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Research Article

# On quantile-based dynamic survival extropy and its applications

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## Abstract

The cumulative residual extropy is an uncertainty measure that parallels extropy in an absolutely continuous cumulative distribution function. The dynamic version of this measure is known as dynamic survival extropy. In this paper, we study some properties of the dynamic survival extropy using quantile function approach. Unlike the dynamic survival extropy, the quantile-based dynamic survival extropy determines the quantile density function uniquely through a simple relationship. We also extend the definition of quantile-based dynamic survival extropy into order statistics. Finally, an application of new quantile-based uncertainty measure as a risk measure is derived.

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**Keywords.** Characterization, dynamic survival extropy, quantile function, reliability measures, order statistics, risk measure, stochastic orders, uncertainty

## 1. Introduction

Shannon [27] by developing information theory, introduced a criterion for measurement of uncertainty which is called "entropy". Shannon's entropy has found a special place in the sciences, including economics, physics, computer, telecommunications, communication theory, reliability and so on. If $X$ is a non-negative continuous random variable (rv) with probability density function (pdf) $f_X(x)$, then Shannon's entropy of $X$ is given by $H(X) = \int_0^{+\infty} f_X(x) \log f_X(x) dx$. Lad et al. [12] provided a completion to theories of information based on entropy. They showed that Shannon's entropy function has a complementary dual function which is called "extropy". The extropy of discrete rv $X$ is given by $J(X) = -\sum_{i=1}^{N}(1 - p_i)\log(1 - p_i)$, where $p_i = P(X = x_i)$. When the range of possibilities for discrete rv $X$ increases, the extropy measure $J(X)$ can be closely approximated by $1 - \frac{1}{2}\sum_{i=1}^{N} p_i^2$, which led to the definition of differential extropy. The extropy of non-negative continuous rv $X$ with pdf $f_X(x)$ is given by $J(X) = -\frac{1}{2}\int_0^{+\infty} f_X^2(x)dx$.

Extropy has several applications. For example, $(i)$ extropy is used to score the forecasting distributions using the total scoring rule [4]; $(ii)$ extropy is interpreted as a measure of the amount of uncertainty represented by the distribution for rv, that is, if the extropy of $X$ is less than that of another rv $Y$, that is, $J(X) \leq J(Y)$, then $X$ is said to have more

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uncertainty than Y [21]; (*iii*) extropy is used to compare two mixed systems with same signature but with different components [21].

For some studies on extropy and its applications, we refer to [6, 7, 9, 17, 19–22, 24, 34].

If a component is known to have survived to age t then extropy is no longer useful to measure the uncertainty of remaining lifetime of the component. An approach for solving this limitation is use of the residual differential extropy introduced by [20]. Differential extropy of a random lifetime $X$ is defined as

$$\xi J(X;t) = -\frac{1}{2\bar{F}_X^2(t)} \int_t^{+\infty} f_X^2(x)dx, \quad t \geq 0.$$

Replacing pdf in the extropy function $J(X)$ with survival function (sf), Jahanshahi et al. [5] proposed a new measure of uncertainty of non-negative absolutely continuous rv $X$ with sf $\bar{F}_X(x)$, called cumulative residual extropy (CREX). This sf-based uncertainty measure is defined as

$$\xi J(X) = -\frac{1}{2} \int_0^{+\infty} \bar{F}_X^2(x)dx. \tag{1.1}$$

Let two rvs $X$ and $Y$ be lifetimes of two systems $A$ and $B$. Jahanshahi et al. [5] showed that, if CREX of rv $X$ is less than rv $Y$ (i.e. $\xi J(X) \leq \xi J(Y)$), then system $A$ has less uncertainty than system $B$. They also proposed two applications of the CREX to risk measure and independence. In such cases, the information measures are functions of time, and thus they are dynamic. In such situations, either CREX is not suitable and therefore it should be modified to dynamic form. Most recently, Sathar and Nair [25] proposed dynamic version of CREX (called dynamic survival extropy) and studied its important properties. Dynamic survival extropy of a random lifetime $X$ is defined as

$$\xi J(X;t) = -\frac{1}{2\bar{F}_X^2(t)} \int_t^{+\infty} \bar{F}_X^2(x)dx, \quad t \geq 0. \tag{1.2}$$

When a system has completed $t$ units of time, $\xi J(X;t)$ gives information of the extropy for the remaining lifetime of the system. On the other hand, if the dynamic survival extropy of random lifetime $X$ is less than random lifetime $Y$, (i.e. $\xi J(X;t) \leq \xi J(Y;t)$), then we can say that, system $A$ (with lifetime $X$) has less uncertainty than system $B$ (with lifetime $Y$) about the remaining lifetime. It is clear that $\xi J(X;0) = \xi J(X)$.

Since the quantile function (qf) have several properties that are not shared by the cumulative distribution function (cdf), the quantile-based methods have been employed effectively to investigate the information properties of such models. The qf of continuous rv $X$ can be specified in terms of cdf $F_X(x)$ as

$$Q_X(u) = F_X^{-1}(u) = \inf\{x \mid F_X(x) \geq u\}, \quad u \in [0,1].$$

Although both distribution and quantile functions convey the same information about the distribution of a rv, quantile functions (qfs) have several properties that are not shared by cdfs. For example, there are explicit general distribution forms for the qf of order statistics. It is easier to generate random numbers from the qf; there are probability models having no closed form cdfs. However, they have closed form qfs. Also, the use of qfs in the place of cdf provides new models, alternative methodology, easier algebraic manipulations and methods of analysis in certain cases and some new results that are difficult to derive by using distribution function [3, 14].

Accordingly, Sunoj and Sankaran [29] introduced quantile versions of the Shannon's entropy and dynamic version of it. The quantile-based residual entropy of random lifetime $X$ is defined by

$$H(X;Q_X(u)) = \ln(1-u) + (1-u)^{-1} \int_u^1 \ln q_X(p)dp, \quad u \in (0,1), \tag{1.3}$$

where $q_X(u) = Q^{'}(u)$ is the quantile density function (qdf). The measure (1.3) gives the expected uncertainty contained in the conditional density about the predictability of an outcome of $X$ until $100(1-u)\%$ point of distribution. For the usefulness of information measures based on qf, we refer to [1, 8, 16, 23, 30] and the references therein. Recently, the concept of quantile-based information measures is extended for order statistics. For study about this extend of quantile-based information measures, we refer to [11, 28].

Motivated with the usefulness of dynamic survival extropy, in this paper, we investigate some new aspects of it using the quantile function approach. Moreover, we propose a quantile-based dynamic survival extropy of order statistics and prove some of its properties. The quantile-based dynamic survival extropy has several advantages. For example, $(i)$ unlike the dynamic survival extropy, the quantile-based dynamic survival extropy uniquely determines the quantile density function; $(ii)$ we derive quantile-based dynamic survival extropy functions for certain qfs which do not have an explicit form for cdfs; $(iii)$ based on the quantile-based dynamic survival extropy function, we define a new quantile based stochastic order, to compare the uncertainties of residual lives of two random lives $X$ and $Y$ at the age points $Q_X(u)$ and $Q_Y(u)$ at which $X$ and $Y$ possess equally survival probabilities; and $(iv)$ we provide the new characterizations for a family of distribution through simple relationships.

The paper is organized as follows: In Section 2, we recall some preliminary concepts about qf. In this section, we discuss dynamic survival extropy in terms of the qf. Several properties of this uncertainty measure such as characterization results, aging classes and stochastic comparisons are proposed. In Section 3, we proposed the quantile-based dynamic survival extropy of first order statistic and sample maxima and studied some of its properties. In Section 4, we use the absolute value of quantile-based dynamic survival extropy function as risk measure. Some examples are presented for evaluating and comparing this new risk measure with quantile form of the generalized right tail deviation measure. Finally, some concluding remarks are provided in Section 5.

## 2. Quantile-based dynamic survival extropy

In this section, we study a dynamic measure of uncertainty, the quantile version of dynamic survival extropy of non-negative absolutely continuous rv $X$. First, we recall some notations and preliminary concepts of qf [14].

Let $X$ be a non-negative absolutely continuous rv with cdf $F_X(x)$, pdf $f_X(x)$ and qf $Q_X(u)$. If $F_X(x)$ is right continuous and strictly increasing we have $F_X(Q_X(u)) = u$, so that $F_X(x) = u$ implies $x = Q_X(u)$ and $q_X(u)f_X(Q_X(u)) = 1$, for all $u \in [0, 1]$.

Two primary concepts used to represent the physical properties of the failure patterns are the hazard rate and mean residual. There are extended concepts of the hazard rate and mean residual for quantile function called hazard quantile function (hqf) and mean residual quantile function (mrqf), respectively, defined as, for all $u \in (0, 1)$

$$H_X(u) = h_X(Q_X(u)) = \frac{f_X(Q_X(u))}{\bar{F}_X(Q_X(u))} = [(1-u)q_X(u)]^{-1}, \tag{2.1}$$

$$M_X(u) = m_X\left(Q_X(u)\right) = \frac{1}{(1-u)} \int_u^1 \left(Q_X(p) - Q_X(u)\right) dp = \frac{1}{(1-u)} \int_u^1 \frac{dp}{H_X(p)} dp,$$

where $m_X(x) = E[X - x \mid X > x]$ is the mean residual life function of $X$. We can interpret the hqf explains the conditional probability of failure in the next small interval of time given survival until $100(1-u)\%$ point of distribution. On the other hand, we can interpret mrqf as the mean remaining life of a unit beyond the $100(1-u)\%$ of the distribution. Also, mrqf uniquely determines the qdf by

$$q_X(u) = \frac{M_X(u) - (1-u)\, M_X'(u)}{(1-u)}. \tag{2.2}$$

**Remark 2.1.** Let $X$ be a non-negative absolutely continuous rv with finite mean and qf $Q_X(x)$ such that $Q(0) = 0$. The conditional value-at-risk of X is given by

$$CVaR[X;u] = E[X_{Q_X(u)}] = E[X - Q_X(u)|X > Q_X(u)] \tag{2.3}$$

$$= \frac{1}{(1-u)} \int_{Q_X(u)}^{+\infty} \bar{F}_X(x)dx, \quad u \in (0,1). \tag{2.4}$$

In the context of reliability theory the conditional value-at-risk function is also called mrqf [2].

Since $q_X(u)f_X(Q_X(u)) = 1$, $u \in [0,1]$ and substituting $x = Q_X(p)$ in relation (1.1), we can define the quantile version of dynamic survival extropy of non-negative absolutely continuous rv $X$ as follows [10]:

**Definition 2.2.** Let $X$ be a non-negative absolutely continuous rv with qf $Q_X(x)$ and qdf $q_X(x)$. The quantile-based dynamic survival extropy of $X$ is defined as

$$\xi J\left(X; Q_X(u)\right) = -\frac{1}{2\left(1-u\right)^2} \int_u^1 \left(1-p\right)^2 q_X(p)\, dp, \quad u \in (0,1). \tag{2.5}$$

From relation (2.5) it easily follows that $\xi J(X; Q_X(u))$ takes values in $(-\infty, 0]$. Based on the presented definition of the quantile-based dynamic survival extropy, we can infer the following cases:

(1) The quantile-based dynamic survival extropy measures spectrum of the survival extropy's uncertainty contained in the conditional sf about the predictability of an outcome of $X$ until $100(1-u)\%$ point of distribution.
(2) The quantile-based dynamic survival extropy measures the uncertainty of residual life $X_{Q_X(u)}$, that is, quantile-based dynamic survival extropy measures the uncertainty of $X$ at age point $Q_X(u)$.

From relation (2.1), in terms of $H_X(u)$, $\xi J\left(X; Q_X(u)\right)$ becomes [10]

$$\xi J\left(X; Q_X(u)\right) = -\frac{1}{2\left(1-u\right)^2} \int_u^1 \frac{(1-p)}{H_X(p)}\, dp. \tag{2.6}$$

Also, from relation (2.2), in terms of $M_X(u)$, $\xi J\left(X; Q_X(u)\right)$ becomes

$$\begin{aligned} \xi J\left(X; Q_X(u)\right) &= -\frac{1}{2\left(1-u\right)^2} \int_u^1 (1-p) \left(M_X(p) - (1-p)\, M_X^{'}(p)\right) dp \\ &= -\frac{1}{2\left(1-u\right)^2} \left[\int_u^1 (1-p)\, M_X(p)\, dp - \int_u^1 (1-p)^2\, M_X^{'}(p)\, dp\right]. \end{aligned}$$

Applying integration by parts on the last term yield

$$\xi J\left(X; Q_X(u)\right) = -\frac{M_X(u)}{2} + \frac{1}{2\left(1-u\right)^2} \int_u^1 (1-p)\, M_X(p)\, dp. \tag{2.7}$$

Now, differentiating (2.7) with respect to $u$

$$q_X(u) = 2\left(\xi J^{'}\left(X; Q_X(u)\right) - \frac{2\xi J\left(X; Q_X(u)\right)}{(1-u)}\right). \tag{2.8}$$

It is to be noted that from above relations, by knowing qf, qdf, or hqf, the expression for quantile-based dynamic survival extropy is quite simple to compute. To study the $\xi J(X; Q_X(u))$ value for some rvs that do not have explicitly known cdfs, though it has closed form qfs, we provide the following examples.

**Example 2.3.** Let $X$ be a rv of generalized lambda family with a closed form quantile function as follows:

$$Q_X(u) = \lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} + (1-u)^{\lambda_4}\right), \tag{2.9}$$

where $\lambda_1, \lambda_2, \lambda_4 \in R$ and $\lambda_3 \in Z^+$. Then, quantile-based dynamic survival extropy of rv $X$ is given by

$$\xi J\left(X; Q_X(u)\right) = -\frac{1}{2\lambda_2\left(1-u\right)^2}\left(\lambda_3\bar{\beta}_u(\lambda_3, 3) + \lambda_4\bar{\beta}_u(1, \lambda_4 + 2)\right),$$

where $\bar{\beta}_x(\alpha, \beta) = \int_x^1 z^{\alpha-1}(1-z)^{\beta-1}dz$ is the incomplete beta function.

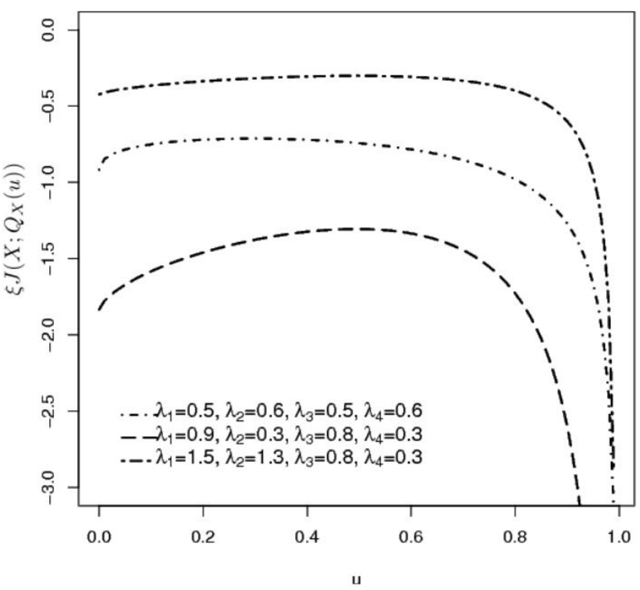


**Figure 1.** Graphs of $\xi J\left(X; Q_X(u)\right)$ for generalized lambda family (2.9) with various choices of parameters.

Figure 1 provides the graphs of $\xi J(X; Q_X(u))$ for various values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, respectively, in the case where $X$ is a rv of generalized lambda family (2.9). Note that $\xi J(X; Q_X(u))$ is nonincreasing function and nondecreasing function of $u$ in terms of various values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

**Example 2.4.** Let $X$ be a rv with a closed form quantile density function as follows:

$$q_X(u) = K(1-u)^{-A}(-\ln(1-u))^{-M}, \tag{2.10}$$

where $K$, $A$ and $M$ are real constants. Using relation (2.5) and by substituting $-\ln(1-p) = z$, quantile-based dynamic survival extropy of rv $X$ is given by

$$\begin{aligned}\xi J\left(X; Q_X(u)\right) &= \frac{K}{2\left(1-u\right)^2}\int_{-\ln(1-u)}^{+\infty} z^{-M}e^{-z(3-A)}dz \\ &= \frac{K}{2\left(1-u\right)^2}\bar{\Gamma}_{-\ln(1-u)}(1-M, 3-A),\end{aligned}$$

where $\bar{\Gamma}_x(\alpha, \beta) = \int_x^{+\infty} z^{\alpha-1}e^{-z\beta}dz$ is the incomplete gamma function.

Figure 2 provides the graphs of $\xi J(X; Q_X(u))$ for $K = 0.2, 2.5, 0.5$, $M = 0.7, 0.9, 0.6$ and $A = 0.1, 0.8, 0.5$, respectively, in the case where $X$ is a rv with qdf given by (2.10). Note that $\xi J(X; Q_X(u))$ is nonincreasing and nondecreasing function of $u$ in terms of various values of $K, M, A$.

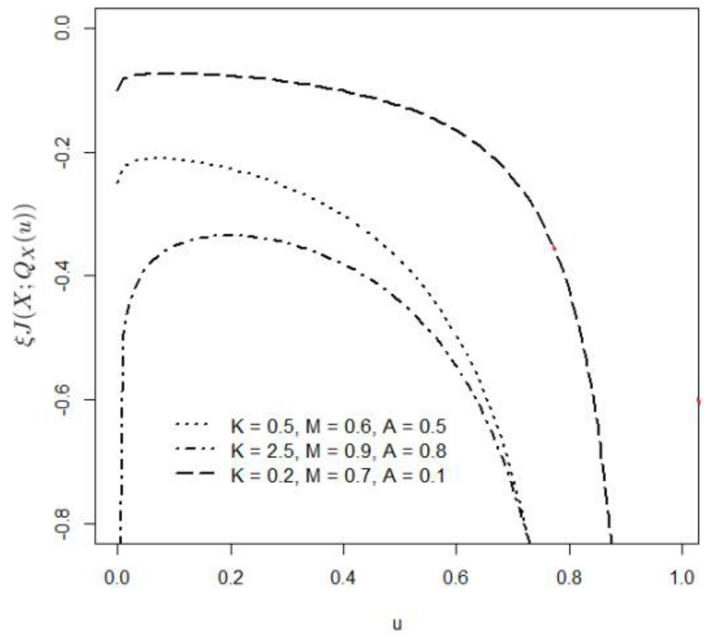


**Figure 2.** Graphs of $\xi J\left(X;Q_X(u)\right)$ for qdf (2.10) with various choices of parameters.

**Remark 2.5.** It is obvious that if $X$ and $Y$ have the same distribution then $\xi J(X;Q_X(u)) = \xi J(Y;Q_X(u))$, the question that arises is: "What about the converse?". Using relation (2.8), If $q_X(u) = q_Y(u)$, then $\xi J(X;Q_X(u)) = \xi J(Y;Q_X(u))$. This implies that underlying quantile density function can be characterized uniquely by quantile-based dynamic survival extropy. Thus, there is a unique characteristic of $\xi J(X;Q_X(u))$ unlike the dynamic survival extropy $\xi J(X;t)$ in (1.2), where no such explicit relationship exists between $\xi J(X;x)$ and $f(x)$.

Sathar and Nair [25] proposed two ageing classes (nonparametric classes of statistical models) increasing dynamic survival extropy (IDSEx) and decreasing dynamic survival extropy (DDSEx). In the following, we define two nonparametric classes of statistical models, using the monotonicity of the quantile-based dynamic survival extropy function.

**Definition 2.6.** We say that $X$ has an increasing (decreasing) quantile-based dynamic survival extropy, shortly written as IQDSEx (DQDSEx), if $\xi J(X;Q_X(u))$ is nondecreasing (nonincreasing) in $u$, $u \geq 0$.

Since qf $Q_X(u)$ is an increasing function, we can say that IDSEx (DDSEx) is equivalent to IQDSEx (DQDSEx). In the following and using the monotonicity of the quantile-based dynamic survival extropy function, we derive upper and lower bounds for quantile-based dynamic survival extropy depending on the hqf. From the relation (2.8) it holds that if $X$ is IQDSEx (DQDSEx), then $\xi J(X;Q_X(u)) \geq (\leq) -\frac{(1-u)q_X(u)}{4}$. Thus, it follows that if $X$ is IQDSEx (DQDSEx), then from (2.1) we have

$$\xi J(X;Q_X(u)) \geq (\leq) -\frac{1}{4H_X(u)}. \tag{2.11}$$

Table 1 gives some applications of monotonicity of the quantile-based dynamic survival extropy function. The table provides some statistical models that belongs to IQDSEx or DQDSEx classes. Also, in the table we have presented upper or lower bounds of the quantile-based dynamic survival extropy for these statistical models.

**Table 1.** The quantile-based dynamic survival extropy of some statistical models, monotone nature of these models and bounds for $\xi J(X;Q_X(u))$.

| Statistical model and qf | $\xi J(X;Q_X(u))$ | Monotone nature | bound |
|---|---|---|---|
| Uniform<br>$Q_X(u)=a+(b-a)u,\ a\geq 0,\ b>a$ | $-\frac{(b-a)(1-u)}{6}$ | IQDSEx<br>IHR | $\geq -\frac{(b-a)(1-u)}{4}$ |
| Rescaled beta<br>$Q_X(u)=\alpha(1-(1-u)^{\frac{1}{\beta}}),\ \alpha>0,\ \beta>0$ | $-\frac{\alpha(1-u)^{\frac{1}{\beta}}}{2(2\beta+1)}$ | IQDSEx<br>DHR | $\geq -\frac{\alpha(1-u)^{\frac{1}{\beta}}}{4\beta}$ |
| Pareto type I<br>$Q_X(u)=\alpha(1-u)^{-\frac{1}{\beta}},\ \alpha>0,\ \beta>0$ | $-\frac{\alpha(1-u)^{-\frac{1}{\beta}}}{2(2\beta-1)}$ | DQDSEx<br>DHR | $\leq -\frac{\alpha}{4\beta(1-u)^{\frac{1}{\beta}}}$ |
| Pareto type II<br>$Q_X(u)=\alpha((1-u)^{-\frac{1}{\beta}}-1),\ \alpha>0,\ \beta>0$ | $-\frac{\alpha(1-u)^{-\frac{1}{\beta}}}{2(2\beta-1)}$ | DQDSEx<br>DHR | $\leq -\frac{\alpha}{4\beta(1-u)^{\frac{1}{\beta}}}$ |
| Generalized Pareto<br>$Q_X(u)=\frac{b}{a}((1-u)^{-\frac{a}{a+1}}-1),\ a>0,\ b>0$ | $-\frac{b(1-u)^{-\frac{a}{a+1}}}{2(a+2)}$ | DQDSEx<br>DHR | $\leq -\frac{b(1-u)\left((1-u)^{-\frac{a}{a+1}}-1\right)}{4a}$ |
| Log logistic<br>$Q_X(u)=\frac{1}{\alpha}(\frac{u}{1-u})^{\frac{1}{\beta}},\ \alpha>0,\ \beta>0$ | $-\frac{\bar{\beta}_u(\frac{1}{\beta},2-\frac{1}{\beta})}{2\alpha\beta(1-u)^2},\ \beta>\frac{1}{2}$ | IQDSEx<br>DHR for $\beta<1$ | $\geq -\frac{u^{\frac{1}{\beta}}(1-u)^{\frac{1}{\beta}-1}}{4\alpha}$ |
| Exponential<br>$Q_X(u)=-\lambda^{-1}\ln(1-u),\ \lambda>0$ | $-\frac{1}{4\lambda}$ | Boundary class<br>Boundary class | $=-\frac{1}{4\lambda}$ |
| Weibull<br>$Q_X(u)=\lambda^{-\frac{1}{\alpha}}\big(-\ln(1-u)\big)^{\frac{1}{\alpha}},\ 0<\alpha<1,\ \lambda>0$ | $-\frac{\lambda^{-\frac{1}{\alpha}}\bar{\gamma}_{-\ln(1-u)}(\frac{1}{\alpha},2)}{2\alpha(1-u)^2}$ | DQDSEx<br>DHR for $\alpha<1$<br>IHR for $\alpha>1$ | $\leq -\frac{\left(-\ln(1-u)\right)^{\frac{1}{\alpha}}}{4\alpha\lambda^{\frac{1}{\alpha}}}$ |

**Remark 2.7.** Since the monotonicity of hazard rate function and quantile hazard function are identical, we can say that $X$ has an increasing (decreasing) hazard rate [IHR (DHR)] if $H_X(u)$ is increasing (decreasing) in $u$. Based on Table 1, some statistical models belongs to IHR (DHR) classes, while they belong to DQDSEx (IQDSEx). Thus, we can say that IHR (DHR) property does not imply IQDSEx (DQDSEx) property.

The following theorem shows that the constant quantile-based dynamic survival extropy characterizes exponential distribution.

**Theorem 2.8.** *The non-negative absolutely continuous rv $X$ has constant quantile-based dynamic survival extropy if and only if $X$ is exponentially distributed.*

***Proof.*** The "if" part is direct from Table 1. To prove the "only if" part, assume that $\xi J\left(X;Q_X(u)\right)=c$, is constant. Using relation (2.8), we obtain $Q_X(u)=4c\ln(1-u)=-\lambda^{-1}\ln(1-u)$, where $\lambda=-\frac{1}{4c}$. Hence the proof is complete. □

The following theorem shows that the linear quantile-based dynamic survival extropy characterizes the linear mean residual quantile distribution [13].

**Theorem 2.9.** *For non-negative absolutely continuous rv $X$, $\xi J\left(X;Q_X(u)\right)=a+bu$, $a,b>0$ holds if and only if $X$ follows a family of distribution with qf*

$$Q_X(u)=4(a+b)\log(1-u)+6bu.$$

***Proof.*** The first part of the proof follows from relation (2.5). Conversely, assume that $\xi J\left(X;Q_X(u)\right)=a+bu,\ \ a,b>0$. Using relation (2.8), we obtain $q_X(u)=\frac{-4(a+b)}{1-u}+6b=Q'_X(u)$, which completes the proof. □

Sathar and Nair [25], compared the uncertainties of two rvs $X$ and $Y$ by comparing their dynamic survival extropy functions at the same time points $t$. Here, we introduce a stochastic order so as to compare the uncertainties of $X$ and $Y$ based on the quantile-based dynamic survival extropy functions at the age points $Q_X(u)$ and $Q_Y(u)$ at which $X$ and $Y$ possess equally survival probabilities.

**Definition 2.10.** The non-negative absolutely continuous rv $X$ is said to be smaller than $Y$ in the

• dynamic survival extropy order denoted by $X \overset{DSEx}{\leqslant} Y$, if $\xi J(X;t) \leq \xi J(Y;t)$ for all $t \geq 0$.

• quantile-based dynamic survival extropy order denoted by $X \overset{QDSEx}{\leqslant} Y$, if $\xi J(X;Q_X(u)) \leq \xi J(Y;Q_Y(u))$ for all $u \in [0,1]$.

Based on the following example, we show that defined stochastic orders do not seem to have been discussed in literature.

**Example 2.11.** Let $X$ and $Y$ have two Pareto type I distribution with survival functions (sfs) $\bar{F}_X(x) = (\frac{1}{x})^b$ and $\bar{G}_X(x) = (\frac{1}{x})^d$, $b, d > 0$, $b \leq d$, respectively. From definition of dynamic survival extropy using relation (1.2), we obtain

$$J(X;t) = -\frac{bt}{2(2-b)} \geq -\frac{dt}{2(2-d)} = J(Y;t), \quad b, d \neq 2, \quad t \geq 1.$$

On the other hand, from Table 1, we have

$$\xi J(X;Q_X(u)) = -\frac{(1-u)^{-\frac{1}{b}}}{2(2b-1)} \leq -\frac{(1-u)^{-\frac{1}{d}}}{2(2d-1)} = \xi J(Y;Q_Y(u)), \quad b, d \neq \frac{1}{2},$$

for all $u \in (0,1)$. Hence, $X \overset{QDSEx}{\leqslant} Y \nRightarrow X \overset{DSEx}{\leqslant} Y$. Also, interchanging the roles of $X$ and $Y$ implies that $X \overset{DSEx}{\leqslant} Y \nRightarrow X \overset{QDSEx}{\leqslant} Y$.

**Definition 2.12.** The rv $X$ is said to be smaller than $Y$ in the hazard quantile function order denoted by $X \overset{QHR}{\leqslant} Y$, if $H_X(u) \geq H_Y(u)$ for all $u \in (0,1)$.

**Remark 2.13.** $(i)$ From definition of hazard quantile function order and using relation (2.6), $X \overset{QHR}{\geqslant} Y$ implies that $X \overset{QDSEx}{\leqslant} Y$. But, $X \overset{QHR}{\leqslant} Y$ dose not imply $X \overset{QDSEx}{\leqslant} Y$. For example, consider two rvs $X$ and $Y$ distributed as $U(0,1)$ and $U(0,3)$, respectively. We have $X \overset{QHR}{\leqslant} Y$ while $X \overset{QDSEx}{\geqslant} Y$.
$(ii)$ Consider the continuous non-negative rvs X and Y. If $X$ and $Y$ have the same lower end of the support and if $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u \in (0,1)$, then $X \overset{st}{\leqslant} Y$ (the usual stochastic order) implies that $X \overset{QHR}{\leqslant} Y$ (see [31]). Hence, from part $(i)$, $X \overset{st}{\geqslant} Y$ imply that $X \overset{QDSEx}{\leqslant} Y$.
$(iii)$ If $X$ or $Y$ is decreasing failure rate (DFR), then $X \overset{hr}{\leqslant} Y$ (the hazard rate order) implies that $X \overset{QHR}{\leqslant} Y$ (see [31]). Hence, From part $(i)$, $X \overset{hr}{\geqslant} Y$ implies that $X \overset{QDSEx}{\leqslant} Y$.

Following theorem shows the quantile-based dynamic survival extropy can be a super-additive functional.

**Theorem 2.14.** *Let $X$ and $Y$ be two independent continuous non-negative rvs with same left-end of the support points and right-end support points $r_X = r_Y < +\infty$. If $X$ and $Y$ have log-concave density and $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u \in (0,1)$, then*
$(i)$ $\xi J(X+Y;Q_{X+Y}(u)) \geq \max\{\xi J(X;Q_X(u)), \xi J(Y;Q_Y(u))\}$.
$(ii)$ $\xi J(X+Y;Q_{X+Y}(u)) \geq \xi J(X;Q_X(u)) + \xi J(Y;Q_Y(u))$.

***Proof.*** From Theorem 3.B.7 of [26], we imply that $X \overset{disp}{\leqslant} X+Y$ (the dispersive order) for any rv $Y$ independent of $X$ with log-concave density. This implies that $X \overset{st}{\geqslant} X+Y$ under the hypothesis $r_X = r_Y < +\infty$ (see Theorem 3.B.13 of [26]). Thus, if $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u \in (0,1)$, Remark 2.13 implies that $\xi J(X+Y; Q_{X+Y}(u)) \geq \xi J(X; Q_X(u))$. Similarly, we have $\xi J(X+Y; Q_{X+Y}(u)) \geq \xi J(Y; Q_Y(u))$ when $Y$ has a log-concave density. Hence, we give the result of part $(i)$. Due to part $(i)$, the result of part $(ii)$ follows by recalling that the quantile-based dynamic survival extropy of a continuous non-negative rv is always non-positive. □

**Remark 2.15.** Suppose that a density $f_X(x)$ can be written as

$$f_X(x) = f_{\varphi_X}(x) = \exp(\varphi_X(x)) = \exp(-(-\varphi_X(x))),$$

where $\varphi_X(x)$ is concave function (and $-\varphi_X(x)$ is convex function). The class of all densities $f_X(x)$ on $\mathbb{R}$, of this form is called the class of log-concave densities.
The class of log-concave densities is rich and contains important densities in probability, statistics and analysis. Gaussian density, Laplace density, uniform density on a convex set, chi density are all log-concave. Moreover, many familiar probability distributions lack closed form cdfs, but have pdfs that are represented by simple algebraic expressions. Conveniently, it turns out that log-concavity of the pdf implies log-concavity of the cdf. For example, the cumulative normal distribution does not have a closed form representation and direct verification of its log-concavity is difficult. But the normal density function is easily seen to be log-concave, since its natural logarithm is a concave quadratic function.

## 3. Quantile-based dynamic survival extropy of order statistics

Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) non-negative rvs that have sfs $\bar{F}(x)$. If $X_{i:n}$ denotes the $i$th order statistics in this sample of size $n$, then the lifetime of a (n-i+1)-out-of-n system is determined by $X_{i:n}$ with sf $\bar{F}_{i:n}(x)$. In spacial cases, the lifetime of two series and parallel systems are determined by $X_{1:n}$ and $X_{n:n}$ with sfs $\bar{F}_{1:n}(x)$ and $\bar{F}_{n:n}(x)$, respectively. The quantile-based uncertainty measures of order statistics is useful to compare the uncertainties of lifetimes of two (n-i+1)-out-of-n systems. In this section, we focus on the quantile-based dynamic survival extropy of lifetime of series (first order statistic) and parallel (sample maxima) systems.
In analogy with relation (1.2), the dynamic survival extropy of $i$th order statistics $X_{i:n}$ is given by

$$\begin{aligned}\xi J(X_{i:n}; t) &= \frac{-1}{2\bar{F}^2_{i:n}(t)} \int_t^{+\infty} \bar{F}^2_{i:n}(x)dx \\ &= \frac{-1}{2\bar{\beta}^2_{F(t)}(i, n-i+1)} \int_t^{+\infty} \bar{\beta}^2_{F(x)}(i, n-i+1)dx.\end{aligned}$$

Since the quantile form of sf of $i$th order statistics is defined as $\bar{F}_{i:n}(x) = \frac{\bar{\beta}_u(i,n-i+1)}{\beta(i,n-i+1)}$, the quantile-based dynamic survival extropy of $i$th order statistics $X_{i:n}$ is given by

$$\xi J(X_{i:n}; Q_{X_{i:n}}(u)) = \frac{-1}{2\bar{\beta}^2_u(i, n-i+1)} \int_u^1 \bar{\beta}^2_p(i, n-i+1) q_X(p)dp.$$

In spacial case and for the series system with lifetime $X_{1:n}$, we have

$$\xi J(X_{1:n}; Q_{X_{1:n}}(u)) = \frac{-1}{2(1-u)^{2n}} \int_u^1 (1-p)^{2n} q_X(p)dp. \tag{3.1}$$

Also for the parallel system with lifetime $X_{n:n}$, we have

$$\xi J(X_{n:n}; Q_{X_{n:n}}(u)) = \frac{-1}{2(1-u^n)^2} \int_u^1 (1-p^n)^2 q_X(p)dp. \tag{3.2}$$

Differentiating (3.1) with respect to $u$ on both sides, we can obtain

$$q_X(u) = 2\xi J^{'}(X_{1:n}; Q_{X_{1:n}}(u)) - \frac{4n\xi J(X_{1:n}; Q_{X_{1:n}}(u))}{(1-u)}. \tag{3.3}$$

Similarity, by differentiating (3.2) with respect to $u$ on both sides, we get

$$q_X(u) = 2\xi J^{'}(X_{n:n}; Q_{X_{n:n}}(u)) - \frac{4n\xi J(X_{n:n}; Q_{X_{n:n}}(u))}{u^{1-n}(1-u^n)}. \tag{3.4}$$

Above equation shows that the quantile-based dynamic survival extropy of $X_{1:n}$ and $X_{n:n}$ uniquely determines the underling distribution function. In following examples, we study the values of $\xi J(X_{1:n}; Q_{X_{1:n}}(u))$ and $\xi J(X_{n:n}; Q_{X_{n:n}}(u))$ for two family of qfs that do not have explicitly known cdfs.

**Example 3.1.** Let $X$ be a rv of generalized lambda family with a closed form quantile function as (2.9). Then, the quantile-based dynamic survival extropy of $X_{1:n}$ and $X_{n:n}$ are given by

$$\xi J(X_{1:n}; Q_{X_{1:n}}(u)) = \frac{-1}{2\lambda_2(1-u)^{2n}}\big(\lambda_3\bar{\beta}_u(\lambda_3, 2n+1) + \lambda_4\bar{\beta}_u(1, \lambda_4+2n)\big),$$

$$\xi J(X_{n:n}; Q_{X_{n:n}}(u)) = \frac{-1}{2\lambda_2(1-u^n)^2}\big(n\lambda_3\bar{\beta}_{u^n}(\frac{\lambda_3}{n}, 3) + \lambda_4(\bar{\beta}_u(1, \lambda_4) + \bar{\beta}_u(2n+1, \lambda_4) - 2\bar{\beta}_u(n+1, \lambda_4))\big).$$

**Example 3.2.** Let $X_{i:n}$ be the $i$th order statistic based on a random sample of size $n$ from no closed quantile density function given by $q_X(u) = Ku^V(1-u)^{-(B+V)}$, $K, B, V \in R$. Then, we obtain the quantile-based dynamic survival extropy of $X_{1:n}$ and $X_{n:n}$ as follows:

$$\xi J(X_{1:n}; Q_{X_{1:n}}(u)) = \frac{-K}{2(1-u)^{2n}}\big(\bar{\beta}_u(V+1, -(B+V)+2n+1)\big),$$

$$\xi J(X_{n:n}; Q_{X_{n:n}}(u)) = \frac{-K}{2(1-u^n)^2}\big(\bar{\beta}_u(V+1, -(B+V)+1) + \bar{\beta}_u(V+2n+1, -(B+v)+1) + \bar{\beta}_u(V+n+1, -(B+V)+1)\big).$$

Now, we find bounds for quantile-based dynamic survival extropy of first order statistic and sample maxima based on the hqf $H_X(u)$. These bounds are useful when the qdf has no closed form or $\xi J(X_{1:n}; Q_{X_{1:n}}(u))$ and $\xi J(X_{n:n}; Q_{X_{n:n}}(u))$ are difficult to compute.

**Theorem 3.3.** *Let $X$ be a continuous non-negative rv with qf $Q_X(u)$ and hqf $H_X(u)$.*
*($i$) If the quantile-based dynamic survival extropy of first order statistic is increasing (decreasing) in u, then*

$$\xi J(X_{1:n}; Q_{X_{1:n}}(u)) \geq (\leq) \frac{-1}{4nH_X(u)}. \tag{3.5}$$

*($ii$) If the quantile-based dynamic survival extropy of sample maxima is increasing (decreasing) in u, then*

$$\xi J(X_{n:n}; Q_{X_{n:n}}(u)) \geq (\leq) \frac{-(1-u^n)}{4nu^n\bar{H}_X(u)}, \tag{3.6}$$

*where $\bar{H}_X(u) = \frac{f(Q_X(u))}{F(Q_X(u))} = \frac{1}{uq_X(u)}$ is the quantile form of reversed hazard rate function.*

***Proof.*** Assume that $\xi J(X_{1:n}; Q_{X_{1:n}}(u))$ is increasing (decreasing), so that $\xi J^{'}(X_{1:n}; Q_{X_{1:n}}(u)) \geq 0 (\leq 0)$. Thus, the lower and upper bounds as (3.5) can be obtained from relation (3.3). Similarly, the bounds as (3.6) can be obtained from relation (3.4). □

The quantile-based dynamic survival extropy of order statistics is useful to compare the uncertainties of lifetimes of $(n-i+1)$-out-of-$n$ systems. The following remark can be viewed as direct application of part (iii) of Remark 2.13 in the area of order statistics.

**Remark 3.4.** Let $X_1, X_2, ..., X_n$ be iid non-negative DFR rvs having continuous qfs $Q_X(u)$. Then,

$(a)$ $X_{i:n} \overset{QDSEx}{\geqslant} X_{i+1:n}$. That is $\xi J(X_{i:n}; Q_{X_{i:n}}(u))$ is a decreasing function of $i$.

$(b)$ $X_{1:n} \overset{QDSEx}{\geqslant} X_{1:n-1}$.

$(c)$ $X_{n-1:n-1} \overset{QDSEx}{\geqslant} X_{n:n}$.

$(d)$ We know that $X \overset{hr}{\leqslant} X_{1:n}$. Thus, $X \overset{QDSEx}{\geqslant} X_{1:n}$. Also, since $X_{n:n} \overset{hr}{\leqslant} X$, we have $X_{n:n} \overset{QDSEx}{\geqslant} X$.

$(e)$ If $X_1 \overset{hr}{\leqslant} Y_1$ then $X_{i:n} \overset{QDSEx}{\geqslant} Y_{i:n}$.

**Remark 3.5.** Consider the Cox proportional hazard model, defined by $h_Y(x) = \theta h_X(x)$, $\theta > 0$. The quantile-based dynamic survival extropy of first order statistic $Y_{1:n}$ is given by

$$\begin{aligned}\xi J(Y_{1:n}; Q_{Y_{1:n}}(u)) &= \frac{-1}{2(1-u)^{2n}} \int_u^1 (1-p)^{2n} q_Y(p) dp. \\ &= \frac{-1}{2(1-u)^{2n}} \int_u^1 (1-p)^{2n} \frac{(1-p)^{\frac{1}{\theta}-1}}{\theta} q_X(1-(1-p)^{\frac{1}{\theta}}) dp \\ &= \frac{-1}{2(1-u)^{2n}} \int_{1-(1-u)^{\frac{1}{\theta}}}^1 (1-z)^{2n\theta} q_X(z) dz, \qquad (3.7)\end{aligned}$$

where the last equation is obtained by taking $z = 1-(1-p)^{\frac{1}{\theta}}$. On the other hand, relation (3.7) gives

$$\begin{aligned}\xi J(Y_{1:n}; Q_{Y_{1:n}}(u)) &\leq \frac{-1}{2(1-u)^{2n}} \int_u^1 (1-z)^{2n} q_X(z) dz \\ &\leq \xi J(X_{1:n}; Q_{X_{1:n}}(u)).\end{aligned}$$

Similarly, for the quantile-based dynamic survival extropy of sample maxima, we can find $\xi J(Y_{n:n}; Q_{Y_{n:n}}(u)) \leq \xi J(X_{n:n}; Q_{X_{n:n}}(u))$.

Next, we prove a characterization theorem for some well-known distributions using the relationship between the quantile-based dynamic survival extropy and the hqf of the first order statistics.

**Theorem 3.6.** *Let $X_{1:n}$ denote the first order statistics with hqf $H_{X_{i:n}}(u)$. Then, the quantile-based dynamic survival extropy $\xi J(X_{1:n}; Q_{X_{1:n}}(u))$ is given by*

$$\xi J(X_{1:n}; Q_{X_{1:n}}(u)) = -C\ H^{-1}_{X_{i:n}}(u), \qquad (3.8)$$

*if and only if $X$ is distributed as*

*$(i)$ uniform distribution, if $C = \frac{n}{2(2n+1)}$.*

*$(ii)$ exponential distribution, if $C = \frac{1}{4}$.*

*$(iii)$ Pareto type I distribution, if $C = \frac{n\alpha}{2(2n\alpha-1)}$.*

***Proof.*** Assume that the relationship (3.8) holds. Using (3.1) we have

$$\int_u^1 (1-p)^{2n} q_X(p) dp = 2C(1-u)^{2n} H^{-1}_{X_{i:n}}(u). \qquad (3.9)$$

Substituting $H_{X_{i:n}}(u) = \frac{f_{i:n}(Q_X(u))}{\bar{F}_{1:n}(Q_X(u))} = \frac{n}{(1-u)q_X(u)}$ and simplifying, (3.9) gives

$$\int_u^1 (1-p)^{2n} q_X(p) dp = \frac{2C(1-u)^{2n+1} q_X(u)}{n}. \qquad (3.10)$$

Differentiating from (3.10) with respect to $u$ and after some algebraic simplification, we get

$$\frac{q'_X(u)}{q_X(u)} = \frac{2C(2n+1)-n}{2C(1-u)}. \tag{3.11}$$

Integrating (3.11) with respect to $u$ and simplifying, we obtain

$$q_X(u) = (1-u)^{-\frac{2C(2n+1)-n}{2C}} e^L,$$

where $L$ is the constant of integration. Now, if $C = \frac{n}{2(2n+1)}$ and $L = \ln(b-a)$; $b > 1$, which implies that $Q(u) = a + (b-a)u$. Thus, we have the uniform distribution $U(a,b)$. If $C = \frac{1}{4}$ and $L = -\ln\lambda$, which implies that $Q(u) = \lambda^{-1}\ln(1-u)$. Thus, we have the exponential distribution with parameter $\lambda > 0$. If $C = \frac{n\alpha}{2(2n\alpha-1)}$ and $L = \ln\left(\frac{\alpha}{\beta}\right)$; $\alpha > 0$, $\beta > 0$, we have $\alpha(1-u)^{-\frac{1}{\beta}}$. This means, we have the Pareto type I distribution. Only the if part of the theorem is easy to be proved. □

## 4. Application

This is well-known that the qf $Q_X(u)$ of rv X plays a very important role in comparing risks. In fact, in this context it is known as value-at-risk and is denoted by $Var[X;u] = Q_X(u)$, $u \in (0,1)$. It is employed to infer that the amount of capital needed to keep the probability of going bankrupt is at least $1-u$. As the value-at-risk at given level $u$ gives only a local information about the underlying risk, a more refined measure is known as conditional value-at-risk and is denoted by (see Remark 2.1)

$$CVaR[X;u] = E[X - Var[X;u] \mid X > Var[X;u]], \quad u \in (0,1).$$

Hence, the conditional value-at-risk is the average losses that exceed the threshold at value-at-risk confidence level $u$. It can be notice that $CVaR[X;u] = M_X(u)$ (see Remark 2.1). Using relation (2.7), we obtain

$$\xi J\left(X; Q_X(u)\right) = -\frac{CVaR[X;u]}{2} + \frac{1}{2\left(1-u\right)^2}\int_u^1 \left(1-p\right) CVaR[X;p]\, dp.$$

Thus $CVaR[X;u]$ determines $\xi J\left(X; Q_X(u)\right)$ uniquely. This coincidence allows the application of absolute value of quantile-based dynamic survival extropy function $|\xi J(X; Q_X(u))| \in [0, +\infty)$ as risk measure. In addition, $|\xi J(X; Q_X(u))|$ provides an account of the uncertainty associated with the losses at each percentile $u$. Furthermore, $|\xi J(X; Q_X(u))|$ preserves some basic properties of a risk measure as follows:

(1) Using Remark 2.13, we can consider a monotonicity property for the absolute value of quantile-based dynamic survival extropy under the hypothesis of usual stochastic order.

(2) Relation $|\xi J(aX+b; Q_X(u))| = a|\xi J(X; Q_X(u))|$, $a, b > 0$, shows the absolute value of quantile-based dynamic survival extropy is a shift-independent measure.

(3) Using Theorem 2.14, we can obtain a subadditive property of $|\xi J(X; Q_X(u))|$.
There are some sf-based information measures considered as a risk measure in literature. Yang [33] proposed the cumulative residual entropy function as a risk measure for heavy-tailed distribution when variance dose not exist. Psarrakos and Toomaj [18] studied a generalized form of cumulative residual entropy and dynamic form of it as risk measures. Recently, Jahanshahi et al. [5] proposed the absolute value of CREX function as risk measure.

We can not use sf-based information measures as a risk measure for some rvs having no closed form sfs. In such situations, we suggest quantile-based information measures.

Most Recently, Nair et al. [15] point out the use of quantile-based dynamic cumulative residual entropy as a risk measure. In this section, we consider the absolute value of quantile-based dynamic survival extropy function $|\xi J(X;Q_X(u))|$ at fixed value $u=u_0$ as risk measure in real-life problems.

From real-life problems and for fixed time value $t_0$, Psarrakos and Toomaj [18] defined the generalized right tail deviation measure as

$$R(X;r,t_0)=\int_{t_0}^{+\infty}\left(\frac{\bar F(x)}{\bar F(t_0)}\right)^r dx-\int_{t_0}^{+\infty}\frac{\bar F(x)}{\bar F(t_0)}dx,\quad 0<r\le 1. \tag{4.1}$$

Substituting $x=Q_X(p)$ in relation (4.1), we can propose a quantile form of the generalized right tail deviation measure as

$$R(X;r,u_0)=\int_{u_0}^{1}\left(\frac{1-p}{1-u_0}\right)^r q_X(p)dp-\int_{u_0}^{1}\left(\frac{1-p}{1-u_0}\right)q_X(p)dp,\quad 0<u_0\le 1. \tag{4.2}$$

Note that for $r=\frac{1}{2}$, above relations give the right tail deviation measure $D(X)$ [32] and quantile form of it $D(X;u_0)$, respectively. In following examples we consider some statistical models to comparing $|\xi J(X;Q_X(u_0))|$, $R(X;r,u_0)$ and $D(X;u_0)$.

**Example 4.1.** Consider some statistical models in Table 1. For uniform distribution we have

$$|\xi J(X;Q_X(u_0))|=\frac{(b-a)(1-u_0)}{6},\quad R(X;r,u_0)=(b-a)\left(\frac{1-u_0}{r+1}-\frac{1-u_0}{2}\right),$$
$$D(X;u_0)=\frac{(b-a)(1-u_0)}{6}.$$

For Pareto type II distribution we obtain

$$|\xi J(X;Q_X(u_0))|=\frac{\alpha(1-u_0)^{-\frac{1}{\beta}}}{2(2\beta-1)},\quad R(X;r,u_0)=\alpha((1-u_0)^{-\frac{1}{\beta}})\left(\frac{1}{\beta r-1}-\frac{1}{\beta-1}\right),$$
$$D(X;u_0)=\alpha((1-u_0)^{-\frac{1}{\beta}})\left(\frac{2}{\beta-2}-\frac{1}{\beta-1}\right).$$

Figures 3 to 7 show some plots of the functions $|\xi J(X;Q_X(u_0))|$, $R(X;r,u_0)$ and $D(X;u_0)$ for some choices of parameters. For uniform distribution, we can see that $R(X;r,u_0)$ decreases when $r$ grows, and generally when $u_0$ becomes larger. Moreover, Figure 3 gives following results for uniform distribution

$$|\xi J(X;Q_X(u_0))|<R(X;r,u_0),\quad 0<r<\frac{1}{2},$$
$$|\xi J(X;Q_X(u_0))|>R(X;r,u_0),\quad \frac{1}{2}<r\le 1,$$
$$|\xi J(X;Q_X(u_0))|=D(X;u_0).$$

For Pareto type II distribution, we can see that $R(X;r,u_0)$ is a nondecreasing function of $u_0$. But, it decreases when $r$ grows. Moreover, Table 2 gives some results for Pareto type II distribution from the Figures 4 to 6.

**Example 4.2.** Let X be a rv with no closed qdf in Example 3.2. After some calculations, we get

$$|\xi J(X;Q_X(u_0))|=\frac{K}{2(1-u_0)^2}(\bar\beta_{u_0}(V+1,-(V+B)+3)),$$

and

$$R(X;r,u_0)=K\left(\frac{\bar\beta_{u_0}(V+1,-(V+B)+r+1)}{(1-u_0)^r}-\frac{\bar\beta_{u_0}(V+1,-(V+B)+2)}{(1-u_0)}\right),$$

while $D(X;u_0)=R(X;\frac{1}{2},u_0)$.

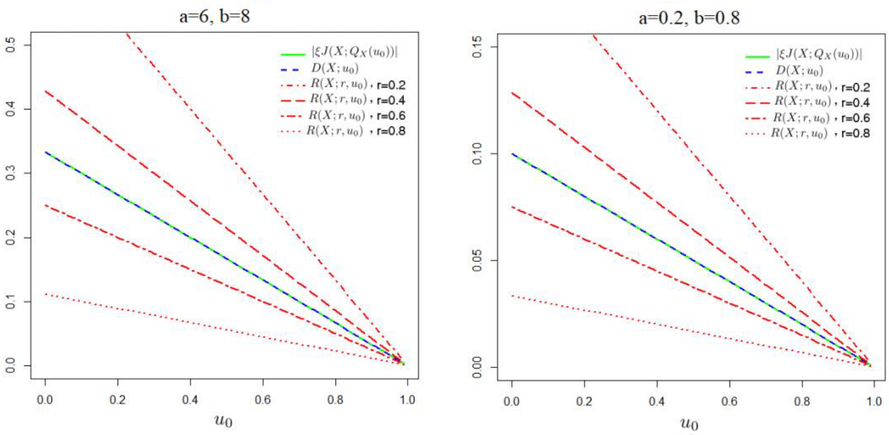


**Figure 3.** Graphs of $|\xi J(X;Q_X(u_0))|$, $D(X;u_0)$ and $R(X;r,u_0)$ for uniform distribution in Example 4.1 with parameters as indicated.

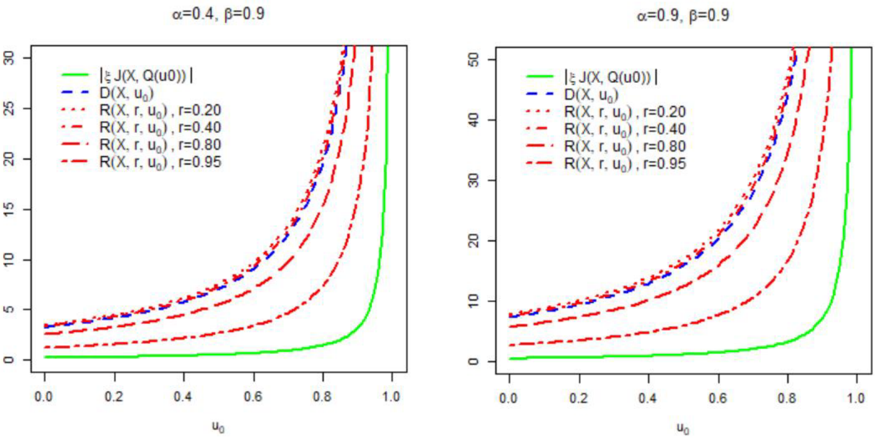


**Figure 4.** Graphs of $|\xi J(X;Q_X(u_0))|$, $D(X;u_0)$ and $R(X;r,u_0)$ for Pareto type II distribution in Example 4.1 with parameters as indicated.

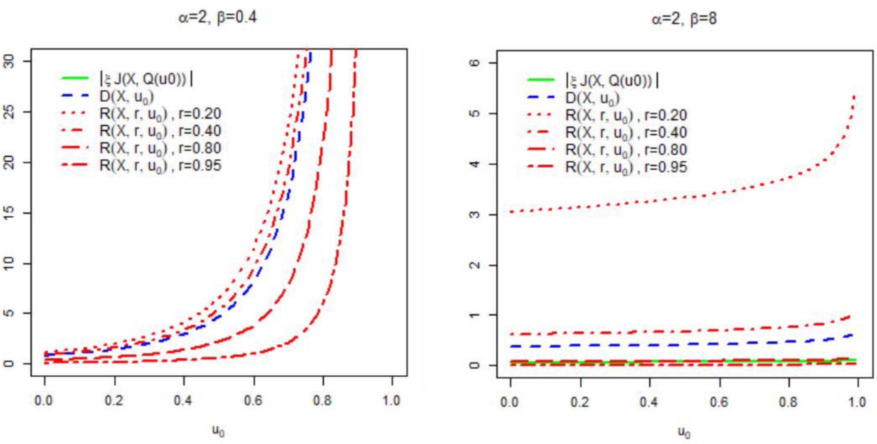


**Figure 5.** Graphs of $|\xi J(X;Q_X(u_0))|$, $D(X;u_0)$ and $R(X;r,u_0)$ for Pareto type II distribution in Example 4.1 with parameters as indicated.

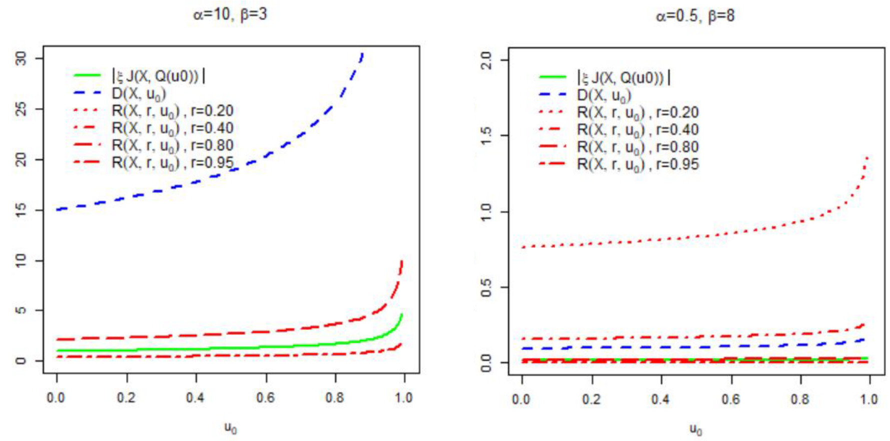


**Figure 6.** Graphs of $|\xi J(X; Q_X(u_0))|$, $D(X; u_0)$ and $R(X; r, u_0)$ for Pareto type II distribution in Example 4.1 with parameters as indicated.

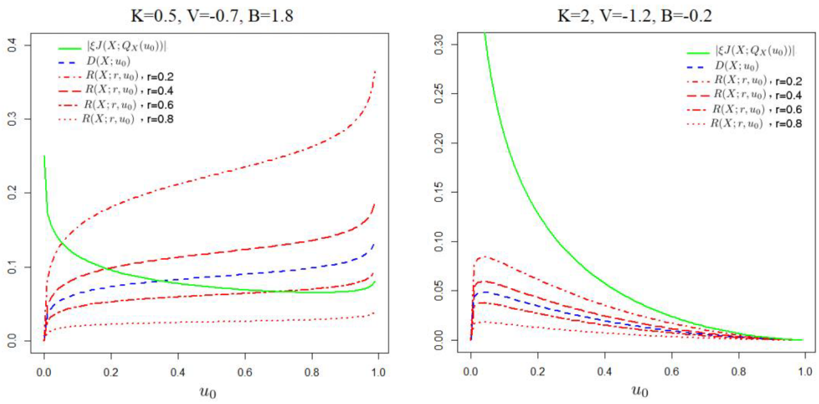


**Figure 7.** Graphs of $|\xi J(X; Q_X(u_0))|$, $D(X; u_0)$ and $R(X; r, u_0)$ in Example 4.2 for parameters as indicated.

**Table 2.** Comparison between $|\xi J(X; Q_X(u_0))|$ and $R(X; r, u_0)$ of Pareto type II distribution in Example 4.1 for some choices of parameters.

| Parameters | Conclusions |
|---|---|
| $\alpha = 0.9, \beta = 0.9$ | $\mid \xi J(X; Q_X(u_0)) \mid < R(X; r, u_0)$, for $0 < r < 1$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.95, u_0)$ |
| $\alpha = 0.4, \beta = 0.9$ | $\mid \xi J(X; Q_X(u_0)) \mid < R(X; r, u_0)$, for $0 < r < 1$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.95, u_0)$ |
| $\alpha = 2, \beta = 0.4$ | $\mid \xi J(X; Q_X(u_0)) \mid < (>) R(X; r, u_0)$, for $0 < r < 0.75$ $(0.75 < r < 1)$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.75, u_0)$ |
| $\alpha = 2, \beta = 8$ | $\mid \xi J(X; Q_X(u_0)) \mid < R(X; r, u_0)$, for $0 < r < 1$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.9, u_0)$ |
| $\alpha = 10, \beta = 3$ | $\mid \xi J(X; Q_X(u_0)) \mid < (>) R(X; r, u_0)$, for $0 < r < 0.75$ $(0.75 < r < 1)$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.75, u_0)$ |
| $\alpha = 0.5, \beta = 8$ | $\mid \xi J(X; Q_X(u_0)) \mid < R(X; r, u_0)$, for $0 < r < 1$<br>$\mid \xi J(X; Q_X(u_0)) \mid$ is relatively close to $R(X; 0.9, u_0)$ |

Figure 7 provides the graphs of $|\xi J(X; Q_X(u_0))|$, $R(X; r, u_0)$ and $D(X; u_0)$ for some choices of parameters. For $K = 0.5, V = -0.7, B = 1.8$, we see that $R(X; r, u_0)$ increases when $r$ grows, and generally when $u_0$ becomes larger. We observe that $|\xi J(X; Q_X(u_0))|$

is a nonincreasing function of $u_0$. Moreover, we find that

$$|\xi J(X; Q_X(0.05375))| = R(X; 0.2, 0.05375) = 0.13259$$
$$|\xi J(X; Q_X(0.18735))| = R(X; 0.4, 0.18735) = 0.09762$$
$$|\xi J(X; Q_X(0.69758))| = R(X; 0.6, 0.69758) = 0.06659.$$

For $K = 2, V = -1.2, B = -0.2$, we see that $R(X; r, u_0)$ decreases when $r$ grows, and generally when $u_0$ becomes larger. We observe that $|\xi J(X; Q_X(u_0))|$ is a nonincreasing functions of $u_0$. Moreover, we find that $|\xi J(X; Q_X(u_0))| > R(X; r, u_0)$ for all $0 < r \leq 1$.

**Example 4.3.** To study the application of our results in risk analysis, we have a set of 35 observations on the hurricane loss during the period 1949-1980 given in [18]. To examine behaviour of some risk measures on hurricane loss data we consider $C_Q(u)$, $CVaR(X; u)$ and $\xi J(X; u)$ measures which are based on quantile function. The $C_Q(u)$ risk measure was proposed and studied by [16]. Figure 8 gives the estimates of considered risk measures. It is seen that all three risk measures are increasing for hurricane loss data.

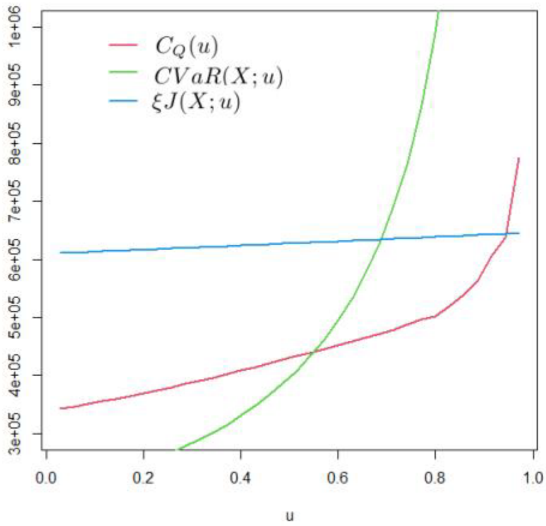


**Figure 8.** Estimate of $C_Q(u)$, $CVaR(X; u)$ and $\xi J(X; u)$ risk measures for hurricane loss data.

## 5. Conclusion

The quantile approach is a worthy tool in information theory. In this paper, we present a quantile form of dynamic survival extropy and study some properties of it. Some highlights of the proposed quantile-based dynamic survival extropy are as follows:

- It is a quantile measure of uncertainty, which can measure the uncertainty of residual life $X_{Q_X(u)}$.
- It can be used for statistical models that do not have explicitly known cdfs, though it has closed form qfs.
- The quantile-based dynamic survival extropy uniquely determines the quantile density function.
- We present quantile-based dynamic survival extropy of $X_{j:n}$, $j = 1, n$, which can be used to compare the uncertainties of residual lives of two series (parallel) systems at the age points $Q_{X_{j:n}}(u)$ and $Q_{Y_{j:n}}(u)$ at which $X_{j:n}$ and $Y_{j:n}$ possess equally survival probabilities.

- Unlike sf-based information measures, the absolute value of quantile-based dynamic survival extropy can be used as risk measure for some random variables having no closed form sfs.

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