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Spectral Properties of the Antiperiodic Boundary-Value-Transition Problems

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Abstract − This work is concerned with the boundary-value-transition problem consisting of a two-interval Sturm-Liouville equation

$$Lu := -u''(x) + q(x)u(x) = \lambda u(x)\,, x \in [-1,0) \cup (0,1]$$

together with antiperiodic boundary conditions, given by

$$u(-1) = -u(1)$$
$$u'(-1) = -u'(1)$$

and transition conditions at the interior point $x = 0$, given by

$$u(+0) = Ku(-0)$$
$$u'(+0) = \frac{1}{K}u'(-0)$$

where $q(x)$ is a continuous function in the intervals $[-1,0)$ and $(0,1]$ with finite limit values $q(\pm 0)$, $K \neq 0$ is the real number, and λ is the complex eigenvalue parameter. In this study, we shall investigate some properties of the eigenvalues and eigenfunctions of the considered problem.

Keywords–*Antiperiodic Sturm-Liouville problem, eigenvalue, eigenfunction, transition condition*

1. Introduction

A simple model for the movement of electrons in a crystal lattice, consisting of the ions in the crystal lattice and crystal with a periodic potential time-independent Schrödinger equation that describes the effects of forces from other electrons. The wave function of the electron meets the one-dimensional Schrödinger equation with the periodic potential. $T(x)$. Let t be a period that is $T(x + t) = T(x)$. By changing the variable

$$u(x) = \varphi\left(\frac{x}{t}\right),\ q(x) = \frac{2mt^2}{\hbar}T\left(\frac{x}{ta}\right), \text{and}\ \ \lambda = \frac{2mE}{\hbar^2}$$

we have

$$-u'' + q(x)u = \lambda u \tag{1.1}$$

where u is the normalized wavefunction, λ is energy parameter, $q(x + 1) = q(x)$. The spectrum of (1.1) is absolutely continuous and occurs a combination of closed intervals or 'bands' separated by 'gaps'. The presence of these bands and gaps has important implications for the conductivity properties of crystals.

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The study published by Birkhoff [1] investigated the asymptotic behaviour of the solutions of linear differential equations depending on the eigenvalue parameter given by

$$\frac{d^n y}{dx^n}+\lambda a_{n-1}(x,\lambda)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+\lambda^n a_0(x,\lambda)y=0$$

In this work, asymptotic formulas of solutions of the considered linear differential equations related to the eigenvalue parameter have been studied, and it is defined as the concept of regular boundary conditions. In the literature, such conditions are called regular boundary conditions in the sense of Birkhoff. He proved the theorem associate with the completeness of systems consisting of eigenfunctions and associated functions (i.e., root functions) of the differential operator corresponding to the problem.

In the study of Tamarkin [2], it is found the asymptotic of basic solutions for linear differential equations dependent on parameters. He defined the concept of strong regular conditions, and he studied the properties of eigenfunctions under these boundary conditions and the expansion in the series of eigenfunctions. In later years, the investigation of new concrete problems posed by physics led to the rapid development of Sturm-Liouville theory. Today, the Sturm-Liouville problems remain one of the most current issues needed by spectral theory.

In the study of Lee [3] showed the periodic analogues of spectral and oscillation theory concerned with the standard Sturm-Liouville problem.

Berghe et al. [4] investigated the eigenvalues of boundary value problems under periodic and quasi-periodic boundary conditions and explained that a simple linearly dependent multistep method could reduce the error of approximate eigenvalues.

Liu [5] prove existence for the solutions of the periodic Sturm-Liouville problem consisting of the n-th order functional differential equation with impulses effects, given by

$$\begin{cases} x^n(s)=f\left(s,x(s),x(\alpha_1(s))\dots,x(\alpha_m(s))\right), & s\in[0,S]\\ \Delta x^i(s_k)=I_{i,k}(x(s_k),\dots x^{n-1}(s_k)), & k=1,2,\dots,r\end{cases}$$

and the periodic boundary conditions, given by

$$x^i(0)=x^i(S),\qquad i=0,1,\dots,n-1$$

This method is based on Mawhin's theory and some technical inequalities.

In the study of Wang [6], by using a fixed-point theorem for operators on a cone, some results of first-order periodic Sturm-Liouville problem of impulsive dynamic equations with time scales are established. Examples are provided to show the results in this paper.

The article by Malathi et al. [7] discusses the shooting algorithm and the Floquet theory. In the shooting algorithm for an eigenvalue of a Sturm-Liouville problem, the equation is solved as an initial value problem on the interval $[a,b]$. Floquet theory is used to show a non-trivial solution of boundary value problems, and the application of shooting techniques approximates the eigenvalues. The numerical results of Sturm-Liouville eigenvalue problems with periodic boundary conditions are given.

In the studies [8-12], boundary value transmission problems are discussed for the two-linked regular Sturm-Liouville equations.

This study investigates some properties of eigenvalues and characteristic function of the antiperiodic Sturm-Liouville value transition problem together with boundary-transition conditions on $[-1,0)\cup(0,1]$.

2. Eigenvalues and Corresponding Eigenfunctions of The Problem

In this study, in the Hilbert space $L_2(-1,0)\oplus L_2(0,1)$ we shall examine some spectral properties of a boundary-value-transition problem consisting of a two-interval Sturm-Liouville equation

$$Lu:=-u''(x)+q(x)u(x)=\lambda u(x),\qquad x\in[-1,0)\cup(0,1] \tag{2.1}$$

together with antiperiodic boundary conditions, given by

$$u(-1) = -u(1), \qquad u'(-1) = -u'(1) \tag{2.2}$$

and transition conditions at the interior point $x = 0$, given by

$$u(+0) = Ku(-0), \quad u'(+0) = \frac{1}{K}u'(-0) \tag{2.3}$$

where$q(x)$ is a continuous function in the intervals $[-1,0)$ and $(0,1]$ with finite limit values $q(\pm 0)$, $K \neq 0$ is the real number, and λ is the complex eigenvalue parameter.

Theorem 2.1. All eigenvalues of the boundary-value-transition problem $(2.1) - (2.3)$ are real.

Proof. Let (λ, u) be an eigenvalue-eigenfunction pair, $\bar{u}$ be the complex conjugate of u, $\bar{\lambda}$ be the complex conjugate of λ. Since K is a real number and $q(x)$ is a real-valued function, we get

$$-\bar{u}''(x) + q(x)\bar{u}(x) = \bar{\lambda}\bar{u}(x) \tag{2.4}$$

$$\bar{u}(-1) = -\bar{u}(1), \qquad \bar{u}'(-1) = -\bar{u}'(1)$$

$$\bar{u}(+0) = K\bar{u}(-0), \qquad \bar{u}'(+0) = \frac{1}{K}\bar{u}'(-0)$$

Now, multiplying the equation (2.1) by $\bar{u}$ and the equation (2.4) by u we have

$$-u''\bar{u} + q(x)u\bar{u} = \lambda u\bar{u}$$

and

$$-u\bar{u}'' + q(x)u\bar{u} = \bar{\lambda}u\bar{u}$$

respectively. Subtracting these two equalities gives

$$u\bar{u}'' - u''\bar{u} = (\lambda - \bar{\lambda})u\bar{u}$$

Taking in view the identity $u\bar{u}'' - u''\bar{u} = (u\bar{u}' - u'\bar{u})'$ we have

$$(u\bar{u}' - u'\bar{u})' = (\lambda - \bar{\lambda})u\bar{u}$$

Now integrating over $[-1, 0)$ we obtain

$$\int_{-1}^{-0} (u\bar{u}' - u'\bar{u})'\,dx = \int_{-1}^{-0} (\lambda - \bar{\lambda})u\bar{u}dx$$

Hence,

$$u(-0)\bar{u}'(-0) - u'(-0)\bar{u}(-0) - u(-1)\bar{u}'(-1) + u'(-1)\bar{u}(-1) = \int_{-1}^{-0} (\lambda - \bar{\lambda})u\bar{u}dx$$

Similarly, we can show that

$$u(1)\bar{u}'(1) - u'(1)\bar{u}(1) - u(+0)\bar{u}'(+0) + u'(+0)\bar{u}(+0) = \int_{+0}^{1} (\lambda - \bar{\lambda})u\bar{u}dx$$

Since $u(x)$ satisfies the transition conditions (2.3), we have

$$u(+0) = Ku(-0)$$

$$u'(+0) = \frac{1}{K}u'(-0)$$

Similarly, we get

$$\bar{u}(+0) = K\bar{u}(-0)$$

$$\bar{u}'(+0) = \frac{1}{K}\bar{u}'(-0)$$

and

$$u(1)\bar{u}'(1) - u'(1)\bar{u}(1) - Ku(-0)\frac{1}{K}\bar{u}'(-0) + \frac{1}{K}u'(-0)K\bar{u}(-0) = \int_{+0}^{1}(\lambda - \bar{\lambda})u\bar{u}dx$$

Thus, we get that

$$0 = (\lambda - \bar{\lambda})\left[\int_{-1}^{-0} u\bar{u}dx + \int_{+0}^{1} u\bar{u}dx\right] = (\lambda - \bar{\lambda})\|u\|_H^2$$

Since the eigenfunction u is nonzero, the last equality gives $\lambda = \bar{\lambda}$. Consequently, λ is real, which completes the proof.

Theorem 2.2. Let (λ_m, u_m) and (λ_n, u_n) be two eigenpairs of the boundary-value-transition problem (2.1) − (2.3). If $\lambda_m \neq \lambda_n$ then the eigenfunctions u_m and u_n are orthogonal in the Hilbert space $H \coloneqq L_2(-1,0) \oplus L_2(0,1)$. That is,

$$\int_{-1}^{-0} u_m(x)u_n(x)dx + \int_{+0}^{1} u_m(x)u_n(x)dx = 0$$

Proof. Since u_m and u_n are eigenfunctions corresponding to the eigenvalues λ_m and λ_n, respectively, we get the following equalities,

$$-u_m{}'' + q(x)u_m = \lambda_m u_m$$

$$-u_n{}'' + q(x)u_n = \lambda_n u_n$$

Multiplying the first equality by u_n and the second equality by u_m and taking the difference yields

$$u_m u_n{}'' - u_m{}'' u_n = (\lambda_m - \lambda_n)u_m u_n$$

Applying the well-known Lagrange's formulae commonly known as Green's identity we get

$$u_m(-0)u_n{}'(-0) - u_m{}'(-0)u_n(-0) - u_m(-1)u_n{}'(-1) + u_m{}'(-1)u_n = \int_{-1}^{-0} (\lambda_m - \lambda_n)u_m u_n dx \quad (2.5)$$

By the boundary conditions (2.2) we have

$$u_m(-1) = -u_m(1),\ u_m{}'(-1) = -u_m{}'(1)$$

and

$$u_n(-1) = -u_n(1),\ u_n{}'(-1) = -u_n{}'(1)$$

Substituting these into the equation (2.5), we get

$$u_m(-0)u_n{}'(-0) - u_m{}'(-0)u_n(-0) - u_m(1)u_n{}'(1) + u_m{}'(1)u_n(1) = \int_{-1}^{-0} (\lambda_m - \lambda_n)u_m u_n dx$$

Similarly, we can show that

$$u_m(1)u_n{}'(1) - u_m{}'(1)u_n(1) - u_m(+0)u_n{}'(+0) + u_m{}'(+0)u_n(+0) = \int_{+0}^{1} (\lambda_m - \lambda_n) u_m u_n dx \qquad (2.6)$$

Since u_m and u_n satisfy the transition conditions (2.3), we get

$$u_m(+0) = Ku_m(-0), u_m{}'(+0) = \frac{1}{K} u_m{}'(-0)$$

and

$$u_n(+0) = Ku_n(-0),\ u_n{}'(+0) = \frac{1}{K} u_n{}'(-0)$$

Substituting these into the equation (2.6), we obtain

$$u_m(1)u_n{}'(1) - u_m{}'(1)u_n(1) - Ku_m(-0)\frac{1}{K}u_n{}'(-0) + \frac{1}{K}u_m{}'(-0)Ku_n(-0) = \int_{+0}^{1} (\lambda_m - \lambda_n) u_m u_n dx$$

from which it follows that

$$0 = (\lambda_m - \lambda_n)\left[\int_{-1}^{-0} u_m u_n dx + \int_{+0}^{1} u_m u_n dx\right]$$

Since $\lambda_m \neq \lambda_n$, we get that

$$\int_{-1}^{-0} u_m u_n dx + \int_{+0}^{1} u_m u_n dx = 0,$$

That is $\langle u_m , u_n \rangle = 0$, which completes the proof.

2.1. Construction of the Hilbert Space and Differential Operator for Given Boundary-Value-Transition Problem

Let us define the inner product of $\varphi(x),\ \omega(x) \in H$ by the equality

$$\langle \varphi, \omega \rangle = \int_{-1}^{-0} \varphi(x)\overline{\omega(x)}dx + \int_{+0}^{1} \varphi(x)\overline{\omega(x)}dx$$

where

$$H = \{\varphi(x) \mid \varphi(x) \in L_2(-1,0) \oplus L_2(0,1)\}$$

We can show that the inner-product axioms are obviously satisfied.

Lemma 2.1.1. The inner product space $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Proof. It is sufficient to show that every Cauchy sequence in the space H is convergent to some limit point in H. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H. Then for any $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\|\varphi_n - \varphi_m\| < \varepsilon^2$ whenever $n, m \geq n_0(\varepsilon)$. Since

$$\begin{aligned}\|\varphi_n - \varphi_m\|_H^2 &= \langle \varphi_n - \varphi_m , \varphi_n - \varphi_m \rangle_H \\ &= \|\varphi_n - \varphi_m\|_{L_2(-1,0)}^2 + \|\varphi_n - \varphi_m\|_{L_2(0,1)}^2 < \varepsilon^2\end{aligned}$$

we have

$$\|\varphi_n - \varphi_m\|_{L_2(-1,0)}^2 < \varepsilon^2,\ \|\varphi_n - \varphi_m\|_{L_2(0,1)}^2 < \varepsilon^2$$

Consequently, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in both Hilbert spaces $L_2(-1,0)$ and $L_2(0,1)$. Since the spaces $L_2(-1,0)$ and $L_2(0,1)$ are complete, any Cauchy sequence taken from these spaces are convergent sequences. So, there are $\varphi_l \in L_2(-1,0)$ and $\varphi_r \in L_2(0,1)$ such that

$$\|\varphi_n - \varphi_l\|^2_{L_2(-1,0)} \longrightarrow 0 \ (n \to \infty), \qquad \|\varphi_n - \varphi_r\|^2_{L_2(0,1)} \longrightarrow 0 \ (n \to \infty)$$

Consequently

$$\|\varphi_n - \tilde{\varphi}\|^2_H = \|\varphi_n - \varphi_l\|^2_{L_2(-1,0)} + \|\varphi_n - \varphi_r\|^2_{L_2(0,1)} \longrightarrow 0 \ (n \to \infty)$$

where $\tilde{\varphi} := \begin{cases} \varphi_l, & x \in [-1,0) \\ \varphi_r, & x \in (0,1] \end{cases} \in H$. Therefore, the completeness of the inner-product space H is proved.

Now we will define a linear operator $A : H \to H$ associated with the boundary value transition problem $(2.1) - (2.3)$ as follows:

Let the domain $D(A)$ be define as follows:

$$D(A) = \{ \varphi \in H \mid \text{The functions } \varphi_1(x), \varphi_2(x),\ \varphi_1'(x) \text{ and } \varphi_2'(x) \text{ are absolute continuous in the intervals } [-1,0] \text{ and } [0,1], \text{ there are finite limit values } \varphi(\pm 0) \text{ and } \varphi'(\pm 0), \text{ and } -\varphi_1'' + q(x)\varphi_1 \in L_1(-1,0),\ -\varphi_2'' + q(x)\varphi_2 \in L_2(0,1),\ \varphi_1(-1) = -\varphi_2(1),\ \varphi_1'(-1) = -\varphi_2'(1),\ \varphi_1(0) = K\varphi_2(0),\ \varphi_1'(0) = \frac{1}{K}\varphi_2'(0) \} \tag{2.7}$$

and the operator $A: D(A) \to H$ be defined by

$$A\varphi := -\varphi'' + q(x)\varphi \tag{2.8}$$

where

$$\varphi_1(x) = \begin{cases} \varphi(x), & x \in [-1,0) \\ \varphi(-0), & x = 0 \end{cases} \text{ and } \varphi_2(x) = \begin{cases} \varphi(x), & x \in (0,1] \\ \varphi(+0), & x = 0 \end{cases}$$

The eigenvalues and the eigenfunctions of the boundary value transition problem are defined as the eigenvalues and eigenfunctions of the operator A, respectively.

The following lemma is easy to prove.

Lemma 2.1.2. The operator A is the linear operator.

Theorem 2.1.2. The linear operator A defined by $(2.7) - (2.8)$ is symmetric in the Hilbert space

$H = L_2(-1,0) \oplus L_2(0,1)$.

Proof. Let $\varphi, \omega \in D(A) \subset H$. By the definition of A we have

$$\langle A\varphi, \omega \rangle_H = -\int_{-1}^{-0} \varphi''(x)\overline{\omega(x)}\,dx + \int_{-1}^{-0} q(x)\varphi(x)\overline{\omega(x)}\,dx - \int_{+0}^{1} \varphi''(x)\overline{\omega(x)}\,dx + \int_{+0}^{1} q(x)\varphi(x)\overline{\omega(x)}\,dx \tag{2.9}$$

Integrating by parts twice, we obtain

$$\int_{-1}^{-0} \varphi''(x)\overline{\omega(x)}\,dx = \varphi'(x)\overline{\omega(x)}\Big|_{-1}^{-0} - \varphi(x)\overline{\omega'(x)}\Big|_{-1}^{-0} + \int_{-1}^{-0} \varphi(x)\,\overline{\omega''(x)}dx,$$

and therefore,

$$\int_{-1}^{-0} \left(-\varphi''(x) + q(x)\varphi(x)\right)\overline{\omega(x)}\,dx = \int_{-1}^{-0} \varphi(x)\left(\overline{-\omega''(x) + q(x)\omega(x)}\right)dx + W(\varphi,\overline{\omega};-0) - W(\varphi,\overline{\omega};-1) \tag{2.10}$$

By similar technique as above, one can show that

$$\int_{+0}^{1}\left(-\varphi''(x)+q(x)\varphi(x)\right)\overline{\omega(x)}\,dx=\int_{+0}^{1}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0) \quad (2.11)$$

Substituting (2.10) and (2.11) into (2.9), we obtain

$$\begin{aligned}\langle A\varphi,\omega\rangle_H=&\int_{-1}^{-0}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx+\int_{+0}^{1}\varphi(x)\left(\overline{-\omega''(x)+q(x)\omega(x)}\right)dx\\&+W(\varphi,\overline{\omega};-0)-W(\varphi,\overline{\omega};-1)+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0)\end{aligned} \quad (2.12)$$

Hence, (2.12) takes the form

$$\langle A\varphi,\omega\rangle_H-\langle\varphi,A\omega\rangle_H=W(\varphi,\overline{\omega};-0)-W(\varphi,\overline{\omega};-1)+W(\varphi,\overline{\omega};1)-W(\varphi,\overline{\omega};+0)$$

Since $\varphi,\omega\in D(A)$, this yield

$$\begin{aligned}&\varphi(-0)\overline{\omega'(-0)}-\varphi'(-0)\overline{\omega(-0)}-\varphi(-1)\overline{\omega'(-1)}+\varphi'(-1)\overline{\omega(-1)}+\varphi(1)\overline{\omega'(1)}-\varphi'(1)\overline{\omega(1)}\\&-\varphi(+0)\overline{\omega'(+0)}+\varphi'(+0)\overline{\omega(+0)}=0\end{aligned}$$

Then the equality

$$\langle A\varphi,\omega\rangle_H=\langle\varphi,A\omega\rangle_H$$

is valid for all $\varphi,\omega\in D(A)$. This completes the proof of Theorem.

2.2. Some Auxiliary Initial Value Problems and Solutions

In this section, we will use solutions of some auxiliary initial value problems, given only on the sub-intervals $[-1,0]$ and $[0,1]$, which are closely related to the boundary value transition problem $(2.1)-(2.3)$. The initial value problem

$$-u''(x)+q(x)u(x)=\lambda u(x), \qquad x\in[-1,0]$$

$$u(-1)=1$$

$$u'(-1)=0$$

has a unique solution $u=\phi_1(x,\lambda)$ for each $\lambda\in\mathbb{C}$ for the theory of ordinary differential equations and this solution is analytical in the whole complex plane concerning the variable λ for each $x\in[-1,0]$. (See, [13])

Let $\phi_2(x,\lambda)$ be the solution of the initial-value problem given by

$$-u''(x)+q(x)u(x)=\lambda u(x), \qquad x\in[0,1]$$

$$u(1)=1$$

$$u'(1)=0$$

This solution is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[0,1]$. (See, [13])

Similarly, for each $\lambda\in\mathbb{C}$, the initial-value problem

$$-u''(x)+q(x)u(x)=\lambda u(x), \qquad x\in[0,1]$$

$$u(1)=0$$

$$u'(1)=1$$

has a unique solution $u=\chi_2(x,\lambda)$ and the initial-value problem

$$-u''(x)+q(x)u(x)=\lambda u(x), \qquad x\in[0,1]$$

$$u(-1) = 0$$
$$u'(-1) = 1$$

has a unique solution $u = \chi_1(x,\lambda)$. These solutions are also analytical in the whole complex plane concerning the variable λ for each fixed x, that is, $\chi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ are entire functions of $\lambda \in \mathbb{C}$ for each fixed x(see, [13]).

2.3. The Characteristic Function

Theorem 2.3.1. The eigenvalues of the boundary-value-transition problem $(2.1) - (2.3)$ are coincide with the zeros of the characteristic function

$$\Delta(\lambda) = \left[\phi_2(+0,\lambda) - K\phi_1(-0,\lambda)\right]\left[\chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda)\right] - \left[\chi_2(+0,\lambda) - K\chi_1(-0,\lambda)\right]\left[\phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda)\right] \quad (2.13)$$

Proof. Since for each $\lambda \in \mathbb{C}$ the Wronskian $W(\phi_1,\chi_1;x)$ is independent on $x \in [-1,0]$ and $W(\phi_1,\chi_1;-1) = 1 \neq 0$, the functions $\phi_1(x,\lambda)$, $\chi_1(x,\lambda)$ are linearly independent solutions of the equation (2.1) in the interval $[-1,0]$. Therefore, the general solution of the equation (2.1) on the left interval $[-1,0]$ can be expressed in the form

$$y = c_1\phi_1(x,\lambda) + c_2\chi_1(x,\lambda)$$

Similarly, the general solution of the same differential equation on the right interval $[0,1]$ can be expressed in the form

$$y = c_3\phi_2(x,\lambda) + c_4\chi_2(x,\lambda)$$

Thus, the general solution of the differential equation (2.1) on the interval $[-1,0) \cup (0,1]$ can be written in the form

$$y = \begin{cases} c_1\phi_1(x,\lambda) + c_2\chi_1(x,\lambda), & x \in [-1,0) \\ c_3\phi_2(x,\lambda) + c_4\chi_2(x,\lambda), & x \in (0,1] \end{cases}$$

Applying the antiperiodic boundary conditions (2.2) we obtain

$$c_1\phi_1(-1,\lambda) + c_2\chi_1(-1,\lambda) = c_3\phi_2(1,\lambda) + c_4\chi_2(1,\lambda)$$
$$c_1\phi_1'(-1,\lambda) + c_2\chi_1'(-1,\lambda) = c_3\phi_2'(1,\lambda) + c_4\chi_2'(1,\lambda)(2.3.2) \quad (2.14)$$

By the definition of the solutions ϕ_1, χ_1, ϕ_2 and χ_2 we get

$$\phi_1(-1,\lambda) = 1,\ \phi_1'(-1,\lambda) = 0,\ \phi_2(1,\lambda) = 1,\ \phi_2'(1,\lambda) = 0$$
$$\chi_1(-1,\lambda) = 0,\ \chi_1'(-1,\lambda) = 1,\ \chi_2(1,\lambda) = 0,\ \chi_2'(1,\lambda) = 1$$

Substituting these equalities into (2.14), we obtain that $c_1 = c_3 = A$ and $c_2 = c_4 = B$. Then, the general solution can be written in the form

$$y = A\phi(x,\lambda) + B\chi(x,\lambda)$$

Substituting this into transition conditions (2.3), we obtain the following linear system of equations concerning the variables A and B, given by

$$\left(\phi_2(+0,\lambda) - K\phi_1(-0,\lambda)\right)A + \left(\chi_2(+0,\lambda) - K\chi_1(-0,\lambda)\right)B = 0$$
$$\left(\phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda)\right)A + \left(\chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda)\right)B = 0$$

This homogeneous system of linear equations has a nontrivial solution $(A,B) \neq (0,0)$ if the determinant of this system is equal to zero, i.e.

$$\begin{vmatrix} \phi_2(+0,\lambda) - K\phi_1(-0,\lambda) & \chi_2(+0,\lambda) - K\chi_1(-0,\lambda) \\ \phi_2'(+0,\lambda) - \frac{1}{K}\phi_1'(-0,\lambda) & \chi_2'(+0,\lambda) - \frac{1}{K}\chi_1'(-0,\lambda) \end{vmatrix} = 0$$

Hence, $\Delta(\lambda) = 0$. This completes the proof.

Theorem 2.3.2. If $K^2 \neq 1$, then for the characteristic function $\Delta(\lambda)$ the following asymptotic formulas hold

$$\Delta(\lambda) = (1-K)\left(1-\frac{1}{K}\right)\cos^2\sqrt{\lambda} + (1+K)\left(1+\frac{1}{K}\right)\sin^2\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}e^{2t}\right)$$

as $|\lambda| \to \infty$, where $t = Im\sqrt{\lambda}$.

Proof. By applying well-known properties of Volterra integral equations, we can derive the following asymptotic formulas

$$\phi_1(x,\lambda) = \cos\left(\sqrt{\lambda}(x+1)\right) + O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x+1|}\right)$$

$$\phi_1{}'(x,\lambda) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x+1)\right) + O\left(e^{|t||x+1|}\right)$$

$$\phi_2(x,\lambda) = \cos\left(\sqrt{\lambda}(x-1)\right) + O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x-1|}\right)$$

$$\phi_2{}'(x,\lambda) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x-1)\right) + O\left(e^{|t||x-1|}\right)$$

$$\chi_1(x,\lambda) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x+1)\right) + O\left(\frac{1}{\lambda}e^{|t||x+1|}\right)$$

$$\chi_1'(x,\lambda) = \cos\left(\sqrt{\lambda}(x+1)\right) + O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x+1|}\right)$$

$$\chi_2(x,\lambda) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x-1)\right) + O\left(\frac{1}{\lambda}e^{|t||x-1|}\right)$$

$$\chi_2'(x,\lambda) = \cos\left(\sqrt{\lambda}(x-1)\right) + O\left(\frac{1}{\sqrt{\lambda}}e^{|t||x-1|}\right)$$

as $|\lambda| \longrightarrow \infty$, where O denote the Landau symbol. Substituting these asymptotic formulas into (2.13) we arrive at

$$\begin{aligned}\Delta(\lambda) &= \left[(1-K)\cos\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}e^{|t|}\right)\right]\left[\left(1-\frac{1}{K}\right)\cos\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}e^{|t|}\right)\right] \\ &\quad - \left[(-1-K)\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda} + O\left(\frac{1}{\lambda}e^{|t|}\right)\right]\left[\left(1+\frac{1}{K}\right)\sqrt{\lambda}\sin\sqrt{\lambda} + O\left(e^{|t|}\right)\right] \\ &= (1-K)\left(1-\frac{1}{K}\right)\cos^2\sqrt{\lambda} + (1+K)\left(1+\frac{1}{K}\right)\sin^2\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}e^{2t}\right)\end{aligned}$$

This completes the proof.

Theorem 2.3.3. If $K^2 \neq 1$, then the boundary-value-transition problem $(2.1)-(2.3)$ has a countable set of eigenvalues without finite accumulation point.

Proof. Denote by $\Delta_1(\lambda)$ the leading term of $\Delta(\lambda)$, that is

$$\Delta_1(\lambda) = (1-K)\left(1-\frac{1}{K}\right)\cos^2\sqrt{\lambda} + (1+K)\left(1+\frac{1}{K}\right)\sin^2\sqrt{\lambda}$$

This function has a countable set of zeros $\lambda_n',\ n = 1,2,\cdots$ without a finite accumulation point. Applying now the well-known Rouche's theorem (see, for example, [13]) to the appropriate circles we conclude that the characteristic function $\Delta(\lambda)$ has a countable set zeros $\lambda_n,\ n = 1,2,\cdots$ which satisfies the asymptotic equality $\lambda_n = \lambda_n' + O\left(\frac{1}{n}\right)$. The proof is complete.

3. Conclusions

In this study, antiperiodic Sturm-Liouville problems, including transition conditions, were investigated for the first time in the literature. A Hilbert space suitable for the problem is established. Then, an operator is defined on this Hilbert space that is the same as the problem's eigenvalues. It has been proved that the eigenvalues are real and the eigenfunctions are orthogonal. The problem's characteristic function is defined, and the asymptotic formula is obtained for the characteristic function. Finally, the asymptotic formula for eigenvalues was found using the asymptotic formula of the characteristic function.

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