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Research Article

Sequential Abstract Generalized Right Side Fractional Landau Inequalities

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ABSTRACT. We give uniform and L_p Caputo-Bochner abstract sequential generalized right fractional Landau inequalities over $\mathbb{R}_-$. These estimates the size of second and third sequential abstract generalized right fractional derivatives of a Banach space valued function over $\mathbb{R}_-$. We give an application when the basic fractional order is $\frac{1}{2}$.

Keywords: Sequential abstract generalized right fractional Landau inequality, sequential Caputo abstract generalized right fractional derivative.

2020 Mathematics Subject Classification: 26A33, 26D10, 26D15.

1. INTRODUCTION

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \le C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [5]. The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$, he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{2}$$

are the best constants in (1). In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \le 2 \text{ for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \le 2$ for $p \in [1, \infty)$. In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \le \sqrt{2}$. We need the following concept from abstract fractional calculus. Our integral next is of Bochner type [11]. We need

Definition 1.1. *([4], p. 105) Let $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a, b])$ and increasing, $f \in C([a, b], X)$, $\nu > 0$.*

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We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{b-;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (g(z) - g(x))^{\nu-1} g'(z) f(z) \, dz, \tag{5}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 4.11, p. 101, [4], we get that $\left(J_{b-;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{b-;g}^{0} f := f$ and see that $\left(J_{b-;g}^{\nu} f\right)(b) = 0$.

We also need

Definition 1.2. *([4], p. 107) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We define the right generalized g-fractional derivative X-valued of f of order α as follows:*

$$\left(D_{b-;g}^{\alpha} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)}(g(t)) \, dt, \tag{6}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type. Ordinary vector valued derivative is as in [12], similar to numerical one. If $\alpha \notin \mathbb{N}$, by Theorem 4.11, p. 101, [4], we have that $\left(D_{b-;g}^{\alpha} f\right) \in C([a,b], X)$. We see that

$$\left(J_{b;g}^{n-\alpha}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{b-;g}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{7}$$

We set

$$D_{b-;g}^{n} f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \;\; n \in \mathbb{N}, \tag{8}$$

$$D_{b-;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha} f = D_{b-;id}^{\alpha} f = D_{b-}^{\alpha} f, \tag{9}$$

the usual left X-valued Caputo fractional derivative, see [4], Chapter 2.

By convention, we suppose that

$$\left(D_{x_0-;g}^{\alpha} f\right)(x) = 0, \;\text{ for } x > x_0 \tag{10}$$

for any $x, x_0 \in [a,b]$.

Denote the sequential (also called iterated) generalized left fractional derivative by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} \ldots D_{b-;g}^{\alpha} \quad (n \text{ times}),\, n \in \mathbb{N}. \tag{11}$$

We need the following g-right generalized modified X-valued Taylor's formula.

Theorem 1.1 **([4, p. 120]).** *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$ strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Let $F_k := D_{b-;g}^{k\alpha} f$, $k = 1, \ldots, n$, that fulfill $F_k \in C^1([a,b], X)$, and $F_{n+1} \in C([a,b], X)$. Then,*

$$f(x) = \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f\right)(b) + \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t) \, dt, \tag{12}$$

$\forall\, x \in [a,b]$.

We make

Remark 1.1 (to Theorem 1.1). *When* $0<\alpha<1$, *by* (6), *we get*

$$\left(D_{b-;g}^{\alpha}f\right)(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{x}^{b}(g(t)-g(x))^{-\alpha}g'(t)\left(f\circ g^{-1}\right)'(g(t))\,dt, \tag{13}$$

$\forall\, x\in[a,b]$.
Hence,

$$\begin{aligned}\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| &\leq \frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}(g(t)-g(x))^{-\alpha}g'(t)\left\|\left(f\circ g^{-1}\right)'(g(t))\right\|dt\\ &\leq \frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)}\left(\int_{x}^{b}(g(t)-g(x))^{-\alpha}g'(t)\,dt\right)\\ &=\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)}(g(b)-g(x))^{1-\alpha}.\end{aligned} \tag{14}$$

That is

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\|\leq\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)}(g(b)-g(x))^{1-\alpha}<\infty, \tag{15}$$

$\forall\, x\in[a,b]$, $0<\alpha<1$. *Hence, it holds*

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(b)\right\|=0,$$

i.e.

$$\left(D_{b-;g}^{\alpha}f\right)(b)=0, \tag{16}$$

when $0<\alpha<1$.

The author has already done an extensive amount of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However, there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]). Usually there the domains, where $[A,+\infty)$ or $(-\infty,B]$, with $A,B\in\mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.
In this work with less assumptions, we establish uniform and L_p type right Caputo-Bochner abstract sequential generalized fractional Landau inequalities over $\mathbb{R}_-$. The method of proving is based on right Caputo-Bochner sequential generalized fractional Taylor's formula with integral remainder, see Theorem 1.1.
We give also an application for $\alpha=\frac{1}{2}$. Clearly we are also inspired by [3], [4].

2. Main Results

We present the following abstract sequential generalized fractional Landau inequalities over $\mathbb{R}_-$.

Theorem 2.2. *Let* $g\in C^1(\mathbb{R}_-)$ *strictly increasing, with* $g^{-1}\in C^1(g(\mathbb{R}_-))$. *Let* $0<\alpha<1$, $f\in C^1(\mathbb{R}_-,X)$ *with* $\|\|f\|\|_{\infty,\mathbb{R}_-}$, $\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}_-}<\infty$. *For* $k=1,2,3$, *we assume that* $D_{b-;g}^{k\alpha}f\in C^1((-\infty,b],X)$ *and* $D_{b-;g}^{4\alpha}f\in C((-\infty,b],X)$, $\forall\, b\in\mathbb{R}_-$. *We further assume that*

$$\overline{K_g}:=\left\|\left\|D_{b-;g}^{4\alpha}f(t)\right\|\right\|_{\infty,\mathbb{R}_-^2}<\infty, \tag{17}$$

where $(b,t) \in \mathbb{R}_-^2$. *Then*

$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^\alpha-1)} \sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^\alpha+1)}{\Gamma(4\alpha+1)} \||f\||_{\infty,\mathbb{R}_-} \overline{K_g}} \tag{18}$$

and

$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \frac{4\sqrt[4]{2}\,\Gamma(3\alpha+1)(\Gamma(4\alpha+1))^{-\frac{3}{4}}(2^{2\alpha}+1)}{\left(\sqrt[4]{3}\right)^3 \left(\sqrt{2}\right)^\alpha (2^\alpha-1)} \||f\||_{\infty,\mathbb{R}_-}^{\frac{1}{4}} \overline{K_g}^{\frac{3}{4}}. \tag{19}$$

That is $\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\|, \sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| < \infty.$

Proof. We notice easily again here that $\left(D_{b-;g}^{\alpha} f \right)(b) = 0$, $\forall\, b \in \mathbb{R}_-$. We make use of Theorem 1.1 for $0 < \alpha < 1$ and $n = 3$, applied for any $b \in \mathbb{R}_-$ and $a = -\infty$. Momentarily, we fix $b \in \mathbb{R}_-$. Let $x_2 < x_1 < b$, then $g(x_2) < g(x_1) < g(b)$, and

$$\begin{aligned} f(x_1) - f(b) = {} & \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f \right)(b) + \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f \right)(b) \\ & + \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t)-g(x_1))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt, \end{aligned} \tag{20}$$

and

$$\begin{aligned} f(x_2) - f(b) = {} & \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f \right)(b) + \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f \right)(b) \\ & + \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (g(t)-g(x_2))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt. \end{aligned} \tag{21}$$

That is

$$\begin{aligned} & \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f \right)(b) + \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f \right)(b) \\ = & f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t)-g(x_1))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt =: A, \end{aligned} \tag{22}$$

and

$$\begin{aligned} & \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} \left(D_{b-;g}^{2\alpha} f \right)(b) + \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \left(D_{b-;g}^{3\alpha} f \right)(b) \\ = & f(x_2) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (g(t)-g(x_2))^{4\alpha-1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt =: B. \end{aligned} \tag{23}$$

We are solving the above system of two equations with two unknowns $\left(D^{2\alpha}_{b-;g}f\right)(b)$, $\left(D^{3\alpha}_{b-;g}f\right)(b)$. The main determinant of system is

$$D := \begin{vmatrix} \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \\ \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} & \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix}$$
$$= \frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}$$
$$\times \left[(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{3\alpha} - (g(b)-g(x_1))^{3\alpha}(g(b)-g(x_2))^{2\alpha}\right]$$
$$= \frac{(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(g(b)-g(x_2))^{\alpha} - (g(b)-g(x_1))^{\alpha}\right] > 0,$$

i.e.

$$D = \frac{(g(b)-g(x_1))^{2\alpha}(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\left[(g(b)-g(x_2))^{\alpha} - (g(b)-g(x_1))^{\alpha}\right] > 0. \tag{24}$$

We obtain the unique solution

$$\left(D^{2\alpha}_{b-;g}f\right)(b) = \frac{\begin{vmatrix} A & \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)} \\ B & \frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)} \end{vmatrix}}{D},$$
$$\left(D^{3\alpha}_{b-;g}f\right)(b) = \frac{\begin{vmatrix} \frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)} & A \\ \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)} & B \end{vmatrix}}{D}. \tag{25}$$

Therefore, we have

$$\left(D^{2\alpha}_{b-;g}f\right)(b) = \frac{\frac{(g(b)-g(x_2))^{3\alpha}}{\Gamma(3\alpha+1)}A - \frac{(g(b)-g(x_1))^{3\alpha}}{\Gamma(3\alpha+1)}B}{D},$$
and
$$\left(D^{3\alpha}_{b-;g}f\right)(b) = \frac{\frac{(g(b)-g(x_1))^{2\alpha}}{\Gamma(2\alpha+1)}B - \frac{(g(b)-g(x_2))^{2\alpha}}{\Gamma(2\alpha+1)}A}{D}. \tag{26}$$

We have the following

$$\|A\| = \left\| f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)}\int_{x_1}^{b}(g(t)-g(x_1))^{4\alpha-1}g'(t)\left(D^{4\alpha}_{b-;g}f\right)(t)\,dt \right\|$$
$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{\left\|\left\|D^{4\alpha}_{b-;g}f(t)\right\|\right\|_{\infty,\mathbb{R}^2_-}}{\Gamma(4\alpha+1)}(g(b)-g(x_1))^{4\alpha} \tag{27}$$

under the assumption $\|\|f\|\|_{\infty,\mathbb{R}_-} < \infty$. That is

$$\|A\| \leq 2\,\|\|f\|\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)}(g(b)-g(x_1))^{4\alpha}, \tag{28}$$

and similarly,

$$\|B\| \leq 2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} (g(b) - g(x_2))^{4\alpha}, \tag{29}$$

where by assumption

$$\overline{K_g} := \left\|\left\|D^{4\alpha}_{b-;g} f(t)\right\|\right\|_{\infty,\mathbb{R}^2_-} < \infty, \tag{30}$$

with $(b,t) \in \mathbb{R}^2_-$. Consequently, we have

$$\begin{aligned} \left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \frac{1}{\Gamma(3\alpha+1) D} \left[(g(b) - g(x_2))^{3\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} (g(b) - g(x_1))^{4\alpha}\right)\right. \\ &\left. + (g(b) - g(x_1))^{3\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} (g(b) - g(x_2))^{4\alpha}\right)\right], \end{aligned} \tag{31}$$

and

$$\begin{aligned} \left\|\left(D^{3\alpha}_{b-;g} f\right)(b)\right\| &\leq \frac{1}{\Gamma(2\alpha+1) D} \left[(g(b) - g(x_1))^{2\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} (g(b) - g(x_2))^{4\alpha}\right)\right. \\ &\left. + (g(b) - g(x_2))^{2\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} (g(b) - g(x_1))^{4\alpha}\right)\right]. \end{aligned} \tag{32}$$

Set now $g(x_1) := g(b) - h$, $g(x_2) := g(b) - 2h$, where $h > 0$, so that $g(b) - g(x_1) = h$, $g(b) - g(x_2) = 2h$. Hence, we get

$$D = \frac{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} > 0. \tag{33}$$

Therefore, we derive (from (26))

$$\begin{aligned} \left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} \left[2^{3\alpha} h^{3\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} h^{4\alpha}\right)\right. \\ &\left. + h^{3\alpha} \left(2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} 2^{4\alpha} h^{4\alpha}\right)\right] \end{aligned} \tag{34}$$

$$\begin{aligned} &= \frac{\Gamma(2\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} \left[2 \left\|\|f\|\right\|_{\infty,\mathbb{R}_-} \left(2^{3\alpha} + 1\right) h^{3\alpha} + \frac{\overline{K_g}}{\Gamma(4\alpha+1)} \left(2^{3\alpha} + 2^{4\alpha}\right) h^{7\alpha}\right] \\ &= \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2^\alpha - 1)}\right) \left[\frac{2\left(2^{3\alpha} + 1\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha} (2^\alpha + 1)}{\Gamma(4\alpha+1)} \overline{K_g} h^{2\alpha}\right]. \end{aligned} \tag{35}$$

That is

$$\begin{aligned} \left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2^\alpha - 1)}\right) \\ &\times \left[\frac{2\left(2^{3\alpha} + 1\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha} (2^\alpha + 1)}{\Gamma(4\alpha+1)} \overline{K_g} h^{2\alpha}\right], \end{aligned} \tag{36}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\begin{aligned} \sup_{b \in \mathbb{R}_-} \left\|\left(D^{2\alpha}_{b-;g} f\right)(b)\right\| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2^\alpha - 1)}\right) \\ &\times \left[\frac{2\left(2^{3\alpha} + 1\right) \left\|\|f\|\right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{2^{3\alpha} (2^\alpha + 1)}{\Gamma(4\alpha+1)} \overline{K_g} h^{2\alpha}\right] < \infty, \end{aligned} \tag{37}$$

$\forall\, h>0$, $0<\alpha<1$. By (26), we derive

$$\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[h^{2\alpha}\left(2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}2^{4\alpha}h^{4\alpha}\right)\right.$$
$$\left.+2^{2\alpha}h^{2\alpha}\left(2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}h^{4\alpha}\right)\right]$$
$$(38)\qquad =\frac{\Gamma(3\alpha+1)}{2^{2\alpha}h^{5\alpha}(2^{\alpha}-1)}\left[2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}\left(2^{2\alpha}+1\right)h^{2\alpha}+\frac{\overline{K_g}}{\Gamma(4\alpha+1)}\left(2^{4\alpha}+2^{2\alpha}\right)h^{6\alpha}\right]$$
$$=\left(\frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)}\right)\left[\frac{2\left(2^{2\alpha}+1\right)\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\left(2^{2\alpha}+1\right)}{\Gamma(4\alpha+1)}\overline{K_g}h^{\alpha}\right]$$
$$=\frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}\left[\frac{2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right].$$

That is

$$(39)\qquad \left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}\times\left[\frac{2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right],$$

$\forall\, b\in\mathbb{R}_-$, $\forall\, h>0$. I.e., it holds

$$(40)\qquad \sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\| \leq \frac{\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{2^{2\alpha}(2^{\alpha}-1)}\times\left[\frac{2\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}}+\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}h^{\alpha}\right]<\infty,$$

$\forall\, h>0$, $0<\alpha<1$. Call

$$(41)\qquad \mu := 2\left(2^{3\alpha}+1\right)\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_-},$$

$$(42)\qquad \theta = \frac{2^{3\alpha}\left(2^{\alpha}+1\right)\overline{K_g}}{\Gamma(4\alpha+1)},$$

both are greater than zero. Set also $\rho:=2\alpha$; $0<\rho<2$. We consider the function

$$(43)\qquad y(h) := \mu h^{-\rho}+\theta h^{\rho},\quad \forall\, h>0.$$

We have

$$(44)\qquad y'(h) = -\rho\mu h^{-\rho-1}+\rho\theta h^{\rho-1}=0,$$

then

$$\theta h^{2\rho}=\mu,$$

with a unique solution

$$(45)\qquad h_0 := h_{crit.no.} = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}.$$

We have that

$$(46)\qquad y''(h) = \rho(\rho+1)\mu h^{-\rho-2}+\rho(\rho-1)\theta h^{\rho-2}.$$

We see that

$$y''(h_0) = y''\left(\left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}\right) = \rho(\rho+1)\mu\left(\frac{\mu}{\theta}\right)^{\frac{-\rho-2}{2\rho}} + \rho(\rho-1)\theta\left(\frac{\mu}{\theta}\right)^{\frac{\rho-2}{2\rho}}$$
$$= \rho\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}}\left[(\rho+1)\sqrt{\mu\theta} + (\rho-1)\sqrt{\mu\theta}\right]$$
$$= \rho\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}}\left(2\rho\sqrt{\mu\theta}\right) = 2\rho^2\sqrt{\mu\theta}\left(\frac{\theta}{\mu}\right)^{\frac{1}{\rho}} > 0.$$

Therefore, y has a global minimum at $h_0 = \left(\frac{\mu}{\theta}\right)^{\frac{1}{2\rho}}$, which is

$$y(h_0) = \mu\left(\frac{\mu}{\theta}\right)^{-\frac{1}{2}} + \theta\left(\frac{\mu}{\theta}\right)^{\frac{1}{2}} = \mu\left(\frac{\theta}{\mu}\right)^{\frac{1}{2}} + \sqrt{\theta\mu} = 2\sqrt{\theta\mu}.$$

We have proved that (see (37))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D_{b-;g}^{2\alpha}f\right)(b)\right\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^\alpha-1)}$$
$$\times\sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^\alpha+1)}{\Gamma(4\alpha+1)}\||f\||_{\infty,\mathbb{R}_-}\overline{K_g}}. \tag{47}$$

Call

$$\xi := 2\||f\||_{\infty,\mathbb{R}_-},$$
$$\psi := \frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}, \tag{48}$$

both are greater than zero. We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^\alpha, \quad \forall\, h > 0. \tag{49}$$

We have

$$\gamma'(h) = -3\alpha\xi h^{-3\alpha-1} + \alpha\psi h^{\alpha-1} = 0,$$

then

$$\psi h^{4\alpha} = 3\xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}. \tag{50}$$

We have that

$$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \alpha(\alpha-1)\psi h^{\alpha-2}. \tag{51}$$

We see

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\xi}{\psi}\right)^{\frac{-3\alpha-2}{4\alpha}} + \alpha(\alpha-1)\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}$$
$$= \alpha\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}\left[3(3\alpha+1)\xi\frac{\psi}{3\xi} + (\alpha-1)\psi\right]$$
$$= \alpha\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}}(4\alpha\psi) = 4\alpha^2\psi\left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} > 0. \tag{52}$$

Therefore, γ has a global minimum at $h_0 = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}$, which is

$$\gamma(h_0) = \xi\left(\frac{3\xi}{\psi}\right)^{-\frac{3}{4}} + \psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}$$

$$= \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}\left(\xi\frac{\psi}{3\xi} + \psi\right) = \frac{4}{3}\psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}. \tag{53}$$

Consequently,

$$\gamma(h_0) = \frac{4}{3}\psi\left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} = \frac{4}{\left(\sqrt[4]{3}\right)^3}\psi^{\frac{3}{4}}\xi^{\frac{1}{4}}. \tag{54}$$

We have proved that (see (40))

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{3\alpha}_{b-;g}f\right)(b)\right\| \le \frac{4\Gamma(3\alpha+1)\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{2\alpha}\left(2^{\alpha}-1\right)}$$

$$\times\left(2\,\|\|f\|\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{4}}\left(\frac{2^{2\alpha}\overline{K_g}}{\Gamma(4\alpha+1)}\right)^{\frac{3}{4}}$$

$$= \frac{4\sqrt[4]{2}\Gamma(3\alpha+1)\Gamma(4\alpha+1)^{-\frac{3}{4}}\left(2^{2\alpha}+1\right)}{\left(\sqrt[4]{3}\right)^3 2^{\frac{\alpha}{2}}\left(2^{\alpha}-1\right)}\|\|f\|\|^{\frac{1}{4}}_{\infty,\mathbb{R}_-}\overline{K_g}^{\frac{3}{4}}. \tag{55}$$

The theorem is proved. □

We continue with abstract L_p right sequential generalized fractional Landau inequalities over $\mathbb{R}_-$.

Theorem 2.3. *Let $g \in C^1(\mathbb{R}_-)$ strictly increasing, with $g^{-1} \in C^1(g(\mathbb{R}_-))$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha < 1$. Let $f \in C^1(\mathbb{R}_-, X)$ with $\|\|f\|\|_{\infty,\mathbb{R}_-}$, $\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,\mathbb{R}_-} < \infty$. For $k = 1, 2, 3$, we assume that $D^{k\alpha}_{b-;g}f \in C^1((-\infty, b], X)$ and $D^{4\alpha}_{b-;g}f \in C((-\infty, b], X)$, $\forall\, b \in \mathbb{R}_-$. We further assume that*

$$\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D^{4\alpha}_{b-;g}f\right\|\right\|_{p,\mathbb{R}_-}\right) < \infty. \tag{56}$$

Then
1) under $\frac{1}{2p} < \alpha < 1$, we get

$$\sup_{b\in\mathbb{R}_-}\left\|\left(D^{2\alpha}_{b-;g}f\right)(b)\right\| \le \left[\left(\frac{2^{\alpha}\Gamma(2\alpha)\left(4\alpha-\frac{1}{p}\right)}{2^{\alpha}-1}\right)\left(\frac{4\alpha\left(1+2^{-3\alpha}\right)}{2\alpha-\frac{1}{p}}\right)^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\right.$$

$$\left.\times\left(\frac{1+2^{\alpha-\frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\right]\|\|f\|\|^{\left(\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}_{\infty,\mathbb{R}_-}$$

$$\times\left(\sup_{b\in\mathbb{R}_-}\left\|\left\|D^{4\alpha}_{b-;g}f\right\|\right\|_{p,\mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)} < \infty. \tag{57}$$

2) under $\frac{1}{p} < \alpha < 1$, we get

$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \left[\left(\frac{\Gamma(3\alpha)\left(4\alpha - \frac{1}{p}\right)}{2^\alpha - 1} \right) \left(\frac{6\alpha\left(1 + 2^{-2\alpha}\right)}{\alpha - \frac{1}{p}} \right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \right.$$

$$\left. \times \left(\frac{1 + 2^{2\alpha - \frac{1}{p}}}{\Gamma(4\alpha)\left(q(4\alpha - 1) + 1\right)^{\frac{1}{q}}} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} \right] \|\|f\|\|_{\infty, \mathbb{R}_-}^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}$$

$$\times \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} < \infty. \tag{58}$$

That is $\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\|, \sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| < \infty.$

Proof. As in the proof of Theorem 2.2, we have that

$$\|A\| \overset{(22)}{=} \left\| f(x_1) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t) - g(x_1))^{4\alpha - 1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt \right\|$$

$$\leq 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \int_{x_1}^{b} (g(t) - g(x_1))^{4\alpha - 1} g'(t) \left\| \left(D_{b-;g}^{4\alpha} f \right)(t) \right\| dt$$

$$\leq 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \frac{(g(b) - g(x_1))^{\frac{(q(4\alpha - 1) + 1)}{q}}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right) \tag{59}$$

$$\overset{(g(b) - g(x_1) =: h > 0)}{=} 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{1}{\Gamma(4\alpha)} \frac{h^{\left(4\alpha - \frac{1}{p}\right)}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right),$$

with $\frac{1}{4p} < \alpha < 1$. That is

$$\|A\| \leq 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right), \tag{60}$$

where $\frac{1}{4p} < \alpha < 1$. We also have

$$\|B\| \overset{(23)}{=} \left\| f(x_2) - f(b) - \frac{1}{\Gamma(4\alpha)} \int_{x_2}^{b} (g(t) - g(x_2))^{4\alpha - 1} g'(t) \left(D_{b-;g}^{4\alpha} f \right)(t)\, dt \right\| \tag{2.1}$$

$$\leq 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{(g(b) - g(x_2))^{\frac{(q(4\alpha - 1) + 1)}{q}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right) \tag{61}$$

$$\overset{(g(b) - g(x_2) =: 2h)}{=} 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right).$$

That is

$$\|B\| \leq 2 \|\|f\|\|_{\infty, \mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{b \in \mathbb{R}_-} \left\|\left\| D_{b-;g}^{4\alpha} f \right\|\right\|_{p, \mathbb{R}_-} \right), \tag{62}$$

where $\frac{1}{4p} < \alpha < 1$. We have assumed that

$$\overline{M_g} := \left(\sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-;g}^{4\alpha} f \right\| \right\|_{p,\mathbb{R}_-} \right) < \infty. \tag{63}$$

For convenience, we call

$$c := \Gamma (4\alpha) \left(q (4\alpha - 1) + 1 \right)^{\frac{1}{q}} > 0. \tag{64}$$

So, we have

$$\|A\| \leq 2 \left\| \|f\| \right\|_{\infty,\mathbb{R}_-} + \frac{h^{4\alpha - \frac{1}{p}}}{c} \overline{M_g} \quad \text{and} \quad \|B\| \leq 2 \left\| \|f\| \right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} \overline{M_g}, \tag{65}$$

where $\frac{1}{4p} < \alpha < 1$. Next, we estimate the (26)-quantities and we have

$$\begin{aligned} \left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\| &\leq \frac{1}{D\Gamma (3\alpha + 1)} \left[2^{3\alpha} h^{3\alpha} \|A\| + h^{3\alpha} \|B\| \right] \\ &\overset{(33)}{=} \frac{h^{3\alpha} \Gamma (2\alpha + 1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} \left[2^{3\alpha} \|A\| + \|B\| \right] \\ &\overset{(65)}{\leq} \frac{\Gamma (2\alpha + 1)}{2^{2\alpha} (2^\alpha - 1) h^{2\alpha}} \left[2^{3\alpha+1} \left\| \|f\| \right\|_{\infty,\mathbb{R}_-} + \frac{2^{3\alpha} h^{4\alpha - \frac{1}{p}}}{c} \overline{M_g} \right. \\ &\left. + 2 \left\| \|f\| \right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} \overline{M_g} \right] \end{aligned} \tag{66}$$

$$\begin{aligned} &= \frac{\Gamma (2\alpha + 1)}{2^{2\alpha} (2^\alpha - 1)} \left[\frac{\left(2^{3\alpha+1} + 2 \right) \left\| \|f\| \right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(2^{3\alpha} + 2^{4\alpha - \frac{1}{p}} \right)}{c} \overline{M_g} h^{2\alpha - \frac{1}{p}} \right] \\ &= \frac{2^\alpha \Gamma (2\alpha + 1)}{(2^\alpha - 1)} \left[\frac{2 \left(1 + 2^{-3\alpha} \right) \left\| \|f\| \right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1 + 2^{\alpha - \frac{1}{p}} \right) \overline{M_g}}{c} h^{2\alpha - \frac{1}{p}} \right]. \end{aligned} \tag{67}$$

That is

$$\left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\| \leq \left(\frac{2^\alpha \Gamma (2\alpha + 1)}{2^\alpha - 1} \right) \left[\frac{2 \left(1 + 2^{-3\alpha} \right) \left\| \|f\| \right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1 + 2^{\alpha - \frac{1}{p}} \right) \overline{M_g}}{c} h^{2\alpha - \frac{1}{p}} \right], \tag{68}$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\begin{aligned} \sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f \right)(b) \right\| &\leq \left(\frac{2^\alpha \Gamma (2\alpha + 1)}{2^\alpha - 1} \right) \\ &\times \left[\frac{2 \left(1 + 2^{-3\alpha} \right) \left\| \|f\| \right\|_{\infty,\mathbb{R}_-}}{h^{2\alpha}} + \frac{\left(1 + 2^{\alpha - \frac{1}{p}} \right) \overline{M_g}}{c} h^{2\alpha - \frac{1}{p}} \right], \end{aligned} \tag{69}$$

$\forall\, h > 0$, under $\frac{1}{4p} < \alpha < 1$. Again from (26), we get

$$\left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \frac{1}{\Gamma(2\alpha+1) D} \left[h^{2\alpha} \|B\| + 2^{2\alpha} h^{2\alpha} \|A\| \right]$$

(70)
$$= \frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^{\alpha}-1)} \left[\|B\| + 2^{2\alpha} \|A\| \right]$$

$$\leq \left(\frac{h^{2\alpha}\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^{\alpha}-1)} \right) \left[2 \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{4\alpha-\frac{1}{p}} h^{4\alpha-\frac{1}{p}}}{c} \overline{M_g} \right.$$
$$\left. + 2^{2\alpha+1} \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-} + \frac{2^{2\alpha} h^{4\alpha-\frac{1}{p}}}{c} \overline{M_g} \right]$$

(71)
$$= \frac{\Gamma(3\alpha+1)}{2^{2\alpha}(2^{\alpha}-1)} \left[\frac{\left(2+2^{2\alpha+1}\right) \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(2^{2\alpha}+2^{4\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right]$$

$$= \frac{\Gamma(3\alpha+1)}{(2^{\alpha}-1)} \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right].$$

That is

(72)
$$\left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1} \right) \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right],$$

$\forall\, b \in \mathbb{R}_-$, $\forall\, h > 0$. I.e., it holds

$$\sup_{b\in\mathbb{R}_-} \left\| \left(D_{b-;g}^{3\alpha} f \right)(b) \right\| \leq \left(\frac{\Gamma(3\alpha+1)}{2^{\alpha}-1} \right)$$

(73)
$$\times \left[\frac{2\left(1+2^{-2\alpha}\right) \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-}}{h^{3\alpha}} + \frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{c} \overline{M_g} h^{\alpha-\frac{1}{p}} \right],$$

$\forall\, h > 0$, $\frac{1}{4p} < \alpha < 1$. Call

(74)
$$\mu := 2\left(1+2^{-3\alpha}\right) \left\|\left\| f \right\|\right\|_{\infty,\mathbb{R}_-},$$
$$\theta := \frac{\left(1+2^{\alpha-\frac{1}{p}}\right)\overline{M_g}}{c},$$

both are greater than zero. We consider the function

(75)
$$y(h) = \mu h^{-2\alpha} + \theta h^{2\alpha-\frac{1}{p}}, \quad \forall\, h > 0.$$

We have

(76)
$$y'(h) = -2\alpha\mu h^{-2\alpha-1} + \left(2\alpha - \frac{1}{p}\right) \theta h^{2\alpha-\frac{1}{p}-1} = 0,$$

then

$$\left(2\alpha - \frac{1}{p}\right) \theta h^{2\alpha-\frac{1}{p}-1} = 2\alpha\mu h^{-2\alpha-1},$$

i.e.,

$$\left(2\alpha-\frac{1}{p}\right)\theta h^{4\alpha-\frac{1}{p}}=2\alpha\mu,$$

with a unique solution

$$h_0:=h_{crit.no.}=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}} \tag{77}$$

(assuming $\frac{1}{2p}<\alpha<1$). We have that

$$y''(h)=2\alpha(2\alpha+1)\mu h^{-2\alpha-2}+\left(2\alpha-\frac{1}{p}\right)\left(2\alpha-\frac{1}{p}-1\right)\theta h^{2\alpha-\frac{1}{p}-2}. \tag{78}$$

We see that

$$y''(h_0)=2\alpha(2\alpha+1)\mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}}$$

$$+\left(2\alpha-\frac{1}{p}\right)\left(2\alpha-\frac{1}{p}-1\right)\theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}}$$

$$=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}}\left[2\alpha(2\alpha+1)\mu+2\alpha\mu\left(2\alpha-\frac{1}{p}-1\right)\right] \tag{79}$$

$$=2\alpha\mu\left(\frac{\left(2\alpha-\frac{1}{p}\right)\theta}{2\alpha\mu}\right)^{\left(\frac{2(\alpha+1)}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha-\frac{1}{p}\right)>0.$$

Therefore, y has a global minimum at

$$h_0=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

$$y(h_0)=\mu\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}}+\theta\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}} \tag{80}$$

$$=\left(\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}}\left[\mu+\theta\frac{2\alpha\mu}{\left(2\alpha-\frac{1}{p}\right)\theta}\right]$$

$$=\mu\left(\frac{\left(2\alpha-\frac{1}{p}\right)\theta}{2\alpha\mu}\right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}\left(1+\frac{2\alpha}{2\alpha-\frac{1}{p}}\right)$$

$$= \frac{\left(4\alpha - \frac{1}{p}\right)}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}} \left(2\alpha - \frac{1}{p}\right)^{\frac{-2\alpha + \frac{1}{p}}{4\alpha - \frac{1}{p}}} \mu^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \theta^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}. \tag{81}$$

That is

$$y(h_0) = \frac{\left(4\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p}\right)^{\left(\frac{-2\alpha + \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}} \mu^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \theta^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)}. \tag{82}$$

Therefore, we derive (see (69))

$$\sup_{b \in \mathbb{R}_-} \left\| \left(D_{b-;g}^{2\alpha} f\right)(b) \right\| \leq \left(\frac{2^{\alpha}\Gamma(2\alpha)}{2^{\alpha} - 1}\right) \left(\frac{2\alpha}{2\alpha - \frac{1}{p}}\right)^{\frac{\left(2\alpha - \frac{1}{p}\right)}{\left(4\alpha - \frac{1}{p}\right)}} \left(4\alpha - \frac{1}{p}\right)$$

$$\times \left(2\left(1 + 2^{-3\alpha}\right)\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{\left(1 + 2^{\alpha - \frac{1}{p}}\right)}{\Gamma(4\alpha)\left(q(4\alpha - 1) + 1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)} \| \| f \| \|_{\infty, \mathbb{R}_-}^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}$$

$$\times \left(\sup_{b \in \mathbb{R}_-} \left\| \left\| D_{b-;g}^{4\alpha} f \right\| \right\|_{p, \mathbb{R}_-}\right)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}}\right)} < \infty, \tag{83}$$

where $\frac{1}{2p} < \alpha < 1$. Call

$$\xi := 2\left(1 + 2^{-2\alpha}\right) \| \| f \| \|_{\infty, \mathbb{R}_-},$$

$$\psi := \frac{\left(1 + 2^{2\alpha - \frac{1}{p}}\right)}{c} \overline{M_g}, \tag{84}$$

both are greater than zero. We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^{\alpha - \frac{1}{p}}, \ \forall\, h > 0. \tag{85}$$

We have

$$\gamma'(h) = -3\alpha\xi h^{-3\alpha - 1} + \left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 0,$$

then

$$\left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 3\alpha\xi h^{-3\alpha - 1}$$

and

$$\left(\alpha - \frac{1}{p}\right) \psi h^{4\alpha - \frac{1}{p}} = 3\alpha\xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha - \frac{1}{p}}}. \tag{86}$$

(assuming $\frac{1}{p} < \alpha < 1$). We have that

(87) $$\gamma''(h) = 3\alpha(3\alpha+1)\xi h^{-3\alpha-2} + \left(\alpha - \frac{1}{p}\right)\left(\alpha - \frac{1}{p} - 1\right)\psi h^{\alpha - \frac{1}{p} - 2}.$$

We observe

$$\gamma''(h_0) = 3\alpha(3\alpha+1)\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}$$

$$+\left(\alpha-\frac{1}{p}\right)\left(\alpha-\frac{1}{p}-1\right)\psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}\right)}$$

(88) $$= \left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}\left[3\alpha(3\alpha+1)\xi + \left(\alpha-\frac{1}{p}-1\right)3\alpha\xi\right]$$

$$= 3\alpha\xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha-2}{4\alpha-\frac{1}{p}}\right)}\left(4\alpha-\frac{1}{p}\right) > 0.$$

Therefore, y has a global minimum at

$$h_0 = \left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

(89) $$\gamma(h_0) = \xi h_0^{-3\alpha} + \psi h_0^{\alpha-\frac{1}{p}} = h_0^{-3\alpha}\left(\xi + \psi h_0^{4\alpha-\frac{1}{p}}\right)$$

$$= \left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\xi + \psi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)\right)$$

$$= \xi\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right).$$

That is

(90) $$\gamma(h_0) = \xi\left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)\left(\frac{3\alpha\xi}{\left(\alpha-\frac{1}{p}\right)\psi}\right)^{\left(\frac{-3\alpha}{4\alpha-\frac{1}{p}}\right)}$$

$$= \left(\frac{4\alpha-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)\xi^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(\frac{\left(\alpha-\frac{1}{p}\right)}{3\alpha}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\psi^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}.$$

I.e., we have found

(91)
$$\gamma\left(h_{0}\right)=\frac{\left(4\alpha-\frac{1}{p}\right)}{\left(\alpha-\frac{1}{p}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(3\alpha\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}}\xi^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\psi^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}.$$

We have proved that (see (73))

(92)
$$\sup_{b\in\mathbb{R}_{-}}\left\|\left(D_{b-;g}^{3\alpha}f\right)(b)\right\|\leq\left(\frac{\left(4\alpha-\frac{1}{p}\right)\Gamma\left(3\alpha\right)}{2^{\alpha}-1}\right)\left(\frac{3\alpha}{\alpha-\frac{1}{p}}\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$
$$\times\left(2\left(1+2^{-2\alpha}\right)\right)^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}\left(\frac{\left(1+2^{2\alpha-\frac{1}{p}}\right)}{\Gamma\left(4\alpha\right)\left(q\left(4\alpha-1\right)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_{-}}^{\left(\frac{\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}\right)}$$
$$\times\left(\sup_{b\in\mathbb{R}_{-}}\left\|\left\|D_{b-;g}^{4\alpha}f\right\|\right\|_{p,\mathbb{R}_{-}}\right)^{\left(\frac{3\alpha}{4\alpha-\frac{1}{p}}\right)}<\infty,$$

where $\frac{1}{p}<\alpha<1$. The theorem is proved. □

We give an application when $\alpha=\frac{1}{2}$ and $g\left(t\right)=e^{t}|_{\mathbb{R}_{-}}$.

Corollary 2.1. *Let $f\in C^{1}\left(\mathbb{R}_{-},X\right)$ with $\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_{-}}$, $\left\|\left\|\left(f\circ\ln\right)'\circ e^{t}\right\|\right\|_{\infty,\mathbb{R}_{-}}<\infty$, where $\left(X,\left\|\cdot\right\|\right)$ is a Banach space. For $k=1,2,3$, we assume that $D_{b-;e^{t}}^{k\frac{1}{2}}f\in C^{1}\left(\left(-\infty,b\right],X\right)$ and $D_{b-;e^{t}}^{4\frac{1}{2}}f\in C\left(\left(-\infty,b\right],X\right)$, $\forall\, b\in\mathbb{R}_{-}$. We further assume that*

(93)
$$\left\|\left\|D_{b-;e^{t}}^{4\frac{1}{2}}f\left(t\right)\right\|\right\|_{\infty,\mathbb{R}_{-}^{2}}<\infty,$$

where $\left(b,t\right)\in\mathbb{R}_{-}^{2}$. Then,

(94)
$$\sup_{b\in\mathbb{R}_{-}}\left\|\left(D_{b-;e^{t}}^{2\frac{1}{2}}f\right)(b)\right\|\leq\left(\frac{\sqrt{12+6\sqrt{2}}}{\sqrt{2}-1}\right)\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_{-}}^{\frac{1}{2}}\left(\left\|\left\|D_{b-;e^{t}}^{4\frac{1}{2}}f\left(t\right)\right\|\right\|_{\infty,\mathbb{R}_{-}^{2}}\right)^{\frac{1}{2}}<\infty$$

and

(95)
$$\sup_{b\in\mathbb{R}_{-}}\left\|\left(D_{b-;e^{t}}^{3\frac{1}{2}}f\right)(b)\right\|\leq\left(\frac{9\sqrt{\pi}}{\left(2-\sqrt{2}\right)\sqrt[4]{2}\left(\sqrt[4]{3}\right)^{3}}\right)\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}_{-}}^{\frac{1}{4}}\left(\left\|\left\|D_{b-;e^{t}}^{4\frac{1}{2}}f\left(t\right)\right\|\right\|_{\infty,\mathbb{R}_{-}^{2}}\right)^{\frac{3}{4}}<\infty.$$

That is $\sup_{b\in\mathbb{R}_{-}}\left\|\left(D_{b-;e^{t}}^{2\frac{1}{2}}f\right)(b)\right\|,\ \sup_{b\in\mathbb{R}_{-}}\left\|\left(D_{b-;e^{t}}^{3\frac{1}{2}}f\right)(b)\right\|<\infty.$

Proof. By Theorem 2.2. □

REFERENCES

[1] A. A. Aljinovic, Lj. Marangunic and J. Pecaric: *On Landau type inequalities via Ostrowski inequalities*, Nonlinear Funct. Anal. Appl., **10** (4) (2005), 565-579.

[2] G. A. Anastassiou, *Fractional Differentiation inequalities*, Research monograph, Springer, New York, (2009).

[3] G. A. Anastassiou: *Advances on Fractional Inequalities*, Springer, New York, (2011).

[4] G. A. Anastassiou: *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, (2018).

[5] N. S. Barnett, S. S. Dragomir: *Some Landau type inequalities for functions whose derivatives are of locally bounded variation*, Tamkang Journal of Mathematics, **37**, (4), 301-308, winter (2006).

[6] Z. Ditzian: *Remarks, questions and conjectures on Landau-Kolmogorov-type inequalities*, Math. Inequal. Appl., **3** (2000), 15-24.

[7] G. H. Hardy, J.E. Littlewood: *Some integral inequalities connected with the calculus of variations*, Quart. J. Math. Oxford Ser. **3** (1932), 241-252.

[8] G. H. Hardy, E. Landau and J.E. Littlewood: *Some inequalities satisfied by the integrals or derivatives of real or analytic functions*, Math. Z., **39** (1935), 677-695.

[9] R. R. Kallman, G. C. Rota: *On the inequality* $\|f'\|^2 \leq 4\|f\| \cdot \|f''\|$, in *Inequalities*, Vol. II, (O. Shisha, Ed.), 187-192, Academic Press, New York, (1970).

[10] E. Landau: *Einige Ungleichungen für zweimal differentzierban funktionen*, Proc. London Math. Soc., **13** (1913), 43-49.

[11] J. Mikusinski: *The Bochner integral*, Academic Press, New York, (1978).

[12] G. E. Shilov: *Elementary Functional Analysis*, Dover Publications Inc., New York, (1996).

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