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Inertial Hybrid Self-adaptive subgradient extragradient method for Fixed Point of Quasi$-\phi-$nonexpansive multivalued mappings and Equilibrium problem

Murtala H. Harbau[a]

[a]*Department of Science and Technology Education, Bayero University, Kano, Nigeria.*

Abstract

In this paper, we propose a new inertial self-adaptive subgradient extragradient algorithm for approximating common solution in the set of pseudomonotone equilibrium problems and the set of fixed point of finite family of quasi$-\phi-$nonexpansive multivalued mappings in real uniformly convex Banach spaces and uniformly smooth Banach spaces. Strong convergence of the iterative scheme is established. Our results generalizes and improves several recent results annouced in the literature.

Keywords: Pseudomonotone Equilibrium problem Inertial self adaptive hybrid method Multivalued quasi$-\phi-$nonexpansive mapping Banach Spaces.
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1. Introduction

Let E be a real Banach space and E^* be the dual of E. Let C be a nonempty closed and convex subset of E. The equilibrium problem is to find $\bar{z} \in C$ such that

$$g(\bar{z}, y) \geq 0 \quad \forall \quad y \in C, \tag{1}$$

where $g : C \times C \to \mathbb{R}$ is a bifunction with property $g(x, x) = 0 \; \forall \; x \in C$. The equilibrium problem (1) was introduced by Blum and Oettli [5]. We denote by $EP(g, C)$ to be the set solutions of equilibrium problem (1), i.e.

$$EP(g, C) = \{\bar{z} \in C : g(\bar{z}, y) \geq 0 \;\; \forall \;\; y \in C\}.$$

Email address: murtalaharbau@yahoo.com (Murtala H. Harbau)

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Equilibrium problem (1) generalizes many important problems such as variational inequality problem, optimization problem, complementarity problem, fixed point problem, see, for example, [5, 28].

A map $T : E \to E$ is said to be nonexpansive if $||Tx-Ty|| \leq ||x-y|| \quad \forall\, x, y \in E$. A point $x \in E$ is said to be a fixed point of T if $x = Tx$. The set of fixed points of T is denoted by $F(T)$, i.e $F(T) = \{x \in E : x = Tx\}$. T is called quasi nonexpansive $||Tx - z|| \leq ||x - z|| \quad \forall\, x \in E,\ z \in F(T)$.

Let $CB(E)$ be a family of nonempty closed and bounded subsets of E and $T : E \to CB(E)$ be a multivalued mapping. A point $z \in E$ is called a fixed point of T if $z \in Tz$. We denote by $F(T)$ the set of all fixed points of T i.e $F(T) = \{z \in E : z \in Tz\}$. A point $z \in F(T)$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in E such that $x_n \rightharpoonup z$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. The set of all asymptotic fixed points of T is denoted by $\tilde{F}(T)$.

A multi-valued mapping $T : E \to CB(E)$ is called *relatively nonexpansive* if $F(T) \neq \emptyset, \quad F(T) = \tilde{F}(T)$ and $\phi(z, p) \leq \phi(z, x) \quad \forall x \in E, \quad p \in Tx, \quad z \in F(T)$.
T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(z, p) \leq \phi(z, x) \quad \forall x \in E, \quad p \in Tx, \quad z \in F(T)$.
T is said to be *closed* if for any sequence $\{x_n\}$ in E with $x_n \to x$ and $\{w_n\} \subset T(x_n)$ with $w_n \to y$, then $y \in T(x)$.

Remark 1.1. *Observe that from the above definitions, the class of quasi-ϕ-nonexpansive multi-valued mappings contains the class of relatively nonexpansive multi-valued mappings which require a strong restriction $\tilde{F}(T) = F(T)$. Furthermore if E is a real Hilbert space H, the class of quasi-ϕ-nonexpansive mappings coincides with the class of quasi nonexpansive mappings which inturn contains the class of nonexpansive mappings.*

Due to their importance, various methods have been imployed to approximate solutions of equilibrium and fixed point problems (see, for example, [3, 18, 19, 35, 36] and the references contained therein). One of the common methods use is the proximal point method in which the convergence analysis has been considered when the bifunction g is monotone see [26]. However the proximal point method is not valid when the underlying bifunction g is pseudomonone see Wen, [41].

Another method use is the extragradient-like method [1, 17, 22, 23, 25, 34, 39] which involved two strongly convex optimization problem defined over the constrained set C and the Lipschitz-type condition imposed on the bifunction g. Moreover to solve the two strongly convex problem over the constrained set C in each iteration can be complicated especially if C is not simple. Motivated by this, Censor et al. [7] introduced a method called subgradient extragradient for approximating solutions of variational inequality problem in a real Hilbert space H, in which one projection was taken over constructed subpace which can easily be computed. Hieu [21] extended the subgradient extragradient method equilibrium problems in a real Hilbert spaces H, the author proposed the following algorithm;

$$\begin{cases} x_0 \in H \\ y_n = \operatorname{argmin}\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ z_n = \operatorname{argmin}\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in T_n\}, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n,\ n \geq 0, \end{cases} \tag{2}$$

where $T_n = \{v \in H : \langle (x_n - \lambda w_n) - y_n, v - y_n\rangle \leq 0\}$, $w_n \partial_2 f(x_n, y_n)$ and $\lambda,\ \alpha_n$ satisfy the following conditions;

1. $0 < \lambda < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$
2. $0 < \alpha_n < 1,\ \lim\limits_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = +\infty.$

The author proved strong convergence of the iterative sequence (2) to the solution of the equilibrium problem. Recently, Dadashi et al. [14] used subgradient extragradient method to approximate solution of pseudomonotone equilibrium problem in real Hilbert spaces.

One problem of the aforemention results was the computation of the Lipschitz constants c_1, c_2 of the bifunction f which sometimes is difficult to estimate. Motivated by this, very recently, Yang and Liu [42] introduced a new step size, in the subgradient extragradient method for pseudomonotone equilibrium problem and fixed point of quasi nonexpansive mapping in a real Hilbert space. They proved strong convergence of the following iterative sequence without the prior knowledge of the Lipschitz-type constants of the bifunction f.

$$\begin{cases} x_0 \in H \\ y_n = \underset{y \in C}{\operatorname{argmin}}\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ T_n = \{v \in H : \langle (x_n - \lambda_n w_n) - y_n, v - y_n\rangle \leq 0\}, \\ z_n = \underset{y \in T_n}{\operatorname{argmin}}\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ t_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ x_{n+1} = \beta_n z_n + (1-\beta_n) S t_n, \ n \geq 0 \end{cases} \tag{3}$$

where S is quasi nonexpansive map, $w_n \in \partial_2 f(x_n, y_n)$, λ_0, $\mu \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying some conditions and

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||x_n - y_n||^2 + ||z_n - y_n||^2)}{2(f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n))}, \lambda_n\}, & f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0, \\ \lambda_n, & \text{Otherwise.} \end{cases}$$

They proved strong convergence of (3) to common point in the set of fixed point of quasi nonexpansive mapping and set of pseudomonotone equilibrium problems.

Recently, inertial method which was introduced by Polyak [30] to speed up the rate of convergence of the iteration methods has been considerably attracting interest of reseachers, (see, for example, [4, 8, 9, 11, 12, 13, 16, 27, 29, 31, 37, 40] and the references contained therein).

Motivated by the above results, the purpose of this paper is to propose an inertial self-adaptive subgradient extragradient algorithm for approximating common solution in the set of pseudomonotone equilibrium problem and the set of fixed point of finite family of quasi$-\phi-$nonexpansive multivalued mappings in real uniformly convex Banach spaces and uniformly smooth Banach spaces. The step size η_n is chosen self adaptively and estimates of Lipschizt-type constants are dispensed with.

2. Preliminaries

Let E be a real Banach space and E^* be the dual of E. Let C be a nonempty closed and convex subset of E. We denote by $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle ., .\rangle$ denotes the duality pairing between the element of E and that of E^*. It is well known that $J(x)$ is nonempty for each $x \in E$, see [36]. We denote weak and strong convergence by $\rightharpoonup$ and $\to$ respectively.

Let $S(E)$ be a unit sphere centered at the origin. A Banach space is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x,\ y \in S(E)$ and $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(t) = \inf\Big\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1 = \|y\|,\ \|x-y\| \geq \epsilon\Big\},\ \forall\ t \in [0,2].$$

E is called uniformly convex if $\delta_E(t) \geq 0\ \forall\ t \in [0,2]$ and p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(t) \geq c_p t^p\ \forall\ t \in [0,2]$. Note that every p-uniformly convex Banach space is uniformly convex and every uniformly convex is strictly convex and reflexive. The modulus of smoothness $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \sup\Big\{\frac{\|x+\tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\Big\}.$$

E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$ and E is $q-$uniformly smooth if there exists $d_q > 0$ such that $\rho_E(\tau) \leq d_q \tau^q$. It is well known that if E is $q-$uniformly smooth, then $q \leq 2$ and E uniformly smooth. Furthermore every uniformly smooth Banach space is smooth. We know that (see, for example, [10]) if E is smooth, strictly convex and reflexive, then J is single-valued, one-to-one and onto respectively and J^{-1} is also single-valued, one-to-one, onto and it is the duality mapping from E^* into E. In addition if E is uniformly smooth, then the norm on E is fr$\acute{e}$chet differentiable and J is uniformly norm-to-norm continuous on bounded subsets of E and E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a smooth Banach space and C be a closed convex subset of E. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall\ x, y \in E, \tag{4}$$

is called Lyapunov bifunction introduced by Alber [2], where J is the normalized duality mapping. Observe from the definition of ϕ in (4) above, we have that,

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,\ y,\ z \in E, \quad \textit{and} \tag{5}$$

$$\Big(\|x\| - \|y\|\Big)^2 \leq \phi(x,y) \leq \Big(\|x\| + \|y\|\Big)^2, \quad \forall x,\ y \in E \tag{6}$$

Follwing Alber [2], the generalized projection $\Pi_C : E \to C$ is a mapping defined by

$$\Pi_C(x) = arg \min_{y \in C} \phi(y,x) \quad \forall x \in E.$$

Remark 2.1. *(1) If E is a Hilbert space, then $\phi(y,x) = \|y - x\|^2$, and the generalized projection reduces to metric projection P_C of E onto C.*

(2) If E is smooth and strictly convex, then $\phi(x,y) = 0$ if and only if $x = y$ $\forall x, y \in E$, see, for example, [36]

Definition 2.2. *(see [6, 24]) The subdifferential of f, ∂f is the mapping $\partial f : E \to 2^{E^*}$ defined by*

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^*\rangle\ \forall\ y \in E\} \text{ for all } x \in E.$$

Remark 2.3. *It is known that if the function f is proper, lower semicontinuous and convex, then for each $x \in D(f)$ the subdifferential $\partial f(x)$ is a nonempty closed convex set, where $D(f)$ is the domain of f.*

Definition 2.4. *A bifunction $g : C \times C \to \mathbb{R}$ is said to be;*

1. *γ-strongly monotone on C if there exists $\gamma > 0$ such that*

$$g(x,y) + g(y,x) \leq -\gamma||x - y||^2 \quad \forall \quad x, y \in C,$$

2. *Monotone if*

$$g(x,y) + g(y,x) \leq 0 \quad \forall \quad x, y \in C,$$

3. *Pseudomonotone if*

$$g(x,y) \geq 0 \quad \Rightarrow \quad g(y,x) \leq 0\ \forall\ x, y \in C,$$

It is clear from Definition 2.4, that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To solve the equilibrium problem, we assume the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions;

(D1) $g(x,x) = 0$ for every $x \in C$;

(D2) $g(x,.)$ is convex, lower semicontinuous and subdifferentiable on E;

(D3) g is pseudomonotone on C;

(D4) g is jointly continuous on $E \times C$ in the sense that if $x \in E$ $y \in C$ and $\{x_n\}$, $\{y_n\}$ are two sequences such that $x_n \to x$, $y_n \to y$, then $g(x_n, y_n) \to g(x, y)$;

(D5) $g(x,y) + g(y,z) \geq g(x,z) - c_1\phi(y,x) - c_2\phi(z,y)$ $\forall$ $x,y,z \in C$ and some c_1, $c_2 > 0$.

In the sequel we will need the following lemmas:

Lemma 2.5. *[43] Let E be a real uniformly smooth and uniformly convex Banach space. Let $T : E \to 2^E$ be a closed quasi-ϕ-nonexpansive multivalued mapping, then $F(T)$ is closed and convex.*

Lemma 2.6. *[39] Assume the bifunction g satisties (D1)-(D4), then the set $EP(g,C)$ of solutions of the equilibrium problems is closed and convex.*

Lemma 2.7. *[38] Let C be a nonempty subset of E and $f : C \to \mathbb{R}$ be a convex and subdifferentiable function, then f is minimized at $x \in C$ if and only if*

$$0 \in \partial f(x) + N_C(x),$$

where $N_C(x)$ is the normal cone to C at $x \in C$, i.e.

$$N_C(x) = \{\zeta \in E^* : \langle y - x, \zeta\rangle \leq 0 \ \forall \ y \in C\}.$$

Lemma 2.8. *[10] Let E be a reflexive Banach space and $f : E \to \mathbb{R}$, $g : E \to \mathbb{R}$ are two convex functions such that dom $f \cap$ dom $g \neq \emptyset$ and f is continuous, then*

$$\partial(f + g) = \partial f(x) + \partial g(x), \quad \forall \ x \in E.$$

Lemma 2.9. *[2] Let E be a strictly convex, smooth and reflexive Banach space and let K be a nonempty closed and convex subset of E. Let $x \in E$, then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 2.10. *[44] Let E be a uniformly convex Banach space and $r > 0$, then there exists a strictly increasing, continuous and convex function $f : [0, 2r] \to [0, +\infty)$ such that $f(0) = 0$ and*

$$\Big\|\sum_{i=1}^{N} \alpha_i x_i\Big\|^2 \leq \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \alpha_i\alpha_j f(\|x_i - x_j\|),$$

where $\alpha_i \in (0,1)$, $\sum_{i=1}^{N} \alpha_i = 1$ and $x_i \in B_r(0)$, $\forall i \in \{1, 2, \ldots, N\}$,

Lemma 2.11. *[26] Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

3. Main Results

In this section we propose the following inertial hybrid self adaptive subgradient extragradient algorithm in a real uniformly convex Banach space E which is also uniformly smooth;

$$\begin{cases} \eta_1 > 0, \ \mu \in (0,1), \ x_0, \ x_1 \in C_1 = E, \\ \theta_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \underset{y \in C}{\operatorname{argmin}}\{\eta_n g(\theta_n, y) + \frac{1}{2}\phi(y, \theta_n)\}, \\ \Gamma_n = \{z \in E : \langle J\theta_n - \eta_n w_n - Jy_n, z - y_n\rangle \leq 0\}, \\ z_n = \underset{y \in \Gamma_n}{\operatorname{argmin}}\{\eta_n g(y_n, y) + \frac{1}{2}\phi(y, \theta_n)\}, \\ u_n = J^{-1}(\beta_n J z_n + (1 - \beta_n)[\gamma_{n,0} J\theta_n + \sum_{i=1}^{N} \gamma_{n,i} J t_{n,i}]), \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, \theta_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ n \geq 1 \end{cases} \tag{7}$$

where $w_n \in \partial_2 g(\theta_n, y_n)$, $t_{n,i} \in T_i\theta_n$, T_i, $i = 1, 2, 3, \dots, N$ are quasi$-\phi-$nonexpansive multivalued mappings and

$$\eta_{n+1} = \begin{cases} \min\{\frac{\mu(\phi(y_n,\theta_n)+\phi(z_n,y_n))}{2(g(\theta_n,z_n)-g(\theta_n,y_n)-g(y_n,z_n))}, \eta_n\}, & g(\theta_n, z_n) - g(\theta_n, y_n) - g(y_n, z_n) > 0. \\ \eta_n, & \text{Otherwise} \end{cases}$$

Observe, it is obvious from (7) that $C \subseteq \Gamma_n$. Also using algorithm (7), we have the following Lemmas:

Lemma 3.1. *The sequence $\{\eta_n\}$ is a monotone nonincreasing and has a lower bound* $\min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \eta_1\right\}$,

Proof. It is clear that $\{\eta_n\}$ is a monotone nonincreasing sequence. By condition (D5), we get

$$\frac{\mu(\phi(y_n, \theta_n) + \phi(z_n, y_n))}{2(g(\theta_n, z_n) - g(\theta_n, y_n) - g(y_n, z_n))} \geq \frac{\mu(\phi(y_n, \theta_n) + \phi(z_n, y_n))}{2(c_1\phi(y_n, \theta_n) + c_2\phi(z_n, y_n))} \geq \frac{\mu}{2\max\{c_1, c_2\}}.$$

Hence $\{\eta_n\}$ has a lower bound $\min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \eta_1\right\}$. Consequently the $\lim_{n\to\infty} \eta_n$ exists. □

Lemma 3.2. *Let y_n be defined as in algorithm (7). Then $\forall\ n \geq 1$ and $y \in C$ we have*

$$\eta_n g(\theta_n, y) - \eta_n g(\theta_n, y_n) \geq \langle y - y_n, J\theta_n - Jy_n\rangle.$$

Proof. Let $n \geq 0$ and $y \in C$, then by Lemma 2.7 and Lemma 2.8, we get

$$0 \in \eta_n \partial_2 g(\theta_n, y_n) + \frac{1}{2}\nabla_1\phi(y_n, \theta_n) + N_C(y_n).$$

Therefore there exists $w \in \partial_2 g(\theta_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$0 = \eta_n w + Jy_n - J\theta_n + \bar{w}. \tag{8}$$

Since $w \in \partial_2 g(\theta_n, y_n)$, then

$$g(\theta_n, y) \geq g(\theta_n, y_n) + \langle y - y_n, w\rangle. \tag{9}$$

Using (8) and Definition of $N_C(y_n)$, we get

$$\langle y - y_n, -\eta_n w - Jy_n + J\theta_n\rangle \leq 0,$$

so that

$$\eta_n\langle y - y_n, w\rangle \geq \langle y - y_n, J\theta_n - Jy_n\rangle. \tag{10}$$

Hence by (9) and (10), we obtain

$$\eta_n g(\theta_n, y) - \eta_n g(\theta_n, y_n) \geq \langle y - y_n, J\theta_n - Jy_n\rangle.$$

□

Lemma 3.3. *Let C be a nonempty closed convex subset of real uniformly convex and uniformly smooth Banach space E. Let $T_i : E \to 2^E$, $i = 1, 2, 3, \dots, N$ be finite family of quasi$-\phi-$nonexpansive multivalued mappings. Assume g satisfies (D1)-(D5) and $\mathcal{F} = EP(g, C) \cap (\cap_{i=1}^N F(T_i)) \neq \emptyset$. Let $\{\theta_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined as in algorithm (7), then*

$$\phi(x^*, z_n) \leq \phi(x^*, \theta_n) - (1 - \frac{\eta_n}{\eta_{n+1}}\mu)\phi(y_n, \theta_n) - (1 - \frac{\eta_n}{\eta_{n+1}}\mu)\phi(z_n, y_n).$$

Proof. Let $x^* \in \mathcal{F}$, then from Definition of z_n, Lemma 2.7 and Lemma 2.8, we get

$$0 \in \eta_n \partial_2 g(y_n, z_n) + \frac{1}{2}\nabla_1 \phi(z_n, \theta_n) + N_{\Gamma_n}(z_n).$$

Hence $0 = \eta_n \bar{w_n} + Jz_n - J\theta_n + \bar{w}$ for some $\bar{w_n} \in \partial_2 g(y_n, z_n)$ and $\bar{w} \in N_{\Gamma_n}(z_n)$, i.e.

$$\bar{w} = -Jz_n - \eta_n \bar{w_n} + J\theta_n. \tag{11}$$

From Definition of normal cone $N_{\Gamma_n}(z_n)$, we have

$$\langle y - z_n, \bar{w}\rangle \leq 0 \ \forall \ y \in \Gamma_n. \tag{12}$$

By (11) and (12), we obtain

$$\eta_n \langle y - z_n, \bar{w_n}\rangle \geq \langle y - z_n, J\theta_n - Jz_n\rangle \ \forall \ y \in \Gamma_n.$$

Since $x^* \in \mathcal{F} \subset EP(g, C) \subset C \subset \Gamma_n \subset E$, we have

$$\eta_n \langle x^* - z_n, \bar{w_n}\rangle \geq \langle x^* - z_n, J\theta_n - Jz_n\rangle. \tag{13}$$

On the other hand since $\bar{w_n} \in \partial_2 g(y_n, z_n)$, we have

$$g(y_n, y) - g(y_n, z_n) \geq \langle y - z_n, \bar{w_n}\rangle \ \forall \ y \in E. \tag{14}$$

Therefore, combining (13) and (14), we obtain

$$\eta_n (g(y_n, x^*) - g(y_n, z_n)) \geq \langle x^* - z_n, J\theta_n - Jz_n\rangle. \tag{15}$$

As g is pseudomonotone, we have $g(y_n, x^*) \leq 0$. Thus,

$$\begin{aligned} -2\eta_n g(y_n, z_n)) &\geq 2\langle x^* - z_n, J\theta_n - Jz_n\rangle - 2\eta_n g(y_n, x^*) \\ &\geq 2\langle x^* - z_n, J\theta_n - Jz_n\rangle. \end{aligned} \tag{16}$$

Since $w_n \in \partial_2 g(\theta_n, y_n)$, then

$$g(\theta_n, y) - g(\theta_n, y_n) \geq \langle y - y_n, w_n\rangle \ \forall \ y \in E.$$

Letting $y = z_n$ we obtain

$$2\eta_n (g(\theta_n, z_n) - g(\theta_n, y_n)) \geq 2\eta_n \langle z_n - y_n, w_n\rangle. \tag{17}$$

Observe as $z_n \in \Gamma_n$, we get

$$2\langle z_n - y_n, J\theta_n - Jy_n\rangle \leq 2\eta_n \langle z_n - y_n, w_n\rangle. \tag{18}$$

Combining (16), (17), (18) and (5), we obtain

$$\begin{aligned} &2\eta_n\Big(g(\theta_n, z_n) - g(\theta_n, y_n) - g(y_n, z_n)\Big) \\ &\quad \geq 2\langle z_n - y_n, J\theta_n - Jy_n\rangle + 2\langle x^* - z_n, J\theta_n - Jz_n\rangle \\ &\quad = -2\langle z_n - y_n, Jy_n - J\theta_n\rangle - 2\langle x^* - z_n, Jz_n - J\theta_n\rangle \\ &\quad = -(\phi(z_n, \theta_n) - \phi(z_n, y_n) - \phi(y_n, \theta_n)) \\ &\qquad -(\phi(x^*, \theta_n) - \phi(x^*, z_n) - \phi(z_n, \theta_n)) \\ &\quad = \phi(z_n, y_n) + \phi(y_n, \theta_n) - \phi(x^*, \theta_n) + \phi(x^*, z_n). \end{aligned} \tag{19}$$

Thus, from (19) we have

$$\begin{aligned} \phi(x^*, z_n) &\leq \phi(x^*, \theta_n) - \phi(z_n, y_n) - \phi(y_n, \theta_n) \\ &+ 2\eta_n\Big(g(\theta_n, z_n) - g(\theta_n, y_n) - g(y_n, z_n)\Big). \end{aligned}$$

From the Definition of η_n we obtain

$$\begin{aligned}\phi(x^*,z_n) &\leq \phi(x^*,\theta_n)-\phi(z_n,y_n)-\phi(y_n,\theta_n)\\ &+ \frac{2\eta_n}{\eta_{n+1}}\eta_{n+1}\Big(g(\theta_n,z_n)-g(\theta_n,y_n)-g(y_n,z_n)\Big)\\ &\leq \phi(x^*,\theta_n)-\phi(z_n,y_n)-\phi(y_n,\theta_n)\\ &+ \frac{\eta_n}{\eta_{n+1}}\Big(\mu(\phi(y_n,\theta_n)+\phi(z_n,y_n))\Big)\\ &= \phi(x^*,\theta_n)-\Big(1-\frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(y_n,\theta_n)-\Big(1-\frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(z_n,y_n).\end{aligned}$$

□

Theorem 3.4. *Let C be a nonempty closed convex subset of real uniformly convex and uniformly smooth Banach space E. Let $T_i : E \to 2^E$, $i = 1,2,3,\ldots,N$ be finite family of closed quasi–ϕ–nonexpansive multivalued mappings. Assume g satisfies (D1)-(D5) and $\mathcal{F} = EP(g,C)\cap(\cap_{i=1}^N F(T_i)) \neq \emptyset$. Let $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_{n,i}\}$ be real sequences such that $\alpha_n,\beta_n \in (0,1)$, $\gamma_{n,i}\in(\epsilon,1-\epsilon)$ for some $\epsilon\in(0,1)$ and $\gamma_{n,0}+\sum_{i=1}^N\gamma_{n,i} = 1$. Then the sequence $\{x_n\}$ generated by (7) converges strongly to $p^* = \Pi_{\mathcal{F}}x_0$.*

Proof. The proof is divided in to steps;

Step 1: We show $\mathcal{F} = EP(g,C)\cap(\cap_{i=1}^N F(T_i))$ is closed and convex. By Lemma 2.5, $\cap_{i=1}^N F(T_i)$ is closed and convex and by Lemma 2.6, $EP(g,C)$ is closed and convex, therefore $\mathcal{F} = EP(g,C)\cap(\cap_{i=1}^N F(T_i))$ is closed and convex.

Step 2: Here we show C_n, $\forall n\geq 1$ is closed and convex;
Observe $C_1 = C$ is closed and convex. Assume C_n is closed and convex for some $n>1$, then

$$\phi(z,u_n)\leq\phi(z,\theta_n)$$

is equivalent to

$$2\langle z, J\theta_n - Ju_n\rangle \leq \|\theta_n\|^2-\|u_n\|^2.$$

Thus, we obtain C_{n+1} is closed and convex and therefore C_n is closed and convex $\forall n\geq 1$. This shows that the iterative sequence generated by (7) is well defined.

Step 3: We show $\mathcal{F}\subset C_n$ $\forall n\geq 1$.
It is clear that $\mathcal{F}\subset C = C_1$. Suppose $\mathcal{F}\subset C_n$ for some $n>1$. Then for any $x^*\in\mathcal{F}\subset C_n$, we have

$$\begin{aligned}\phi(x^*,u_n) &= \phi(x^*, J^{-1}(\beta_nJz_n+(1-\beta_n)[\gamma_{n,0}J\theta_n+\sum_{i=1}^N\gamma_{n,i}Jt_{n,i}]))\\ &= \|x^*\|^2-2\langle x^*,\beta_nJz_n+(1-\beta_n)\gamma_{n,0}J\theta_n+(1-\beta_n)\sum_{i=1}^N\gamma_{n,i}Jt_{n,i}\rangle\\ &\quad+\|\beta_nJz_n+(1-\beta_n)[\gamma_{n,0}J\theta_n+\sum_{i=1}^N\gamma_{n,i}Jt_{n,i}]\|^2\\ &\leq \|x^*\|^2-2\beta_n\langle x^*,Jz_n\rangle-2(1-\beta_n)\gamma_{n,0}\langle x^*,J\theta_n\rangle\\ &\quad-2(1-\beta_n)\sum_{i=1}^N\gamma_{n,i}\langle x^*,Jt_{n,i}\rangle+\beta_n\|Jz_n\|^2\\ &\quad+(1-\beta_n)\|\gamma_{n,0}J\theta_n+\sum_{i=1}^N\gamma_{n,i}Jt_{n,i}\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \|x^*\|^2 - 2\beta_n\langle x^*, Jz_n\rangle - 2(1-\beta_n)\gamma_{n,0}\langle x^*, J\theta_n\rangle \\
&\quad -2(1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\langle x^*, Jt_{n,i}\rangle + \beta_n\|Jz_n\|^2 \\
&\quad +(1-\beta_n)\gamma_{n,0}\|J\theta_n\|^2 + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\|Jt_{n,i}\|^2 \\
&= \beta_n\phi(x^*, z_n) + (1-\beta_n)\gamma_{n,0}\phi(x^*, \theta_n) + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\phi(x^*, t_{n,i}).
\end{aligned}
$$

Since $t_{n,i} \in T_i\theta_n$ and $T_i,\ i = 1, 2, 3, \ldots, N$ are quasi$-\phi-$nonexpansive multivalued mappings, we obtain

$$
\begin{aligned}
\phi(x^*, u_n) &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\gamma_{n,0}\phi(x^*, \theta_n) + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\phi(x^*, \theta_n) \\
&= \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, \theta_n).
\end{aligned}
$$

By Lemma 3.3, we get

$$
\begin{aligned}
\phi(x^*, u_n) &\leq \beta_n\Big[\phi(x^*, \theta_n) - \Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(y_n, \theta_n) - \Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(z_n, y_n)\Big] \\
&\quad +(1-\beta_n)\phi(x^*, \theta_n) \\
&= \phi(x^*, \theta_n) - \beta_n\Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(y_n, \theta_n) - \beta_n\Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(z_n, y_n).
\end{aligned}
$$

Since $\lim_{n\to\infty}\frac{\eta_n}{\eta_{n+1}}\mu = \mu$ and $0 < \mu < 1$, then there exists a natural number N_0 such that $0 < \frac{\eta_n}{\eta_{n+1}}\mu < 1\ \forall n \geq N_0$. Thus, $\forall n \geq N_0$, we have

$$
\phi(x^*, u_n) \leq \phi(x^*, \theta_n),
$$

which implies $x^* \in C_{n+1}$, that is $\mathcal{F} \subset C_{n+1}$. Hence $\mathcal{F} \subset C_n\ \forall n \geq 1$.

Step 4: We prove $\{x_n\}$ is Cauchy sequence.
Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n\ \forall n \geq 1$, then

$$
\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(x_{n+1}, x_0),\ \forall n \geq 1 \tag{20}
$$

Also by Lemma 2.9, we obtain

$$
\begin{aligned}
\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) &\leq \phi(x^*, x_0) - \phi(x^*, x_n) \\
&\leq \phi(x^*, x_0),\ \forall n \geq 1.
\end{aligned} \tag{21}
$$

From (20) and (21), it follows that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. This implies $\{\phi(x_n, x_0)\}$ is bounded and from (6) we have that $\{x_n\}$ is bounded. Observe from Lemma 2.9

$$
\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n}x_0) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \tag{22}
$$

Therefore

$$
\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0.
$$

By Lemma 2.11, we get

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{23}
$$

From (22) and any $m, n \in \mathbb{N}$ with $m > n$, we obtain

$$
\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n}x_0) \leq \phi(x_m, x_0) - \phi(x_n, x_0). \tag{24}
$$

Since the $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we get

$$\lim_{m,n\to\infty} \phi(x_m, x_n) = 0.$$

Again by Lemma 2.11 we get

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0. \tag{25}$$

It follows from (25) that the sequence $\{x_n\}$ is Cauchy in $C \subset E$.

Step 5: We prove $\lim_{n\to\infty} \|\theta_n - t_{n,i}\| = 0$, $\lim_{n\to\infty} \|z_n - y_n\| = 0$, $\lim_{n\to\infty} \|y_n - \theta_n\| = 0$.

Observe from the scheme (7),

$$\|\theta_n - x_n\| = \alpha_n \|x_n - x_{n-1}\|.$$

Therefore from (23), we obtain

$$\lim_{n\to\infty} \|\theta_n - x_n\| = 0. \tag{26}$$

Also from (23) and (26), we get

$$\lim_{n\to\infty} \|x_{n+1} - \theta_n\| = 0. \tag{27}$$

Now,

$$\begin{aligned}
\phi(x_{n+1}, \theta_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\theta_n\rangle + \|\theta_n\|^2 \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1} - \theta_n, J\theta_n\rangle - \|\theta_n\|^2 \\
&= (\|x_{n+1}\| - \|\theta_n\|)(\|x_{n+1}\| + \|\theta_n\|) - 2\langle x_{n+1} - \theta_n, J\theta_n\rangle \\
&\leq \|x_{n+1} - \theta_n\|(\|x_{n+1}\| + \|\theta_n\|) + 2|\langle x_{n+1} - \theta_n, J\theta_n\rangle| \\
&\leq \|x_{n+1} - \theta_n\|(\|x_{n+1}\| + \|\theta_n\|) + 2\|x_{n+1} - \theta_n\|\|J\theta_n\|.
\end{aligned}$$

Since $\{x_n\}$, $\{\theta_n\}$ are bounded and the duality mapping J is uniformly norm-norm continuous on bounded subsets of E, it follows from (27) that

$$\lim_{n\to\infty} \phi(x_{n+1}, \theta_n) = 0. \tag{28}$$

From the scheme (7), $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$. Hence

$$\phi(x_{n+1}, u_n) \leq (x_{n+1}, \theta_n).$$

Therefore from (28), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$$

and consequently by Lemma 2.11, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{29}$$

From (27) and (29), we get

$$\lim_{n\to\infty} \|u_n - \theta_n\| = 0. \tag{30}$$

Observe that

$$\begin{aligned}
\phi(x^*, \theta_n) - \phi(x^*, u_n) &= \|\theta_n\|^2 - \|u_n\|^2 - 2\langle x^*, J\theta_n - Ju_n\rangle \\
&\leq \|\theta_n - u_n\|(\|\theta_n + u_n\|) + 2\|x^*\|\|J\theta_n - Ju_n\|.
\end{aligned}$$

From (30) and norm-to-norm uniform continuity of J on bounded sets, we obtain

$$\lim_{n\to\infty} (\phi(x^*, \theta_n) - \phi(x^*, u_n)) = 0. \tag{31}$$

Again, from the scheme (7)

$$\begin{aligned}
\phi(x^*,u_n) &= \phi(x^*, J^{-1}(\beta_n Jz_n + (1-\beta_n)[\gamma_{n,0}J\theta_n + \sum_{i=1}^{N}\gamma_{n,i}Jt_{n,i}])) \\
&= \|x^*\|^2 - 2\langle x^*, \beta_n Jz_n + (1-\beta_n)\gamma_{n,0}J\theta_n + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}Jt_{n,i}\rangle \\
&\quad + \|\beta_n Jz_n + (1-\beta_n)[\gamma_{n,0}J\theta_n + \sum_{i=1}^{N}\gamma_{n,i}Jt_{n,i}]\|^2 \\
&\leq \|x^*\|^2 - 2\beta_n\langle x^*, Jz_n\rangle - 2(1-\beta_n)\gamma_{n,0}\langle x^*, J\theta_n\rangle \\
&\quad -2(1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\langle x^*, Jt_{n,i}\rangle + \beta_n\|Jz_n\|^2 \\
&\quad +(1-\beta_n)\|\gamma_{n,0}J\theta_n + \sum_{i=1}^{N}\gamma_{n,i}Jt_{n,i}\|^2. \qquad (32)
\end{aligned}$$

Since $\{\theta_n\}$ is bounded, $t_{n,i} \in T_i\theta_n,\ i = 1,2,\ldots,N$ and T_i are quasi$-\phi-$nonexpansive multivalued mappings, it follows that $\{t_{n,i}\}$ is bounded for each $i \in \{1,2,\ldots,N\}$. Let $r = \max_{1\leq i\leq N}\sup_{n\geq 1}\{\|\theta_n\|,\ \|t_{n,i}\|\}$. Since E is uniformly smooth, then E^* is unifromly convex, therefore, from (32) and Lemma 2.10, we have

$$\begin{aligned}
\phi(x^*,u_n) &\leq \|x^*\|^2 - 2\beta_n\langle x^*, Jz_n\rangle - 2(1-\beta_n)\gamma_{n,0}\langle x^*, J\theta_n\rangle \\
&\quad -2(1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\langle x^*, Jt_{n,i}\rangle + \beta_n\|Jz_n\|^2 \\
&\quad +(1-\beta_n)\gamma_{n,0}\|J\theta_n\|^2 + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\|Jt_{n,i}\|^2 \\
&\quad -(1-\beta_n)\gamma_{n,0}\gamma_{n,i}f(\|J\theta_n - Jt_{n,i}\|) \\
&= \beta_n\phi(x^*,z_n) + (1-\beta_n)\gamma_{n,0}\phi(x^*,\theta_n) + (1-\beta_n)\sum_{i=1}^{N}\gamma_{n,i}\phi(x^*,t_{n,i}) \\
&\quad -(1-\beta_n)\gamma_{n,0}\gamma_{n,i}f(\|J\theta_n - Jt_{n,i}\|) \\
&\leq \beta_n\phi(x^*,z_n) + (1-\beta_n)\phi(x^*,\theta_n) - (1-\beta_n)\gamma_{n,0}\gamma_{n,i}f(\|J\theta_n - Jt_{n,i}\|).
\end{aligned}$$

By Lemma 3.3, we obtain

$$\begin{aligned}
\phi(x^*,u_n) &\leq \phi(x^*,\theta_n) - \beta_n\Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(y_n,\theta_n) - \beta_n\Big(1 - \frac{\eta_n}{\eta_{n+1}}\mu\Big)\phi(z_n,y_n) \\
&\quad -(1-\beta_n)\gamma_{n,0}\gamma_{n,i}f(\|J\theta_n - Jt_{n,i}\|). \qquad (33)
\end{aligned}$$

From (31), (33) and condition $\gamma_{n,i} \in (\epsilon, 1-\epsilon)$, we obtain

$$\lim_{n\to\infty} f(\|J\theta_n - Jt_{n,i}\|) = 0,\ \forall\ i \in \{1,2,\ldots,N\}.$$

By the property of f, we get

$$\lim_{n\to\infty} \|J\theta_n - Jt_{n,i}\| = 0,\ \forall\ i \in \{1,2,\ldots,N\}.$$

Since J^{-1} uniformly norm-to-norm continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|\theta_n - t_{n,i}\| = 0,\ \forall\ i \in \{1,2,\ldots,N\}. \qquad (34)$$

Also form (33), we have

$$\lim_{n\to\infty}\phi(y_n,\theta_n)=0,\ \lim_{n\to\infty}\phi(z_n,y_n)=0.$$

By Lemma 2.11, we obtain

$$\lim_{n\to\infty}\|y_n-\theta_n\|=0,\ \lim_{n\to\infty}\|z_n-y_n\|=0. \tag{35}$$

Hence step 5 is proved.

Since $\{x_n\}$ is Cauchy and E is reflexive Banach space, there exists $p^*\in E$ such that $x_n\to p^*$ as $n\to\infty$. As C is closed, we have $p^*\in C$.

Step 6: We show $p^*\in\mathcal{F}=EP(g,C)\cap(\cap_{i=1}^N F(T_i))$.

Since $x_n\to p^*$ as $n\to\infty$, then from (26),we have

$$\lim_{n\to\infty}\|\theta_n-p^*\|=0. \tag{36}$$

From (34) and (36), we obtain

$$\lim_{n\to\infty}\|t_{n,i}-p^*\|=0,\ \forall\ i\in\{1,2,\dots,N\}. \tag{37}$$

Since $t_{n,i}\in T_i\theta_n$ for each $i\in\{1,2,\dots,N\}$, then from (36), (37) and closedness of T_i, we have $p^*\in F(T_i)\ \forall\ i\in\{1,2,\dots,N\}$, i.e. $p^*\in\cap_{i=1}^N F(T_i)$.

On the other hand

$$\|y_n-p^*\|\le\|y_n-\theta_n\|+\|\theta_n-p^*\|.$$

Therefore, using (35) and (36) we obtain

$$\lim_{n\to\infty}\|y_n-p^*\|=0. \tag{38}$$

From Lemma 3.2, we have

$$\eta_n g(\theta_n,y)-\eta_n g(\theta_n,y_n)\ge\langle y-y_n,J\theta_n-Jy_n\rangle\ \forall\ y\in C. \tag{39}$$

Since $\lim_{n\to\infty}\eta_n>\min\left\{\frac{\mu}{2\max\{c_1,c_2\}},\eta_0\right\}>0$, then from (39), (38), (36), conditions (D1) and (D4), we obtain

$$g(p^*,y)\ge 0,\ \forall\ y\in C,\ i.e.\ p^*\in EP(g,C).$$

Step 7: Finally, we show $p^*=\Pi_{\mathcal{F}}x_0$.

Let $\bar{y}=\Pi_{\mathcal{F}}x_0$, then since $p^*\in\mathcal{F}$, we have

$$\phi(\bar{y},x_0)\le\phi(p^*,x_0). \tag{40}$$

From the scheme (7), $x_n=\Pi_{C_n}x_0$. Since $\bar{y}\in\mathcal{F}\subset C_n$, we have

$$\phi(x_n,x_0)\le\phi(\bar{y},x_0).$$

Also since $\phi(.,y)$ is continuous and $x_n\to p^*$ as $n\to\infty$, we obtain

$$\phi(p^*,x_0)\le\phi(\bar{y},x_0). \tag{41}$$

From (40) and (41), we have $\phi(p^*,x_0)=\phi(\bar{y},x_0)$. Thus, $p^*=\bar{y}=\Pi_{\mathcal{F}}x_0$.

This compeletes the proof. □

Observe that if E is a real Hilbert space, then by Remark 2.1(1) algorithm (7) reduces to the following

$$\begin{cases} \eta_1 > 0,\ \mu \in (0,1),\ x_0,\ x_1 \in C_1 = H \\ \theta_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \underset{y\in C}{\operatorname{argmin}}\{\eta_n g(\theta_n, y) + \frac{1}{2}\|\theta_n - y\|^2\}, \\ \Gamma_n = \{z \in H : \langle \theta_n - \eta_n w_n - y_n, z - y_n\rangle \le 0\}, \\ z_n = \underset{y\in\Gamma_n}{\operatorname{argmin}}\{\eta_n g(y_n, y) + \frac{1}{2}\|\theta_n - y\|^2\}, \\ u_n = \beta_n z_n + (1-\beta_n)[\gamma_{n,0}\theta_n + \sum_{i=1}^N \gamma_{n,i} t_{n,i}], \\ C_{n+1} = \{z \in C_n : \|u_n - z\|^2 \le \|\theta_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 1 \end{cases} \tag{42}$$

where $w_n \in \partial_2 g(\theta_n, y_n)$, $t_{n,i} \in T_i\theta_n$, $T_i,\ i = 1,2,3,\dots,N$ are quasi nonexpansive multivalued mappings and

$$\eta_{n+1} = \begin{cases} \min\{\frac{\mu(\|y_n-\theta_n\|^2 + \|z_n - y_n\|^2)}{2(g(\theta_n,z_n) - g(\theta_n,y_n) - g(y_n,z_n))}, \eta_n\}, & g(\theta_n, z_n) - g(\theta_n, y_n) - g(y_n, z_n) > 0. \\ \eta_n, & \text{Otherwise} \end{cases}$$

Using (42), Theorem 3.4 reduces to the following Corollary;

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : H \to 2^H,\ i = 1,2,3,\dots,N$ be finite family of closed quasi nonexpansive multivalued mappings. Assume g satisfies (D1)-(D5) and $\mathcal{F} = EP(g,C) \cap (\cap_{i=1}^N F(T_i)) \ne \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_{n,i}\}$ be real sequences such that $\alpha_n, \beta_n \in (0,1)$, $\gamma_{n,i} \in (\epsilon, 1-\epsilon)$ for some $\epsilon \in (0,1)$ and $\gamma_{n,0} + \sum_{i=1}^N \gamma_{n,i} = 1$. Then the sequence $\{x_n\}$ generated by (42) converges strongly to $p^* = P_{\mathcal{F}} x_0$.*

Remark 3.6. *Theorem 3.4 extends the results of Yang and Liu [42] from Hilbert space to real uniformly convex and uniformly smooth Banach spaces and from single valued quasi nonexpansive mappings to finite family of multivalued quas$-\phi-$nonexpansive mappings.*

4. Numerical example

In this section, we demonstrate Theorem 3.4
Let $E = \mathbb{R}$ with $\|.\| = |.|$ and $\langle x, y\rangle = xy$. Let $C = [-40, 40]$ and for $i = 1,2,3,4$, let $T_i : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $T_i x = [\frac{x}{i+3}, \frac{x}{i}]$. It is clear that $0 \in \cap_{i=1}^4 F(T_i)$. Let $p \in T_i x$, then $p = ax$ for some a, $\frac{1}{i+3} \le a \le \frac{x}{i}$ and

$$\begin{aligned} \phi(0,p) &= |0|^2 - 2\langle 0, p\rangle + |p|^2 \\ &= |p|^2 = |ax|^2 = a^2|x|^2 \\ &\le \frac{1}{i^2}|x|^2 \\ &\le |x|^2 \\ &= |0|^2 - 2\langle 0, x\rangle + |x|^2 \\ &= \phi(0,x). \end{aligned}$$

Thus, T_i is quasi$-\phi-$nonexpansive multivalued mapping for each $i \in \{1,2,3,4\}$.

Define $g(x,y) = y^2 + 6xy - 7x^2$. It is easy to see $0 \in EP(g,C)$. Also g satisfies (D1), (D2) with $\partial_2 g(x,y) = 2y + 6x$, (D3) and (D4). If $\phi(x,y) = (x-y)^2$, then

$$\begin{aligned} g(x,y) + g(y,z) &= z^2 + 6xy + 6yz - 7x^2 - 6y^2 \\ &= z^2 + 6xy - 7x^2 + 6yz - 6y^2 \\ &= g(x,z) - 3(y-x)^2 - 3(z-y)^2 + 3(z-x)^2 \\ &= g(x,z) - 3\phi(y,x) - 3\phi(z,y) + 3\phi(z,x) \\ &\ge g(x,z) - 3\phi(y,x) - 3\phi(z,y). \end{aligned}$$

Thus, g satisfies (D5) with $c_1 = c_2 = 3$. Furthermore if $\eta_n = \frac{2}{5}$, $\mu = \frac{7}{10}$, $\alpha_n = \frac{3}{10}$, $\beta_n = \frac{n}{3n+2}$, $\epsilon = \frac{1}{10}$, $\gamma_{n,0} = \gamma_{n,1} = \gamma_{n,2} = \gamma_{n,3} = \gamma_{n,4} = \frac{1}{5}$, then ϵ, μ, η_n, α_n, β_n and $\gamma_{n,i}$ satisfy all the conditions of Theorem 3.4. Therefore scheme 7 takes the following form;

$$\begin{cases} \theta_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \frac{1-6\eta_n}{2\eta_n+1}\theta_n, \\ \Gamma_n = \{z \in \mathbb{R} : \langle J\theta_n - \eta_n w_n - Jy_n, z - y_n\rangle \leq 0\}, \\ z_n = \frac{\theta_n - 6\eta_n y_n}{2\eta_n+1}, \\ \frac{\theta_n}{i+3} \leq t_{n,i} \leq \frac{\theta_n}{i},\ i = 1,2,3,4 \\ u_n = \beta_n z_n + (1-\beta_n)[\frac{1}{5}\theta_n + \frac{1}{5}\sum_{i=1}^{4} t_{n,i}]), \\ C_{n+1} = \{z \in C_n : z \leq \frac{\theta_n + u_n}{2}\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 = \frac{\theta_n + u_n}{2} \end{cases} \tag{43}$$

Using (43) the numerical results using MATLAB is given in Figure 1 and Figure 2.

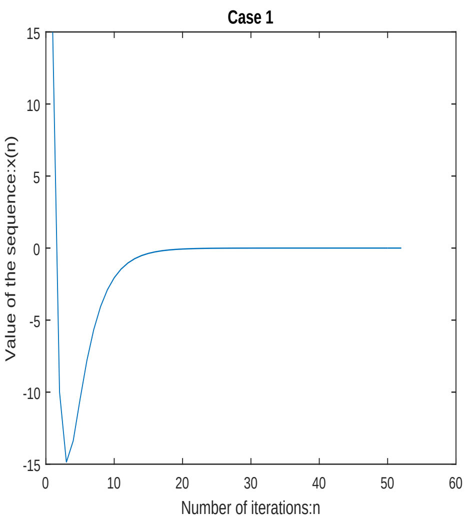

Figure 1: Convergence process of $\{(x_n\}$ with initial points $x_0 = 15,\ x_1 = -10$

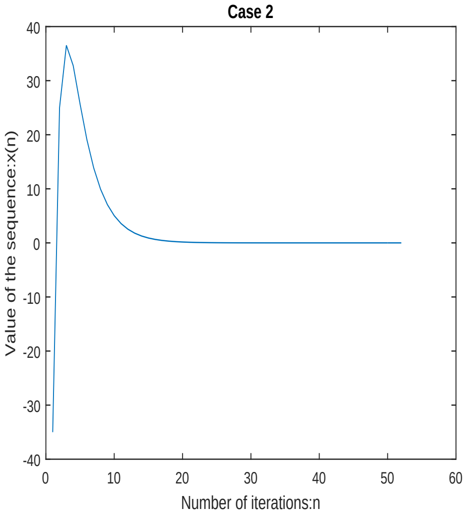

Figure 2: Convergence process of $\{(x_n\}$ with initial points $x_0 = -35,\ x_1 = 25$

5. Conclusion

We studied an inertial hybrid self-adaptive subgradient extragradient algorithm in a real uniformly convex Banach space which is also uniformly smooth. Strong convergence Theorem was proved to approximate solutions of pseudomonotone equilibrium problems and fixed points of quasi$-\phi-$nonexpansive multivalued mappings. Numerical example was presented to show that our iteratative scheme is implementable.

References

[1] P.N. Anh, A hybrid extragradient method extented to fixed point problems and equilibrium problems, Opt., (2011) DOI:10.1080/02331934.2011.607497.
[2] Y.I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, In: Theory and applications of nonlinear operators of accretive and monotone type. Lecture notes, Pure Appl. Math., (1996), pp. 15-50.
[3] B. Ali and M.H. Harbau, Convergence theorems for Bregman K-mappings and mixed equilibrium problems in reflexive Banach spaces, J. Function spaces, (2016) Doi:10.1155/2016/5161682.
[4] B. Ali, J.N. Ezeora and M.S. lawan, Inertial algorithm for solving generalized mixed equilibrium problems in Banah spaces, PanAmerican Journal, 29 (2019), 64-83.
[5] E. Blum and W. Oettli, From Optimization and Variational inequalities to Equilibrium Problems, Mathematics Students, 63 (1994), 123-145.
[6] A. Brondsted and R.T. Rockafellar, On the subdifferentiability of convex functions, Proc. Amer. Math. Soc., 16 (1965), 605-611.
[7] Y. Censor, A. Gibali and S. Riech, The subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 148 (2011), 318-335.
[8] C.E. Chidume, S.I. Ikechukwu and A. Adamu, Inertial algorithm for approximating a common fixed point for finite family of relatively nonexpansive maps, Fixed Point Theory Appl. 9(2018), Doi:10.1186/s13663-018-0634-3.
[9] C.E. Chidume and M.O. Nnakwe, Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and convex feasibility problems, Fixed Point Theory Appl. 16 (2018), Doi:10.1186/s13663-018-0641-4.
[10] I. Cioranescu, Geometry of Banach spaces, duality mappings and nonlinear problems, Kluwer, Dordrecht, (1990) Doi:10.1007/978-94-009-212-4.
[11] W. Cholamjiak, P. Cholamjiak and S. Suantai, An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. Journal of Fixed Point Theory Appl., 20(1), (2018), 1-17.
[12] W. Cholamjiak, S.A. Khan, D. Yambangwai and K.R. Kazmi, Strong convergence analysis of common variational inclusion problems involving an inertial parallel monotone hybrid method for a novel application to image restoration. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 114(2), (2020), 1-20.
[13] W. Cholamjiak, N. Pholasa and S. Suantai, A modified inertial shrinking projection method for solving inclusion problems and quasi-nonexpansive multivalued mappings. Computational and Applied Mathematics, 37(5), (2018), 5750-5774.
[14] V. Dadashi, O.S. Iyiola and Y. Shehu, The subgradient extragradient method for pseudomonotone equilibrium problems, Optim., (2019), Doi:10.1080/02331934.2019.1625899.

[15] P. Daniele, F. Giannessi and A.Maugeri, Equilibrium problems and Variational models, Kluwer Academic, Dordrecht, The Netherlands, 2003.

[16] Q.L. Dong, H.B. Yuan, Y.J. Cho and T.M. Rassias, Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings, Opt. Letters, (2016), Doi:10.1007/s11590-016-1102-9.

[17] G.Z. Eskandani, M. Raeisi and T.M. Rassias, A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance, J. Fixed Point Theory Appl., (2018), Doi:10.1007/s11784-018-0611-9

[18] C. Garodia, A new fixed point algorithm for finding the solution of a delay differential equation, AIMS Mathematics, 5 (4), (2020), 3182-3200.

[19] C. Garodia and I. Uddin, A new iterative method for solving split feasibility problem, J. Applied Anal. Compt., 10 (3), (2020), 986-1004.

[20] K. Goebel and W.A. Kirk, Topics in metric fixed point theory, Cambridge Studies in Ad-vanced Mathematics, 28 Cambridge University Press, Cambridge, 1990.

[21] D.V. Hieu, Halpern subgradient extragradient method extended to equilibrium problems, RACSAM, (2016) Doi:10,1007/s13398-016-0328-9.

[22] D.V. Hieu, Convergence analysis of a new algorithm for strongly pseudomonotone equilibrium problems, Numer. Algorithm, 77(4), (2018), 983-1001.

[23] D.V. Hieu and J. J. Strodoit, Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces, J. Fixed Point Theory Appl. (2018) Doi:10.1007/s11784-018-0608-4

[24] J.B. Hiriart-Urruty, Subdifferential calculus in Convex analysis and Optimization, Pitman, London, (1982), 43-92.

[25] Z. Jouymandi and F. Moradlou, Extragradient and linesearch algorithms for solving equilibrium problems and fixed point problems in Banach spaces, arXiv:1606.01615[Math].

[26] S. Kamimura and W. Takahashi, Strong convergence of proximal-type algorithm in a Banach spaces, SIAM J. Optim., 13 (2002), 938-945.

[27] S.A. Khan, S. Suantai and W. Cholamjiak, Shrinking projection methods involving inertial forward-backward splitting methods for inclusion problems. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemáticas, 113(2), (2019), 645-656.

[28] I. V. Konnov, Equilibrium models and variationa inequalities, Elsevier, Amsterdam, (2007).

[29] A. Phon-on, N. Makaje, A. Sama-Ae, and K. Khongraphan, An inertial S-iteration process, Fixed Point Theory Appl., (2019), Doi;10.1186/s13663-019-0654-7.

[30] B.T. Polyak, Some method of speeding up the convergence of the iteration methods, USSR Comput. Math. Phys., 4 (1964) 1-17.

[31] H. Rehman, P. Kumam, W. Kumam, M. Shutaywi and W. Jirakitpuwan, The inertial subgradient extragradient method for a class of pseudo-monotone equilibrium problems, Symmetry, (2020), Doi;10.3390/sym12030463.

[32] P. Santos and S. Scheimberg, An inexact subgradient algorithms for equilibrium problems, Comput. Math. Appl., 30 (2011), 91-107.

[33] Y. Shehu, Iterative approximations for zeros of sum of accretive operators in Banach spaces, J. Function spaces, (2016), Article ID 5973468.

[34] J.J. Strodiot, V.H. Nguyen and P.T. Vuong, Stong convergence of two hybrid extragradient methods for solving equilibrium problems and fixed point problems, Vietnam J. Math., 40(2), (2013), 371-389.

[35] S. Takahashi and W. Takahashi, Viscosity Approximation methods for Equilibrium problems and fixed point problems in Hilbert Spaces, J. Math. Anal. and Appl., 331(2007), 506-515.

[36] W. Takahashi and K. Zembayash, Strong convergence theorem by new hybrid method for equilibrium problems and reletively nonexpansive mappings, Fixed point Theory and Appl., (2008),Doi. 1155/2008/528476.

[37] B. Tan, S. Xu and S. Li, Modified inertial hybrid and shrinking projection algorithm for solving fixed point problems, Mathematics, (2020), 8, 236, Doi:10.3390/math8020236.

[38] J.V. Tiel, Convex analysis: An introduction, Wiley, New York, (1984).

[39] Q.D. Tran, L. D. Muu and H. V. Nguyen, Extragradient algorithms extended to equilibrium problems, Journal of Optim. **57** (2008) 749-776.

[40] N.T. Vinh, and L. D. Muu, Inertial extragradient algorithm for solving equilibrium problem, Acta Math. Vietnamica, (2019), Doi:10.1007/s40306-019-00338-1.

[41] D.J. Wen, Weak and Strong convergence of hybrid subgradient method for pseudomonotone equilibrium problem and multivalued nonexpansive mappings, Fixed point theory and appli, 2014:232.

[42] J. Yang and H. Liu, The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space, Optim. Letters, (2019), Doi:10.1007/s11590-019-01474-1.

[43] S. Zhang, L. Wang and Y.Zhao, Multi-valued quasi-ϕ-asymptotically nonexpansive semi-groups and strong convergence theorems in Banach spaces, Acta Math. Sci., 33(B), (2013), 589-599.

[44] H. Zegeye, E.U. Ofoedu and N. Shahzad, Convergence theorems for equilibrium problems, variational inequality problem and countably infinite relatively quasi-nonexpansive mappings, Appl. Math. Comput., 216 (2010), 3439-3449.