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Characterization of the Evolute Offset of Ruled Surfaces with B-Darboux Frame

Gul Ugur Kaymanli[1]

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Abstract — In this paper, we first introduce the evolute offset of non-cylindirical ruled surfaces with B-Darboux frame in Euclidean 3-space. Then some geometric properties of this evolute offset of non-cylindirical ruled surface are studied. That is, we examine the striction curve, distribution parameter, orthogonal trajectory of evolute offset of ruled surfaces in terms of B-Darboux frame of given ruled surfaces, and we study the developability of ruled surface generated by B-Darboux vectors in three dimensional Euclidean space.

Keywords — *B-Darboux Frame, Curvatures, Evolute Offset, Ruled Surface*

1. Introduction

In geometry, a ruled surface in three dimensional space is a surface which is stated as the set of points swept by a moving straight line. By using the directions, many offset of this surface such as parallel, Bertrand, Mannheim, involute-evolute, and Smarandache have been defined. In kinematics, mechanism, geophysics, Computer-Aided Geometric Design, and geometric modelling, both offsets and ruled surfaces are extensively worked on. In general, offsets of surfaces are usually more complicated than their origin surfaces. Because of this, analysing offsets surfaces or curves by the help of the properties of the base surface or curve is important. Therefore, many researchers have been working on this subject.

After [1] Pottmann et al. in 1996 studied rational ruled surfaces and their offsets, Kasap et al. in [2], Akyigit et. al. in [3] had a research on the involute-evolute offsets of ruled surface in 2009 and involute-evolute curves in Galilean Space in 2010, respectively. The involute evolute partner of both d-curves in Euclidean 3-space and pseudo null curves in Minkowski 3-space found in [4]- [5]. While Yoon worked on, in 2016, the evolute offsets of ruled surfaces in three dimensional Lorentzian space [6], recently, Senturk and Yuce in [7]- [8] studied evolute offsets of ruled surfaces using Darboux frame in 3 dimensional Euclidean space.

In this paper, after giving necessary definitions and theorems in preliminary section, the evolute offset of ruled surfaces with B-Darboux frame is defined in the following chapter. Some geometric properties of this evolute offset of non-cylindirical ruled surface are studied. That is, we examine the striction curve, distribution parameter and the developability of evolute offset of given ruled surfaces in terms of B-Darboux frame in Euclidean 3- space.

[1]gulugurk@karatekin.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Science, Cankiri Karatekin University, Cankiri, Turkey

2. Preliminary

Let $M(u(s),v(s))$ be an oriented surface and $\alpha(s)$ be a unit speed curve on M in E^3. If $\mathbf{t}$ is the unit tangent vector of α, $\mathbf{U}$ is the unit normal vector of M and $\mathbf{V}=\mathbf{U}\wedge\mathbf{t}$, then $\{\mathbf{t},\mathbf{V},\mathbf{U}\}$ is called the Darboux frame of $\alpha(s)$. Therefore, the Darboux formulas are written as

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{V}\\ \mathbf{U}\end{bmatrix}=\begin{bmatrix}0 & \kappa_g & \kappa_n\\ -\kappa_g & 0 & \tau_g\\ -\kappa_n & -\tau_g & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{V}\\ \mathbf{U}\end{bmatrix} \tag{1}$$

where

$$\kappa_g=\langle \mathbf{t}',\mathbf{V}\rangle,\kappa_n=\langle \mathbf{t}',\mathbf{U}\rangle,\tau_g=\langle \mathbf{V}',\mathbf{U}\rangle \tag{2}$$

are called the normal curvature, geodesic curvature and the geodesic torsion of α, respectively [9]. As an alternative to the Darboux frame, B-Darboux frame is defined as a new adapted frame on the surface [10], [11]. Its mathematical properties derive from the observation that, while the tangent vector $\mathbf{t}$ for a curve on a surface is unique, we can pick any practical arbitrary basis vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ for the remainder of the proposed frame in the normal plane of the surface.

Theorem 2.1. [12] Assume that $r(s)=M(u(s),v(s))$ is a unit speed curve on a surface M in E^3. Let us consider the Darboux frame $\{\mathbf{t},\mathbf{V},\mathbf{U}\}$ along this curve on the surface. Then, the variation equation of the B-Darboux frame $\{\mathbf{t},\mathbf{B}_1,\mathbf{B}_2\}$ on the surface given as

$$\frac{d}{dt}\begin{bmatrix}\mathbf{t}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}=\begin{bmatrix}0 & n_1 & n_2\\ -n_1 & 0 & 0\\ -n_2 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}, \tag{3}$$

where the B-Darboux curvatures are obtained as

$$\begin{aligned} n_1 &= \kappa_g\sin\phi+\kappa_n\cos\phi\\ n_2 &= \kappa_n\sin\phi-\kappa_g\cos\phi, \end{aligned} \tag{4}$$

where the angle ϕ between the vectors $\mathbf{U}$ and $\mathbf{B}_1$ are obtained by

$$\phi-\phi_o=\int\tau_g dt \tag{5}$$

here ϕ_o is an arbitrary integration constant.

A straightforward computation shows that the following relations among the B-Darboux curvatures, the normal curvature and the geodesic curvature holds:

$$\kappa_g^2+\kappa_n^2=n_1^2+n_2^2. \tag{6}$$

If $\alpha(s)$ be a curve and $X(s)$ be a generator vector, then the parametrization of ruled surface $\varphi(s,v)$ is

$$\varphi(s,v)=\alpha(s)+vX(s). \tag{7}$$

The striction curve on the ruled surface consists of the foot of the common perpendicular line of the consecutive rulings on the main ruling. It is written as

$$c(s)=\alpha(s)-\frac{\langle\alpha_s,X_s\rangle}{\langle X_s,X_s\rangle}X(s). \tag{8}$$

Theorem 2.2. The ruled surface is developable if consecutive rulings intersect [13].

The distribution parameter of the ruled surface is determined as (see [14], [15])

$$P_X=\frac{\det(\alpha_s,X,X_s)}{\langle X_s,X_s\rangle}. \tag{9}$$

Theorem 2.3. The ruled surface is named as developable if and only if $P_X=0$ [13].

The ruled surface is called as a non-cylindirical ruled surface if $\langle X_s, X_s \rangle \neq 0$.

A unit direction vector of straight line X is span by the vectors $\{\mathbf{t}, \mathbf{B_1}\}$. Therefore, it is given as

$$X = \cos\phi \mathbf{t} + \sin\phi \mathbf{B_1}. \tag{10}$$

where ϕ is the angle between the vectors $\mathbf{t}$ and X [15]. In [12], the distrubition parameter and the striction curve of ruled surface with B-Darboux frame are determined as

$$P_X = \frac{n_2 \sin\phi \cos\phi}{(\phi' + n_1)^2 + (n_2)^2 (\cos\phi)^2}, \tag{11}$$

and

$$c(s) = \alpha(s) - \frac{(\phi' + n_1)\sin\phi}{(\phi' + n_1)^2 + (n_2)^2 (\cos\phi)^2} X(s). \tag{12}$$

respectively.

The ruled surface with B-Darboux frame is a developable provided that

$$n_2 \cos\phi \sin\phi = 0. \tag{13}$$

In that case, we have the followings:

i) If $n_2 = 0$ then $\frac{\kappa_g}{\kappa_n} = \tan\phi$ which is trivial. Specially, if $\phi = 0$, then α is geodesic curve and never asymptotic line.

ii) If $\cos\phi = 0$ then base curve is orthogonal trajectory.

iii) If $\sin\phi = 0$ then main ruled surface is the tangent developable.

An orthogonal trajectory of a family of curves is a curve which intersect each curve of the family orthogonally. For the ruled surface $\varphi(s, v)$, the orthogonal trajectory is

$$\cos\phi ds = -dv. \tag{14}$$

3. Evolute Offsets of Ruled Surface with B-Darboux Frame

Definition 3.1. Two ruled surfaces $\varphi(s, v)$ with B-Darboux frame $\{\mathbf{t}, \mathbf{B}_1, \mathbf{B}_2\}$ and $\varphi^*(s, v)$ with B-Darboux frame $\{\mathbf{t}^*, \mathbf{B}_1^*, \mathbf{B}_2^*\}$ are given

$$\varphi(s, v) = \alpha(s) + vX(s) \tag{15}$$

$$\varphi^*(s, v) = \alpha^*(s) + vX^*(s). \tag{16}$$

$\varphi(s, v)$ is said to be involute offsets of $\varphi^*(s, v)$ or $\varphi^*(s, v)$ is said to be evolute offset of $\varphi(s, v)$ if frame vectors $\mathbf{t}$ of $\varphi(s, v)$ and $\mathbf{B}_1^*$ of $\varphi^*(s, v)$ are linearly dependent at the points of their corresponding rullings.

A unit direction vector of straight line X^* of φ^* is spanned by the vectors $\{\mathbf{t}^*, \mathbf{B}_1^*\}$. Therefore, it is given

$$X^* = \cos\phi^* \mathbf{t}^* + \sin\phi^* \mathbf{B}_1^*, \tag{17}$$

where ϕ^* is the angle between the vectors $\mathbf{t}^*$ and X^*.

If $\mathbf{t}, \mathbf{B}_1$ and $\mathbf{B}_2$ are the B-Darboux vectors of φ, then the B-Darboux vectors of evolute offset φ^* of φ, as in Figure 1, are written as

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{B}_1^* \\ \mathbf{B}_2^* \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & -\sin\psi \\ 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tag{18}$$

where the ψ is the angle between $\mathbf{B}_2$ and $\mathbf{B}_2^*$.

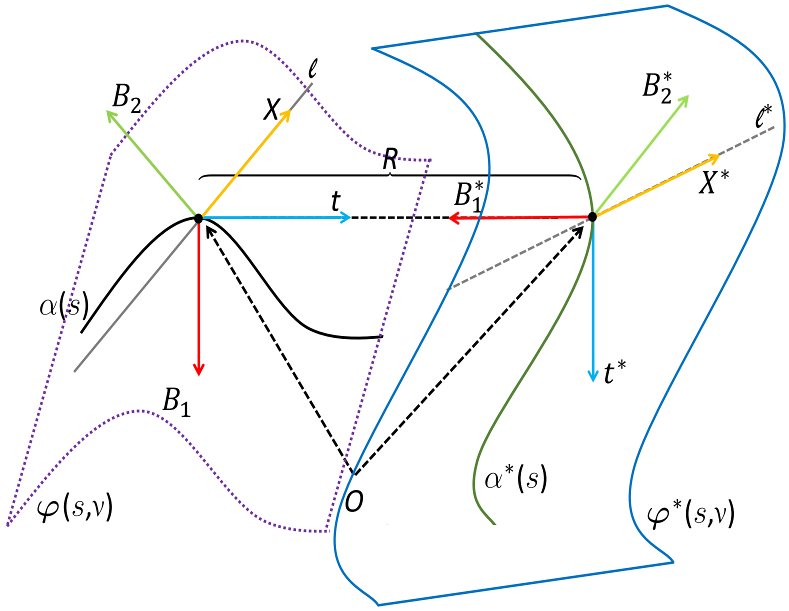

Fig. 1. Relation between φ and φ^*.

It is easy to see that

$$\alpha^*(s) = \alpha(s) + R\mathbf{t} \tag{19}$$

where R is the distance function between the corresponding points of the base curves $\alpha(s)$ and $\alpha^*(s)$, and it is given as $R(s) = c - s$ [4].

$$X^*(s) = \sin\phi^*\mathbf{t} + \cos\phi^*\cos\psi\mathbf{B}_1 - \cos\phi^*\sin\psi\mathbf{B}_2 \tag{20}$$

in terms of B-Darboux frame of φ. The parametrization of the offset ruled surface φ^*, using the equations (19) and (20), can be stated

$$\begin{aligned} \varphi^*(s,v) &= \alpha^*(s) + vX^*(s) \\ &= \alpha(s) + R\mathbf{t} + v[\sin\phi^*\mathbf{t} + \cos\phi^*\cos\psi\mathbf{B}_1 - \cos\phi^*\sin\psi\mathbf{B}_2]. \end{aligned} \tag{21}$$

Taking derivative of the equation (20) with respect to s, we get

$$\begin{aligned} X_s^* &= \cos\phi^*((\phi^*)' - n_1\cos\psi + n_2\sin\psi)\mathbf{t} \\ &\quad +[\sin\phi^*(n_1 - (\phi^*)'\cos\psi) - \psi'\cos\phi^*\sin\psi]\mathbf{B_1} \\ &\quad +[\sin\phi^*(n_2 + (\phi^*)'\sin\psi) - \psi'\cos\phi^*\cos\psi]\mathbf{B_2}. \end{aligned} \tag{22}$$

Striction curve of the ruled surface φ^* in terms of B-Draboux frame of φ is calculated by

$$\begin{aligned} c^*(s) = \alpha(s) + (c-s)\mathbf{t} - \tfrac{c-s}{<X_s^*,X_s^*>} \quad & (\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) \\ & -\psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi))X^*(s). \end{aligned} \tag{23}$$

Distribution parameter of the evolute offset φ^* in terms of B-Draboux frame of ruled surface φ is given as

$$\begin{aligned} P_{X^*} &= \tfrac{c-s}{||X_s^*||^2}(\psi'\cos\phi^*\sin\phi^*(n_1\cos\psi - n_2\sin\psi) - (\phi^*)'(n_1\sin\psi + n_2\cos\psi) \\ &\quad - \cos^2(\phi^*)(n_2\sin\psi - n_1\cos\psi)(n_1\sin\psi + n_2\cos\psi)). \end{aligned} \tag{24}$$

Corollary 3.2. If the tangent vector $\mathbf{t}^*$ and X^* are linearly dependent, then φ^* is developable.

For the distribution parameters of ruled surfaces $\varphi_{\mathbf{t}}$, $\varphi_{\mathbf{B}_1}$, and $\varphi_{\mathbf{B}_2}$, respectively, one can get

$$\begin{aligned} P_{\mathbf{t}} &= \tfrac{\det(\alpha_s,\mathbf{t},\mathbf{t}_s)}{\langle \mathbf{t}_s,\mathbf{t}_s\rangle} &= \tfrac{\det(\mathbf{t},\mathbf{t},\mathbf{t}_s)}{\langle \mathbf{t}_s,\mathbf{t}_s\rangle} &= 0, \\ P_{\mathbf{B}_1} &= \tfrac{\det(\alpha_s,\mathbf{B}_1,\mathbf{B}_{1s})}{\langle \mathbf{B}_{1s},\mathbf{B}_{1s}\rangle} &= \tfrac{\det(\mathbf{t},\mathbf{B}_1,-n_1\mathbf{t})}{\langle -n_1\mathbf{t},-n_1\mathbf{t}\rangle} &= 0, \\ P_{\mathbf{B}_2} &= \tfrac{\det(\alpha_s,\mathbf{B}_2,\mathbf{B}_{2s})}{\langle \mathbf{B}_{2s},\mathbf{B}_{2s}\rangle} &= \tfrac{\det(\mathbf{t},\mathbf{B}_2,-n_2\mathbf{t})}{\langle -n_2\mathbf{t},-n_2\mathbf{t}\rangle} &= 0. \end{aligned} \tag{25}$$

Similarly, the distrubition parameters of evolute offsets of the ruled surface spanned by B-Darboux frame vectors are calculated

$$\begin{aligned} P_{\mathbf{t}^*} &= \tfrac{c-s}{\|\mathbf{t}^*\|^2}(n_1\cos\psi - n_2\sin\psi)(n_1\sin\psi + n_2\cos\psi), \\ P_{\mathbf{B}_1^*} &= 0, \\ P_{\mathbf{B}_2^*} &= \tfrac{c-s}{\|\mathbf{B}_2^*\|^2}(n_2\sin\psi - n_1\cos\psi)(n_1\sin\psi + n_2\cos\psi). \end{aligned} \tag{26}$$

Corollary 3.3. The ruled surface spanning $\mathbf{t}^*$ is a developable only if $\frac{n_1}{n_2} = \pm\cot\psi$ satisfies.

Corollary 3.4. The ruled surface spanning $\mathbf{B}_1^*$ is always developable.

Corollary 3.5. The ruled surface spanning $\mathbf{B}_2^*$ is a developable, either $\frac{n_1}{n_2} = \tan\psi$ or $\frac{n_1}{n_2} = -\cot\psi$ satisfies.

When we take v is constant, we obtain the curve $\beta^*(s) = \alpha^*(s) + vX^*(s)$ on the evolute offsets of ruled surface whose tangent vector field is calculated

$$\begin{aligned} T^* = {} & v\cos\phi^*((\phi^*)' - n_1\cos\psi + n_2\sin\psi)\mathbf{t} \\ & +[Rn_1 + v(\sin\phi^*(n_1 - (\phi^*)'\cos\psi) - \psi'\cos\phi^*\sin\psi)]\mathbf{B_1} \\ & +[Rn_2 + v(\sin\phi^*(n_2 + (\phi^*)'\sin\psi) - \psi'\cos\phi^*\cos\psi)]\mathbf{B_2}. \end{aligned} \tag{27}$$

So, it is easy to get $< T^*, X^* >= R\cos\phi^*(n_1\cos\psi - n_2\sin\psi)$.
The orthogonal trajectory of the evolute offsets φ^* is written as

$$R\cos\phi^*(n_1\cos\psi - n_2\sin\psi)ds = -dv. \tag{28}$$

Theorem 3.6. The shortest distance between the rullings of the evolute offset $\varphi^*(s,v) = \alpha^*(s) + vX^*(s)$ along the orthogonal trajectories is given

$$v = \frac{R(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) - \psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi))}{< X_s^*, X_s^* >}.$$

Proof. Suppose $\alpha^*(s_1)$ and $\alpha^*(s_2)$ are two points in succesive rullings on evolute offset along the orthogonal trajectories. The distance between these two points is given

$$l(v) = \int_{s_1}^{s_2} \|T^*\| ds.$$

Calculating $\|T^*\|$, we get

$$\begin{aligned} l(v) = \int_{s_1}^{s_2} & (R^2(n_1^2 + n_2^2) + 2Rv(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) \\ & -\psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi)) - v^2 < X_s^*, X_s^* >)^{\frac{1}{2}} ds. \end{aligned} \tag{29}$$

In order to find the distance, we need to minimize the integrant. Therefore, using $l'(v) = 0$, we have

$$R(\sin\phi^*(n_1^2 + n_2^2 + (\phi^*)'(n_2\sin\psi - n_1\cos\psi)) - \psi'\cos\phi^*(n_1\sin\psi + n_2\cos\psi)) - v < X_s^*, X_s^* >= 0$$

which proves the theorem. □

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References

[1] H. Pottmann, W. Lü, B. Ravani, *Rational Ruled Surfaces and Their Offsets*, Graphical Models and Image Processing 58(6) (1996) 544-552.

[2] E. Kasap, S. Yuce, N. Kuroglu, *The Involute-Evolute Offsets of Ruled Surface*, Iranian Journal of Science and Technology, Transaction A 33(2) (2009) 195-201.

[3] M. Akyigit, A.Z. Azak, S. Ersoy, *Involute-Evolute Curves in Galilean Space G3*, Scientia Magna (6) (2010) 75-80.

[4] O. Bektas, S. Yuce, *Special Involute-evolute Partner D-curves in* E^3, European Journal of Pure and Applied Mathematics 6(1) (2013) 20-29.

[5] U. Ozturk, E.B. Ozturk, K. Ilarslan, *On the Involute-Evolute of The Pseudo null Curve in Minkowski 3-Space*, Hindawi Publishing Corporation Journal of App. Math. (2013) (2013) 6pp.

[6] D.W. Yoon, *On The Evolute Offsets of Ruled Surfaces in Minkowski 3-space*, Turkish Journal of Mathematics 40 (2016) 594-604.

[7] G.Y. Senturk, S. Yuce, *On The Evolute Offsets of Ruled Surface Using The Darboux Frame*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2) (2019) 1256-1264.

[8] G.Y. Senturk, S. Yuce, *Properties of Integral Invariants of The Involute-evolute Offsets of Ruled Surfaces*, International Journal of Pure and Applied Mathematics 102(4) (2015) 757-768.

[9] W. Kuhnel, *Differential Geometry, Curves-surfaces-manifolds*, American Mathematical Society 2002.

[10] M. Dede, C. Ekici, A. Gorgulu, *Directional q-frame Along a Space Curve*, IJARCSSE, 5(12) (2015) 775-780.

[11] H Tozak, M Dede, C Ekici, *Translation Surfaces According to a New Frame*, Caspian Journal of Mathematical Sciences, 9(1) (2020) 56-67.

[12] G.U. Kaymanli, C. Ekici, M. Dede, *Bertrand Offsets of Ruled Surfaces with B-Darboux Frame*, submitted.

[13] B. Ravani, T.S. Ku, *Bertrand Offsets of Ruled and Developable Surfaces*, Computer Aided Geometric Design 23(2) (1991) 145.

[14] A. Gray, S. Salamon, E. Abbena, *Modern Differentila Geometry of Curves and Surfaces with Mathematica*, Chapman and Hall/CRC, 2006.

[15] G.Y. Senturk, S. Yuce, *Characteristic Properties of The Ruled Surface with Darboux Frame in E3*, Kuwait J. Sci. 42(2) (2015) 14-33.