphic in the region

$$\Omega_2 = w_2 \in \mathbb{C} : -\infty < Re(w_2) < \infty.$$

The complex functions $H_1(w_1)$ and $H_2(w_2)$ are written with the help of idempotent bases e_1 and e_2 as follows:

$$H_1(w_1)e_1 + H_2(w_2)e_2 = \int_{-\infty}^{\infty} cas2\pi w_1 tV(t)dte_1 + \int_{-\infty}^{\infty} cas2\pi w_2 tV(t)dte_2$$

$$= \int_{-\infty}^{\infty} (cos2\pi w_1 t + sin2\pi w_1 t)V(t)e_1 + \int_{-\infty}^{\infty} (cos2\pi w_2 t + sin2\pi w_2 t)V(t)e_2$$

$$= \int_{-\infty}^{\infty} cas2\pi wtV(t)dt = H(w).$$

Since $H_1(w_1)$ and $H_2(w_2)$ are complex holomorphic functions in the regions Ω_1 and Ω_2, respectively, the $H(w)$ bicomplex function is holomorphic in the region

$$\Omega = w \in \mathbb{BC} : w = w_1 e_1 + w_2 e_2, w_1 \in \Omega_1 \ and \ w_2 \in \Omega_2. \tag{16}$$

The complex valued $H_1(w_1)$ and $H_2(w_2)$ holomorphic functions are convergent in the regions Ω_1 and Ω_2, respectively. Then, the convergence region of $H(w)$ is region Ω.
Let $w_1 = x_1 + ix_2,\ w_2 = y_1 + iy_2,\ \ x_1, x_2, y_1, y_2 \in \mathbb{R}$ be with $w_1, w_2 \in \mathbb{C}$.

Where, $-\infty < x_1, y_1 < \infty$ for $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. Using the $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ equations, w is transformed into a 4-component form.

$$w = \frac{x_1 + y_1}{2} + i\frac{x_2 + y_2}{2} + j\frac{y_2 - x_2}{2} + ij\frac{x_1 - y_1}{2} = a_0 + ia_1 + ja_2 + ija_3.$$

Where, $a_0, a_1, a_2, a_3 \in \mathbb{R}$. There are three possible cases:

1. If $x_1 = y_1$, then

$$a_3 = 0.$$

2. If $x_1 > y_1$, then

$$\frac{x_1 - y_1}{2} > 0.$$

That is, $a_3 > 0$.
3. If $x_1 < y_1$,then

$$\frac{x_1 - y_1}{2} < 0.$$

That is, $a_3 < 0$.

Thus, from these three cases we get the following inequalities:

$$-\infty < a_0 < \infty,\ -\infty < a_3 < \infty. \tag{17}$$

So, the region of convergence of $H(w)$ is as follows:

$$\Omega = \{w \in \mathbb{BC} : -\infty < a_0 < \infty, -\infty < a_3 < \infty\}. \tag{18}$$

Let $V(t)$ be a real valued continuous function in the $(-\infty, \infty)$. The Hartley transformation of $V(t)$ is as follows:

$$H(w) = \int_{-\infty}^{\infty} cas2\pi wtV(t)dt, \quad w \in \mathbb{BC}. \tag{19}$$

The Hartley transform $H(w)$ exists and holomorphic for all $w \in \Omega$, where Ω is the convergence region of $H(w)$. We give the following theorems without proof.

Theorem 5. *If the function $V(t)$ is a continuous and real-valued function for $-\infty < t < \infty$, Then the Hartley transform $H(w)$ exists in the region (18).*

Theorem 6. *Let Hartley transformations of $V_1(t)$ and $V_2(t)$ functions be $H_1(w)$ and $H_2(w)$, respectively. If $H_1(w) = H_2(w)$, then $V_1(t) = V_2(t)$.*

4 Conclusion

In future studies, different properties of these special integral transforms, which we have defined and examined here, can be examined.

5 References

1 A. Banerjee, S. K. Datta, M. A. Hoque, *Fourier transform for functions of bicomplex variables*, arXiv preprint arXiv: 1404.4236, (2014).
2 M. Futagawa, *On the theory of functions of a quaternary variable*, Tohoku Mathematical Journal, First Series, **29** (1928), 175-222.
3 M. Futagawa, *On the theory of functions of a quaternary variable (Part II)*, Tohoku Mathematical Journal, First Series, **35**, (1932), 69-120.
4 K. Koklu, *İntegral Dönüşüm ve Uygulamaları*, Papatya Yayıncılık, İstanbul, 2018.
5 M. E. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac, *Bicomplex numbers and their elementary functions.* Cubo (Temuco), **14**(2) (2012), 61-80.
6 G. B. Price, *An introduction to multicomplex spaces and functions.*, M. Dekker, New York, 1991.
7 J. D. Riley, *Contributions to the theory of functions of a bicomplex variable.* Tohoku Mathematical Journal, Second Series, **5**(2) (1953), 132-165.
8 D. Rochon, M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, Anal. Univ. Oradea, fasc. math, **11**(71), (2004), 110.
9 S. Rönn, *Bicomplex algebra and function theory*, (2001). arXiv preprint math/0101200.
10 C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti peralgebrici.* Mathematische Annalen, **40**(3), (1892), 413-467.
11 P. Usta Puhl, *Hartley dönüşümleri ve uygulamaları*, MsC Thesis, Yildiz Technical Uni, 2016.