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*RESEARCH ARTICLE*

# DIFFUSION EQUATION INCLUDING LOCAL FRACTIONAL DERIVATIVE AND NON-HOMOGENOUS DIRICHLET BOUNDARY CONDITIONS

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## ABSTRACT

In this research, we discuss the construction of analytic solution of non-homogenous initial boundary value problem including PDEs of fractional order. Since non-homogenous initial boundary value problem involves local fractional derivative, it has classical initial and boundary conditions. By means of separation of variables method and the inner product defined on $L^2[0,l]$, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including local fractional derivative used in this study. Illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

**Keywords:** *Local fractional derivative, Time-fractional diffusion equation, Initial-boundary-value problems, Spectral method, Non-homogenous Dirichlet boundary conditions*

## 1. INTRODUCTION

Since mathematical models including fractional derivatives play a vital role fractional derivatives draw a growing attention of many researchers in various branches of sciences. Therefore there are many different fractional derivatives such as Caputo, Riemann-Liouville, Atangana-Baleanu. However these fractional derivatives don't satisfy most important properties of ordinary derivative which leads to many diffuculties to analyze or obtain the solution of fractional mathematical models. As a result many scientists focus on defining new fractional derivatives to cover the setbacks of the defined ones. Moreover the success of mathematical modelling of systems or processes depends on the fractional derivative, it involves, since the correct choice of the fractional derivative allows us to model the real data of systems or processes accurately.

In order to the define new fractional derivatives, various methods exists and these ones are classfied based on their features and formation such as nonlocal fractional derivatives and local fractional derivatives. From a physical aspect, the intrinsic nature of the physical system can be reflected to the

JSR A
Journal of Scientific Reports

*Çetinkaya S. and Demir A., Journal of Scientific Reports-A, Number 45, 101-110, December 2020.*

mathematical model of the system by using fractional derivatives. Therefore the solution of the fractional mathematical model is in excellent agreement with the predictions and experimental measurement of it. The systems whose behaviour is non-local can be modelled better by fractional mathematical models and the degree of its non-locality can be arranged by the order of fractional derivative. In order to analyze the diffusion in a non-homogenous medium that has memory effects it is better to analyze the solution of the fractional mathematical model for this diffusion. As a result in order to model a process, the correct choices of fractional derivative and its order must be determined.

Rheology is known as the scientific study of material diffusion. In rheology, mathematical models of diffusion process are employed to analyze the behaviour of materials which allow us to classify and compare them. In this study, we focus on fractional diffusion problem including time fractional derivative. The novelty of this research is that the materials can be classified as liquid, gas and temperature based on the order of time fractional derivative. For instance, fractional diffusion model of materials which behaves slower, the order of time fractional derivative is between zero and one and the order of time fractional derivative for materials which behaves faster is greater than one. Moreover based on the complexity of the material, suitable fractional derivative need to be chosen to facilitate the correct analysis of the material. In this study, we classify non-complex materials therefore local fractional derivative is used. Mathematical model of diffusion problems including local fractional derivatives gives better results than ones including integer order derivatives [1]. There are many published work on the diffusion of various matters in science especially in fluid mechanics and gas dynamics [2-7].

## 2. MAIN RESULTS

The proportional derivative is a newly defined fractional derivative which is generally defined as

$$ {}^{P}D_{\alpha}f(t) = \mathrm{K}_1(\alpha,\mathrm{t})\,\mathrm{f(t)} + \mathrm{K}_0(\alpha,\mathrm{t})\mathrm{f}'(\mathrm{t}), \tag{1}$$

where the functions $K_0$ and $K_1$ satisfy certain properties in terms of limit [8] and $f$ is a differentiable function. Notice that this derivative can be regarded as an extension of conformable derivative and is used in control theory.

In this study we focus on obtaining the solution of following fractional diffusion equation including various proportional derivative operator by making use of the separation of variables method:

$$ {}^{P}D_{t}^{\alpha}u(x,t) = \gamma^2 u_{xx}(x,t), \tag{2}$$
$$ u(0,t) = u_0, u(l,t) = u_1, \tag{3}$$
$$ u(x,0) = f(x) \tag{4}$$

where $0 < \alpha < 1, \gamma \in \mathbb{R}, 0 \leq x \leq l,\ 0 \leq t \leq T,\ u_0$ and $u_1$ are constants. Here we use the following forms of the proportional derivatives:

$$ {}^{P}D_{\alpha}f(t) = K_1(\alpha)\,f(t) + K_0(\alpha)f'(t). \tag{5}$$

Especially we consider the following ones:

$$ {}_{1}^{P}D_{\alpha}f(t) = (1-\alpha)\,f(t) + \alpha f'(t) \tag{6}$$

JSR A
Journal of Scientific Reports

*Çetinkaya S. and Demir A., Journal of Scientific Reports-A, Number 45, 101-110, December 2020.*

and

$$ {}_{2}^{P}D_{\alpha} f(t) = (1-\alpha^2)\, f(t) + \alpha^2 f'(t). \tag{7}$$

Let us consider the following problem including the proportional derivative in (6)

$$ {}^{P}D_t^{\alpha} u(x,t) = \gamma^2 u_{xx}(x,t), \tag{8}$$
$$ u(0,t) = u_0, u(l,t) = u_1, \tag{9}$$
$$ u(x,0) = f(x), \tag{10}$$

where $0 < \alpha < 1, \gamma \in \mathbb{R}, 0 \leq x \leq l,\ 0 \leq t \leq T,\ u_0$ and $u_1$ are constants.

Before investigating the solution of the problem (8)-(10), let us define the function $v(x,t)$ which homogenizes boundary conditions (9) as follows:

$$ v(x,t) = u(x,t) + \frac{x}{l}(u_0 - u_1) - u_0. \tag{11}$$

Via (11), the problem (8)-(10) turns into the following problem (12)-(14).

$$ {}^{P}D_t^{\alpha} v(x,t) = \gamma^2 v_{xx}(x,t), \tag{12}$$
$$ v(0,t) = 0, v(l,t) = 0 \tag{13}$$
$$ v(x,0) = f(x) + \frac{x}{l}(u_0 - u_1) - u_0, \tag{14}$$

where $0 < \alpha < 1, \gamma \in \mathbb{R}, 0 \leq x \leq l,\ 0 \leq t \leq T,\ u_0$ and $u_1$ are constants.

By means of separation of variables method, The generalized solution of above problem is constructed in analytical form. Thus a solution of problem (12)-(14) have the following form:

$$ v(x,t;\alpha) = X(x)T(t;\alpha) \tag{15}$$

where $0 \leq x \leq l,\ 0 \leq t \leq T$.

Plugging (15)into (12) and arranging it, we have

$$ \frac{{}^{P}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2. \tag{16}$$

Equation (16) produce a fractional differential equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (16). Hence with boundary conditions (13), we have the following problem:

$$ X''(x) + \lambda^2 X(x) = 0, \tag{17}$$
$$ X(0) = X(l) = 0. \tag{18}$$

The solution of eigenvalue problem (17)-(18) is accomplished by making use of the exponential function of the following form:

JSR A
Journal of Scientific Reports

Çetinkaya S. and Demir A., Journal of Scientific Reports-A, Number 45, 101-110, December 2020.

$$X(x) = e^{rx}. \tag{19}$$

Hence the characteristic equation is computed in the following form:

$$r^2 + \lambda^2 = 0. \tag{20}$$

Case 1: If $\lambda = 0$, then the characteristic equation have coincident solutions $r_{1,2} = 0$, leading to the general solution of the eigenvalue problem (17)-(18) have the following form:

$$X(x) = k_1 x + k_2.$$

The first boundary condition yields

$$X(0) = k_2 = 0 \Rightarrow k_2 = 0. \tag{21}$$

which leads to the following solution

$$X(x) = k_1 x. \tag{22}$$

Similarly second boundary condition leads to

$$X(l) = k_1 l \ = 0 \Rightarrow k_1 = 0. \tag{23}$$

which implies that

$$X(x) = 0. \tag{24}$$

As a result, the characteristic equation (20) can not have the solution $\lambda = 0$.

Case 2. If $\lambda > 0$, the Eq. (20) have two distinct real roots $r_1$, $r_2$ yielding the general solution of the problem (17)-(18) in the following form:

$$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{25}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 + c_2 = 0. \qquad c_1 = -c_2. \tag{26}$$

From second boundary condition

$$X(l) = (-c_2)e^{r_1 l} + c_2 e^{r_2 l} = c_2(-e^{r_1 l} + e^{r_2 l}) = 0$$

Which indicates that $c_2 = 0$. Hence $c_1 = 0$ which implies that $X(x) = 0$ which implies that there is not any solution for $\lambda > 0$.

Case 3: If $\lambda < 0$, then the characteristic equation have the solutions

$$r_{1,2} = \mp i\lambda \tag{27}$$

which leads to the general solution of the eigenvalue problem (17)-(18) have the following form:

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x). \tag{28}$$

By making use of the first boundary condition we have

$$X(0) = c_1 = 0 \Rightarrow c_1 = 0. \tag{29}$$

Hence the solution becomes

$$X(x) = c_2 \sin(\lambda x). \tag{30}$$

Similarly last boundary condition leads to

$$X(l) = c_2 \sin(\lambda l) = 0 \tag{31}$$

which implies that

$$\sin(\lambda l) = 0. \tag{32}$$

Let $w_n = \lambda_n l$. The solutions of (32) can be denoted by means of $w_n = n\pi$, $n = 0,1,2,3, \dots$ which are eigenvalues of the problem (17)-(18), as follows:

$$\lambda_n = \frac{w_n^2}{l^2}, 0 < w_1 < w_2 < w_3 < \cdots, n = 0,1,2,3, \dots \tag{33}$$

As a result the solution is obtained as follows:

$$X_n(x) = c_n \sin\left(w_n\left(\frac{x}{l}\right)\right) = \sin\left(w_n\left(\frac{x}{l}\right)\right), n = 0,1,2,3, \dots \tag{34}$$

The second equation in (16) for eigenvalue $\lambda_n$ yields the ordinary differential equation below:

$$\frac{{}^{P}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = -\gamma^2\lambda^2, \tag{35}$$

$$\frac{K_1(\alpha)\, T_n(t;\alpha) + K_0(\alpha) T_n{}'(t;\alpha)}{T_n(t;\alpha)} = -\gamma^2\lambda_n^2,$$

$$K_0(\alpha) T_n{}'(t;\alpha) + \left(\gamma^2\lambda_n^2 + K_1(\alpha)\right) T_n(t;\alpha) = 0$$

which yields the following solution

$$T_n(t;\alpha) = exp\left(-\frac{\gamma^2\lambda_n^2 + K_1(\alpha)}{K_0(\alpha)} t\right), n = 0,1,2,3, \dots \tag{36}$$

The solution for every eigenvalue $\lambda_n$ is constructed as

$$v_n(x,t;\alpha) = X_n(x)T_n(t;\alpha) = exp\left(-\frac{\gamma^2\lambda_n^2+K_1(\alpha)}{K_0(\alpha)}t\right)\sin\left(w_n\left(\frac{x}{l}\right)\right), n = 1,2,3,\dots \quad (37)$$

which leads to the following general solution

$$v(x,t;\alpha) = \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right) exp\left(-\frac{\gamma^2\lambda_n^2+K_1(\alpha)}{K_0(\alpha)}t\right). \quad (38)$$

Note that it satisfies boundary condition and fractional differential equation.

The coefficients of general solution are established by taking the following initial condition into account:

$$v(x,0) = f(x) + \frac{x}{l}(u_0 - u_1) - u_0 = \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right). \quad (39)$$

The coefficients $d_n$ for $n = 0,1,2,3,\dots$ determined by the help of inner product defined on $L^2[0,l]$:

$$d_n = \frac{2}{l}\left[\int_0^l \sin\left(\frac{n\pi x}{l}\right) f(x)dx + (u_0 - u_1)\int_0^l \sin\left(\frac{n\pi x}{l}\right)\frac{x}{l}dx - u_0\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx\right]. \quad (40)$$

Substituting (40) in (38) leads to the solution of the problem (12)-(14). By making use of (11) and this solution we obtain the general solution of the problem (8)-(10).

$$u(x,t) = u_0 + \frac{x}{l}(u_1 - u_0) + \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right) exp\left(-\frac{\gamma^2\lambda_n^2+K_1(\alpha)}{K_0(\alpha)}t\right). \quad (41)$$

If $\gamma^2$ is replaced by the fractional diffusion coefficient $c^2\tau_\alpha^{1-\alpha}$ where $c^2$ is ordinary diffusion coefficient and $\tau_\alpha$ is a time constant the solution takes the following form:

$$u(x,t;\alpha) = u_0 + \frac{x}{l}(u_1 - u_0) + \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right) exp\left(-\frac{c^2\tau_\alpha^{1-\alpha}\lambda_n^2+K_1(\alpha)}{K_0(\alpha)}t\right). \quad (42)$$

## 3. ILLUSTRATIVE EXAMPLE

In this section, we first consider the following non-homogenous initial boundary value problem:

$$u_t(x,t) = u_{xx}(x,t),$$
$$u(0,t) = 1\,, u(2,t) = 1,$$
$$u(x,0) = -\sin(\pi x) + 1 \quad (43)$$

which has the solution in the following form:

$$u(x,t) = -\sin(\pi x)\, e^{-\pi^2 t} + 1 \quad (44)$$

where $0 \le x \le 2$, $0 \le t \le T$.

**Example 1.** Now let the following problem called fractional heat-like problem be taken into consideration:

JSR A
Journal of Scientific Reports



$P_t^\alpha(x,t) = u_{xx}(x,t),$ (45)
$u(0,t) = 1, u(2,t) = 1,$ (46)
$u(x,0) = -\sin(\pi x) + 1$ (47)

where $0 < \alpha < 1,\ 0 \leq x \leq 2, 0 \leq t \leq T$.

To make the boundary condition (46) homogenous, we apply the transformation

$$v(x,t) = u(x,t) - 1 \quad (48)$$

to the above problem which leads to the following fractional heat-like problem

$P_t^\alpha(x,t) = v_{xx}(x,t),$ (49)
$v(0,t) = 0\ , v(2,t) = 0,$ (50)
$v(x,0) = -\sin(\pi x)$ (51)

where $0 < \alpha < 1, 0 \leq x \leq 2, 0 \leq t \leq T$. It is clear from Eq. (38) that the solution of above problem can be obtained in the following form:

$$v(x,t;\alpha) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{2}\right) exp\left(-\frac{n^2\pi^2+4-4\alpha}{4\alpha}t\right). \quad (52)$$

Plugging $t = 0$ in to the general solution (52) and making equal to the initial condition (51) we have

$$-\sin(\pi x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{2}\right). \quad (53)$$

The coefficients $d_n$ for $n = 0,1,2,3, \ldots$ are determined by the help of the inner product as follows:

$$d_n = \int_0^2 -\sin\left(\frac{n\pi x}{2}\right) \sin(\pi x)\, dx.$$

$d_n = 0$ for $n \neq 2$. For $n = 2$, $d_2$ is obtained as follows:

$$\Rightarrow d_2 = -\int_0^2 \sin^2(\pi x)\, dx = -\frac{1}{2}\left(x + \frac{\sin(2\pi x)}{4\pi}\right)\Bigg|_{x=0}^{x=2} = -1. \quad (54)$$

Substituting (54) in (52) leads to the solution of the problem (49)-(51).

$$v(x,t;\alpha) = -\sin(\pi x)\, exp\left(-\frac{\pi^2+1-\alpha}{\alpha}t\right). \quad (55)$$

By making use of (48) and the solution (55), we obtain the general solution of the problem (45)-(47) as follows:

$$u(x,t;\alpha) = -\sin(\pi x) \exp\left(-\frac{\pi^2+1-\alpha}{\alpha}t\right) + 1. \quad (56)$$

It is important to note that plugging $\alpha = 1$ in to the solution (56) gives the solution (44) which confirm the accuracy of the method we apply.

JSR A
Journal of Scientific Reports



**Example 2.** Now let the following problem called fractional heat-like problem be taken into consideration:

$$Pt\alpha(x,t)=u_{xx}(x,t), \tag{57}$$
$$u(0,t)=1, u(2,t)=1, \tag{58}$$
$$u(x,0)=-\sin(\pi x)+1 \tag{59}$$

where $0<\alpha<1$, $0\leq x\leq 2, 0\leq t\leq T$. To make the boundary condition (58) homogenous, we apply the transformation

$$v(x,t)=u(x,t)-1 \tag{60}$$

to the above problem which leads to the following fractional heat-like problem

$$2Pt\alpha(x,t)=v_{xx}(x,t), \tag{61}$$
$$v(0,t)=0\ , v(2,t)=0, \tag{62}$$
$$v(x,0)=-\sin(\pi x) \tag{63}$$

where $0<\alpha<1$, $0\leq x\leq 2, 0\leq t\leq T$. It is clear from Eq. (38) that the solution of above problem can be obtained in the following form:

$$v(x,t;\alpha)=\sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{2}\right) exp\left(-\frac{n^2\pi^2+4-4\alpha^2}{4\alpha^2}t\right). \tag{64}$$

As in Example 1, after similar computations the solution can be constructed as follows:

$$\mathrm{u}(x,t;\alpha)=-\sin(\pi x)\exp\left(-\frac{\pi^2+1-\alpha^2}{\alpha^2}t\right)+1. \tag{65}$$

It is important to note that plugging $\alpha=1$ in to the solution (65) gives the solution (44) which confirm the accuracy of the method we apply. The graphics of solutions, obtained by MATLAB 2016b, for Ex.1, Ex. 2 and Problem (43) in 2D and 3D are given in Fig.1 and Fig.2 respectively.

JSR A
Journal of Scientific Reports



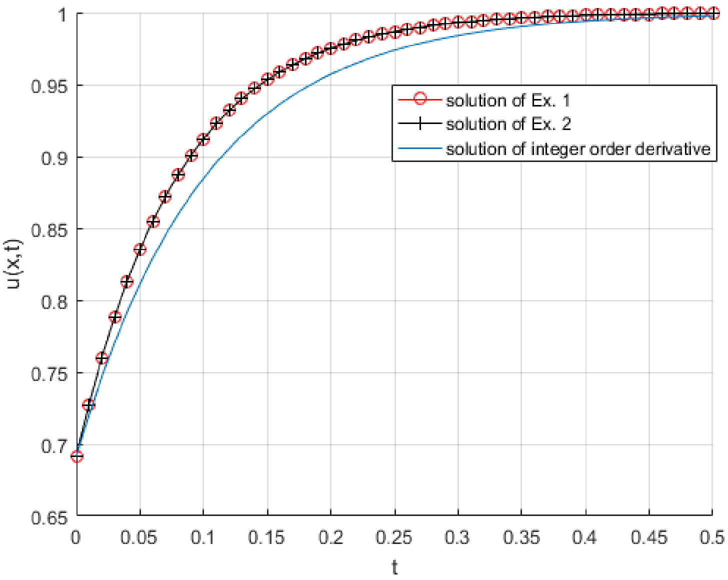


**Figure 1.** The graphics of solutions for Ex. 1 and Ex. 2. in 2D at x=0.1 for $\alpha = 0.8$.

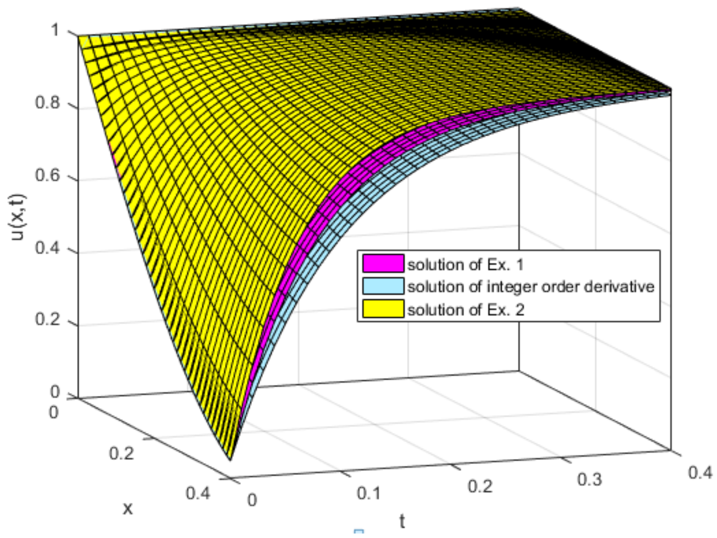


**Figure 2.** The graphics of solutions for Ex. 1 and Ex. 2 in 3D for $\alpha = 0.8$.

## 4. CONCLUSION

In this study, the analytic solution of time fractional diffusion problem including local fractional derivatives in one dimension is constructed analytically in Fourier series form. Taking the separation of variables into account, the solution is formed in the form of a Fourier series with respect to the

JSR A
Journal of Scientific Reports



eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in a proportional sense.

Based on the analytic solution, we reach the conclusion that diffusion processes decays exponential with time until initial condition is reached. As α tends to 0, the rate of decaying increases. This implies that in the mathematical model for diffusion of the matter which has small diffusion rate the value of α must be close to 0. This model can account for various diffusion processes of various methods.

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