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Geometric interpretations and reversed versions of Young's integral inequality

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Abstract

The authors retrospect Young's integral inequality and its geometric interpretation, recall a reversed version of Young's integral inequality, present a geometric interpretation of the reversed version of Young's integral inequality, and conclude a new reversed version of Young's integral inequality.

Keywords: Young's integral inequality; geometric proof; geometric interpretation; generalization; refinement; reversed version.
2010 MSC: 26D15, 26D99.

1. Young's integral inequality

In this section, we retrospect Young's integral inequality and its geometric interpretation.

1.1. *Young's integral inequality*

Let $h(x)$ be a real-valued, continuous, and strictly increasing function on $[0,c]$ with $c>0$. If $h(0)=0$, $a\in[0,c]$, and $b\in[0,h(c)]$, then

$$\int_0^a h(x)\,\mathrm{d}\,x+\int_0^b h^{-1}(x)\,\mathrm{d}\,x\ge ab, \tag{1.1}$$

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where h^{-1} denotes the inverse function of h. The equality in (1.1) is valid if and only if $b = h(a)$.

In the literature, the inequality (1.1) was first stated and proved in [27], so we call it Young's integral inequality. For more information since [27], please refer to [13, Section 2.7], [14, Chapter XIV], the papers [1, 2, 3, 4, 5, 6, 7, 9, 12, 15, 16, 17, 18, 19, 20, 22, 23, 26, 28], and closely related references therein.

1.2. Geometric interpretation

The geometric interpretation of Young's integral inequality (1.1) can be demonstrated by Figures 1 and 2. In Figure 1, we have

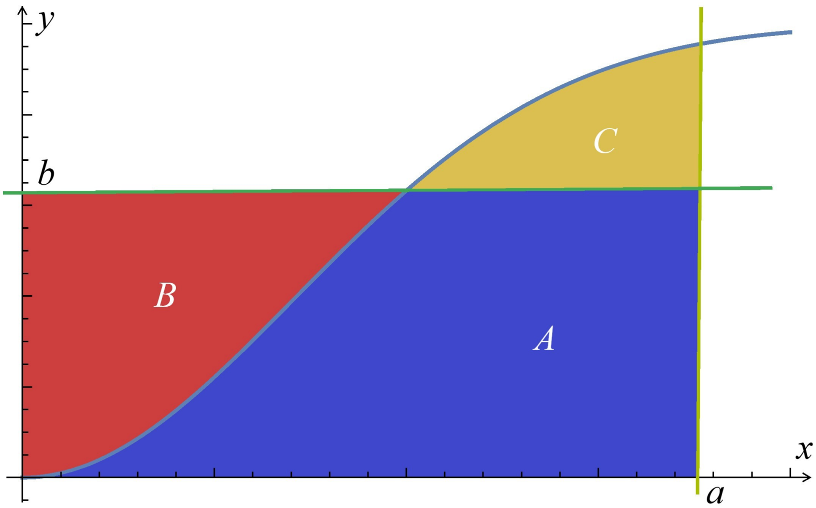

Figure 1: Geometric interpretation of the inequality (1.1)

$$A + C = \int_0^a h(x)\,\mathrm{d}\,x, \quad A + B = ab, \quad B = \int_0^b h^{-1}(x)\,\mathrm{d}\,x,$$
$$A + B + C = \int_0^a h(x)\,\mathrm{d}\,x + \int_0^b h^{-1}(x)\,\mathrm{d}\,x \ge ab = A + B.$$

In Figure 2, we have

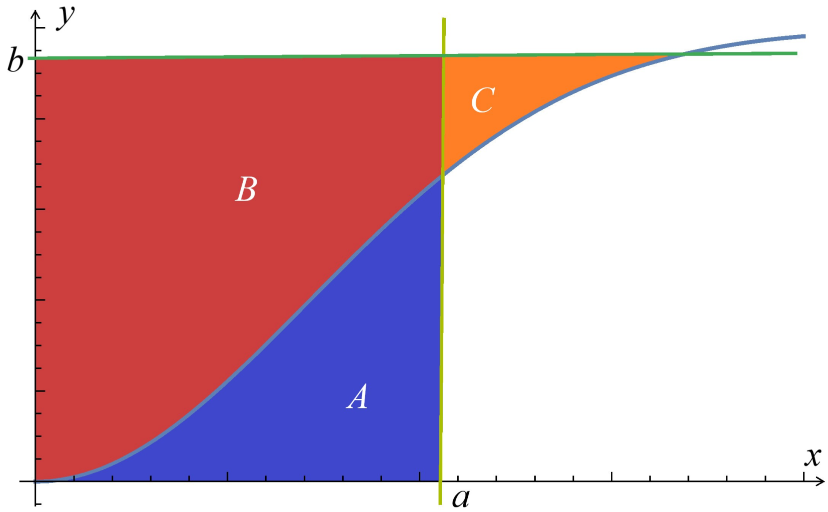

Figure 2: Geometric interpretation of the inequality (1.1)

$$A=\int_0^a h(x)\,\mathrm{d}\,x,\quad A+B=ab,\quad B+C=\int_0^b h^{-1}(x)\,\mathrm{d}\,x,$$
$$A+B+C=\int_0^a h(x)\,\mathrm{d}\,x+\int_0^b h^{-1}(x)\,\mathrm{d}\,x\ge ab=A+B.$$

Therefore, the inequality (1.1) means that the area $C\ge 0$.

In the papers [8, 10, 11, 21, 24], Young's integral inequality (1.1) was refined by estimating the area C and bounding $\int_0^a h(x)\,\mathrm{d}\,x+\int_0^b h^{-1}(x)\,\mathrm{d}\,x$ with lower and upper bounds in terms of derivatives of $h(x)$.

2. Reversed version of Young's integral inequality

In this section, we recall a reversed version of Young's integral inequality, which was analytically established in [25], and supply a geometric interpretation, or say, a geometric proof, for it.

2.1. Reversed version of Young's integral inequality

Under the same conditions as required by Young's integral inequality (1.1), the inequality

$$\min\left\{1,\frac{b}{h(a)}\right\}\int_0^a h(x)\,\mathrm{d}\,x+\min\left\{1,\frac{a}{h^{-1}(b)}\right\}\int_0^b h^{-1}(x)\,\mathrm{d}\,x\le ab, \tag{2.1}$$

where the equality in (2.1) is valid if and only if $b=h(a)$, was established in [25, Theorem 3].

2.2. Geometric interpretation

We now discuss the geometric interpretation of the inequality 2.1. In other words, we now provide a geometric proof of the inequality 2.1.

When $a>h^{-1}(b)$, or say, $h(a)>b$, as showed in Figure 1, the inequality (2.1) becomes

$$\frac{b}{h(a)}\int_0^a h(x)\,\mathrm{d}\,x+\int_0^b h^{-1}(x)\,\mathrm{d}\,x\le ab. \tag{2.2}$$

When $a<h^{-1}(b)$, or say, $h(a)<b$, as showed in Figure 2, the inequality (2.1) becomes

$$\int_0^a h(x)\,\mathrm{d}\,x+\frac{a}{h^{-1}(b)}\int_0^b h^{-1}(x)\,\mathrm{d}\,x\le ab. \tag{2.3}$$

The inequalities (2.2) and (2.3) can be rewritten as

$$\int_0^a \frac{bh(x)}{h(a)}\,\mathrm{d}\,x+\int_0^b h^{-1}(x)\,\mathrm{d}\,x\le ab \tag{2.4}$$

and

$$\int_0^a h(x)\,\mathrm{d}\,x+\int_0^b \frac{ah^{-1}(x)}{h^{-1}(b)}\,\mathrm{d}\,x\le ab \tag{2.5}$$

respectively. These two inequalities (2.4) and (2.5) can be geometrically demonstrated by Figures 3 and 4 respectively.

In Figure 3, by the transform

$$h(x)\to H(x)=\frac{bh(x)}{h(a)},\quad x\in[0,a],$$

the area

$$C+A''+A'=\int_0^a h(x)\,\mathrm{d}\,x$$

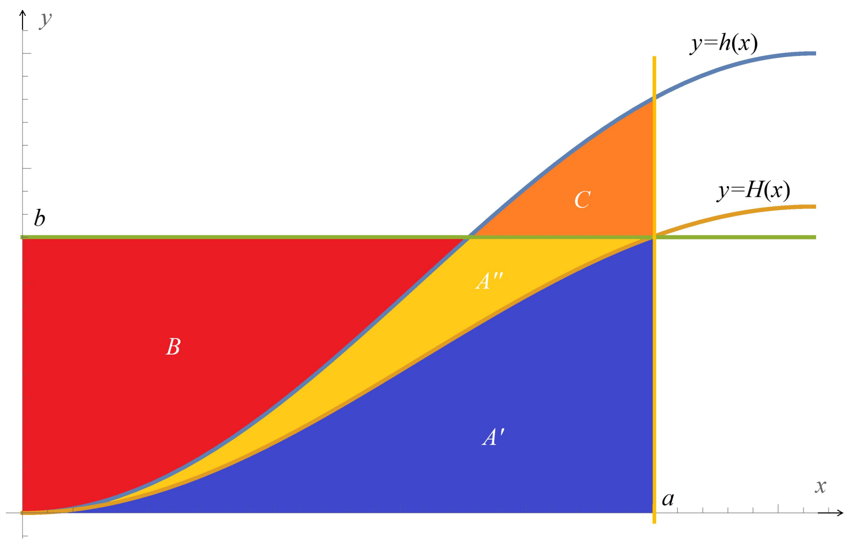

Figure 3: Geometric interpretation of the inequality (2.4)

contracts to

$$A' = \int_0^a H(x)\,\mathrm{d}\,x = \int_0^a \frac{bh(x)}{h(a)}\,\mathrm{d}\,x.$$

Then it is clear that

$$A' + A'' + B = \int_0^a \frac{bh(x)}{h(a)}\,\mathrm{d}\,x + A'' + \int_0^b h^{-1}(x)\,\mathrm{d}\,x = ab,$$

where

$$A'' = \int_0^b \big[H^{-1}(x) - h^{-1}(x)\big]\,\mathrm{d}\,x = \int_0^b \left[\frac{h(a)}{b} - 1\right] h^{-1}(x)\,\mathrm{d}\,x = \left[\frac{h(a)}{b} - 1\right] \int_0^b h^{-1}(x)\,\mathrm{d}\,x \ge 0.$$

Consequently, the inequality (2.4) is valid.

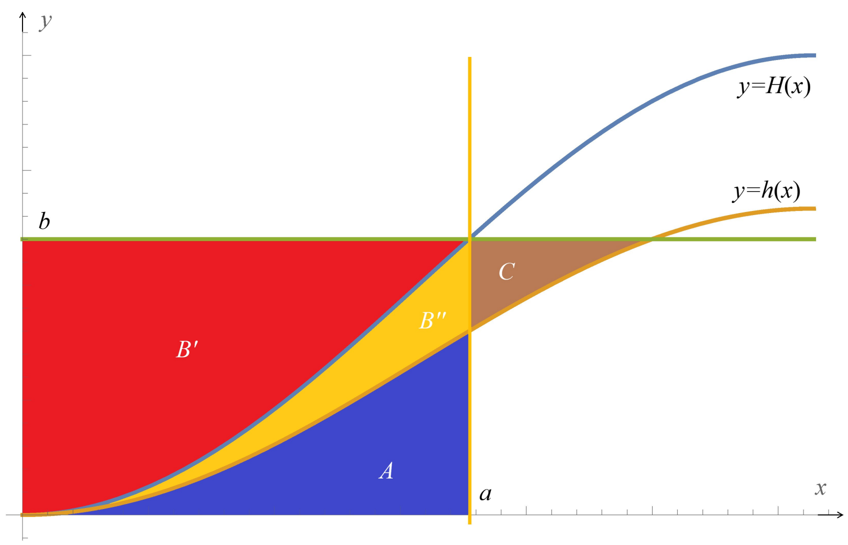

Figure 4: Geometric interpretation of the inequality (2.5)

In Figure 4, by the transform

$$h^{-1}(y) \to H^{-1}(y) = \frac{ah^{-1}(y)}{h^{-1}(b)}, \quad y \in [0, b],$$

the area

$$B' + B'' + C = \int_0^b h^{-1}(y) \, \mathrm{d}\, y$$

contracts to

$$B' = \int_0^b \frac{ah^{-1}(y)}{h^{-1}(b)} \, \mathrm{d}\, y.$$

Then it is obvious that

$$A + B' + B'' = \int_0^a h(x) \, \mathrm{d}\, x + \int_0^b \frac{ah^{-1}(y)}{h^{-1}(b)} \, \mathrm{d}\, y + B'' = ab,$$

where

$$B'' = \int_0^a [H(x) - h(x)] \, \mathrm{d}\, x = \int_0^a \left[\frac{h^{-1}(b)}{a} - 1\right] h(x) \, \mathrm{d}\, x = \left[\frac{h^{-1}(b)}{a} - 1\right] \int_0^a h(x) \, \mathrm{d}\, x \ge 0.$$

Consequently, the inequality (2.5) is valid.

3. A new reversed version of Young's integral inequality

Observing the geometric interpretation in Section 2.2 of the inequality (2.1), we conclude a new reversed version of Young's integral inequality.

Theorem 3.1. *Let $h(x)$ be a continuous and strictly increasing function on $[0, c]$ with $c > 0$, let $h(0) = 0$, $a \in [0, c]$, and $b \in [0, h(c)]$, let h^{-1} denote the inverse function of h, and let*

$$0 \le p(x) \begin{cases} \le \dfrac{b}{h(a)}, & h(a) > b \\ = 1, & h(a) \le b \end{cases}$$

and

$$0 \le p(x) \begin{cases} \le \dfrac{a}{h^{-1}(b)}, & h(a) < b \\ = 1, & h(a) \ge b \end{cases}$$

are continuous functions on $[0, c]$. Then

$$\int_0^a p(x)h(x) \, \mathrm{d}\, x + \int_0^b q(x)h^{-1}(x) \, \mathrm{d}\, x \le ab \tag{3.1}$$

and the equality in (3.1) is valid if and only if $b = h(a)$ and $p(x) = q(x) = 1$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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