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On Matrix Sequences of Narayana and Narayana-Lucas Numbers

Narayana ve Narayana-Lucas Sayılarının Matris Dizileri

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Abstract

In this paper, Narayana and Narayana-Lucas matrix sequences are defined and their properties are investigated.

Keywords: Narayana numbers, Narayana sequence, Narayana matrix sequence, Narayana-Lucas matrix sequence.

Öz

Bu makalede, Narayana ve Narayana-Lucas matris dizileri tanımlandı ve özellikleri incelendi.

Anahtar Kelimeler: Narayana sayıları, Narayana dizisi, Narayana matris dizisi, Narayana-Lucas matris dizisi.

1. Introduction

In this paper, the matrix sequences of Narayana and Narayana -Lucas numbers were defined for the first time in the literature. Then, by giving the generating functions, the Binet formulas, and summation formulas over these new matrix sequences, we will obtain some fundamental properties on Narayana and Narayana-Lucas numbers. Also, we will present the relationship between these matrix sequences.

First, we give some background about Narayana and Narayana-Lucas numbers.

Narayana sequence $\{N_n\}_{n\geq 0}$ (sequence A000930 in Sloane, available: http://oeis.org) and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ (sequence A001609 in Sloane, available: http://oeis.org) are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = N_{n+2} + N_n,\ N_0 = 0,\ N_1 = 1, N_2 = 1, \tag{1}$$

$$U_{n+3} = U_{n+2} + U_n,\ U_0 = 3,\ U_1 = 1, U_2 = 1, \tag{2}$$

The sequences $\{N_n\}_{n\geq 0}, \{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)},$$

$$U_{-n} = -U_{-(n-2)} + U_{-(n-3)},$$

for n=1,2,3,... respectively. Therefore, recurrences (1) and (2) hold for all integer n. For more details on generalized Narayana numbers, see (Soykan 2020).

We can give some relations between $\{N_n\}$ and $\{U_n\}$ as

$$U_n = 3N_{n+4} - 5N_{n+3} + 2N_{n+2}, \tag{3}$$

$$U_n = -2N_{n+3} + 2N_{n+2} + 3N_{n+1}, \tag{4}$$

$$U_n = 3N_{n+1} - 2N_n, \tag{5}$$

$$U_n = N_n + 3N_{n-2}, \tag{6}$$

and

$$31N_n = -3U_{n+4} + U_{n+3} + 11U_{n+2}, \tag{7}$$

$$31N_n = -2U_{n+3} + 11U_{n+2} - 3U_{n+1}, \tag{8}$$

$$31N_n = 9U_{n+2} - 3U_{n+1} - 2U_n, \tag{9}$$

$$31N_n = 6U_{n+1} - 2U_n + 9U_{n-1}, \tag{10}$$

$$31N_n = 4U_n + 9U_{n-1} + 6U_{n-2}. \tag{11}$$

Note that all the above identities hold for all integers n (for more details, see Soykan 2020)). Next, we present the first few values of the Narayana and Narayana-Lucas numbers with positive and negative subscripts:

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Table 1. A few values of the Narayana and Narayana-Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
N_n	0	1	1	1	2	3	4	6	9	13	19	28	41	60
N_{-n}		0	1	0	-1	1	1	-2	0	3	-2	-3	5	1
U_n	3	1	1	4	5	6	10	15	21	31	46	67	98	144
U_{-n}		0	-2	3	2	-5	1	7	-6	-6	13	0	-19	13

It is well known that (see for example (Soykan 2020)) for all integers n, usual Narayana and Narayana-Lucas numbers can be expressed using Binet's formulas

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \quad (12)$$

$$U_n = \alpha^n + \beta^n + \gamma^n, \quad (13)$$

respectively, where, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$. Moreover,

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$

$$\beta = \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$

$$\gamma = \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right).$$

Note that

$$\alpha + \beta + \gamma = 1, \alpha\beta + \alpha\gamma + \beta\gamma = 0, \alpha\beta\gamma = 1.$$

The generating functions for the Narayana sequence $\{N_n\}_{n\geq 0}$ and Narayana-Lucas sequance $\{U_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} N_n x^n = \frac{x}{1-x-x^3} \text{ and } \sum_{n=0}^{\infty} U_n x^n = \frac{3-2x}{1-x-x^3}, \quad (14)$$

respectively.

2. The Matrix Sequences of Narayana and Narayana-Lucas Numbers

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadan (Generalized Fibonacci) numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third-order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. On the other hand, the matrix sequences have taken so much interest for different type of numbers. We present some works on matrix sequences of the numbers in the following Table 2.

Table 2. A few special study on the matrix sequences of the number

Name of sequence	work on the matrix sequences of the numbers
Generalized Fibonacci	(Civciv and Turkmen 2008), (Civciv and Turkmen 2008a), (Gulec and Taskara 2012), (Uslu and Uygun 2013), (Uygun and Uslu 2016), (Uygun 2016), (Uygun 2019), (Yazlik et al. (2012), Wani et al. (2018)).
Generalized Tribonacci	(Cerda-Morales 2019), (Soykan 2020a), (Soykan 2020b), (Yilmaz and Taskara 2013), (Yilmaz and Taskara 2014).
Generalized Tetranacci	(Soykan 2019).

In this section we define Narayana and Narayana-Lucas matrix sequences and investigate their properties.

Definition 1. For any integer $n \geq 0$, the Narayana matrix $(\mathcal{N}_n)$ and Narayana-Lucas matrix $(\mathcal{U}_n)$ are defined by

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, \quad (15)$$

$$\mathcal{U}_n = \mathcal{U}_{n-1} + \mathcal{U}_{n-3}, \quad (16)$$

respectively, with initial conditions

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{N}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_0 = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 0 \\ 0 & 3 & -2 \end{pmatrix}, U_1 = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 3 \\ 3 & -2 & 0 \end{pmatrix}, \mathcal{U}_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

The sequences $\{\mathcal{N}_n\}_{n\geq 0}$ and $\{\mathcal{U}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{N}_{-n} = -\mathcal{N}_{-(n-2)} + \mathcal{N}_{-(n-3)}$$

and

$$\mathcal{U}_{-n} = -\mathcal{U}_{-(n-2)} + \mathcal{U}_{-(n-3)}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (15) and (16) hold for all integers n.

The following theorem gives the nth general terms of the Narayana and Narayana-Lucas matrix sequences.

Theorem 1. For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\mathcal{N}_n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}, \tag{17}$$

$$\mathcal{U}_n = \begin{pmatrix} U_{n+1} & U_{n-1} & U_n \\ U_n & U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-3} & U_{n-2} \end{pmatrix}. \tag{18}$$

Proof. We prove (17) by strong mathematical induction on n. (18) can be proved similarly.

If $n = 0$ then, $N_0 = 0, N_1 = 1, N_2 = 1, N_{-1} = 0, N_{-2} = 1, N_{-3} = 0$, we have

$$\mathcal{N}_0 = \begin{pmatrix} N_1 & N_{-1} & N_0 \\ N_0 & N_{-2} & N_{-1} \\ N_1 & N_{-3} & N_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true and

$$\mathcal{N}_1 = \begin{pmatrix} N_2 & N_0 & N_1 \\ N_1 & N_{-1} & N_0 \\ N_0 & N_{-2} & N_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is true. Asume that the equality holds for $n \leq k$. For $n = k+1$, we have $\mathcal{N}_{k+1} = N_k + N_{k-2}$

$$\begin{aligned}
&= \begin{pmatrix} N_{k+1} & N_{k-1} & N_k \\ N_k & N_{k-2} & N_{k-1} \\ N_{k-1} & N_{k-3} & N_{k-2} \end{pmatrix} + \begin{pmatrix} N_{k-1} & N_{k-3} & N_{k-2} \\ N_{k-2} & N_{k-4} & N_{k-3} \\ N_{k-3} & N_{k-5} & N_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} N_{k-1}+N_{k+1} & N_{k-1}+N_{k-3} & N_k+N_{k-2} \\ N_k+N_{k-2} & N_{k-2}+N_{k-4} & N_{k-1}+N_{k-3} \\ N_{k-1}+N_{k-3} & N_{k-3}+N_{k-5} & N_{k-2}+N_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} N_{k+2} & N_k & N_{k+1} \\ N_{k+1} & N_{k-1} & N_k \\ N_k & N_{k-2} & N_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} N_{k+1+1} & N_{k+1-1} & N_{k+1} \\ N_{k+1} & N_{k+1-2} & N_{k+1-1} \\ N_{k+1-1} & N_{k+1-3} & N_{k+1-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k+1$, this proves (17).

We now give the Binet formulas for the Narayana and Narayana-Lucas matrix sequences.

Theorem 2. For every integer n, the Binet formulas of the Narayana and Narayana-Lucas matrix sequences are given by

$$\mathcal{N}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \tag{19}$$

$$\mathcal{U}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n. \tag{20}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha\mathcal{N}_2 + \alpha(\alpha-1)\mathcal{N}_1 + \mathcal{N}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \\
B_1 &= \frac{\beta\mathcal{N}_2 + \beta(\beta-1)\mathcal{N}_1 + N_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \\
C_1 &= \frac{\gamma\mathcal{N}_2 + \gamma(\gamma-1)\mathcal{N}_1 + \mathcal{N}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)} \\
A_2 &= \frac{\alpha\mathcal{U}_2 + \alpha(\alpha-1)\mathcal{U}_1 + \mathcal{U}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \\
B_2 &= \frac{\beta\mathcal{U}_2 + \beta(\beta-1)\mathcal{U}_1 + \mathcal{U}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \\
C_2 &= \frac{\gamma\mathcal{U}_2 + \gamma(\gamma-1)\mathcal{U}_1 + \mathcal{U}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.
\end{aligned}$$

Proof. We need to prove the theorem only for $n \geq 0$. We prove (19). By the assumption, the characteristic equation of (15) is $x^3 - x^2 - 1 = 0$ and the roots of it are α, β and γ. So it's general solution is given by

$$\mathcal{N}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n.$$

Using initial condition which is given in Definition 1, and also applying lineer algebra operations, we obtain the matrices A_1, B_1, C_1 as desired. This gives the formula for $\mathcal{N}_n$.

Similarly we have the formula (20).

The well known Binet formulas for Narayana and Narayana-Lucas numbers are given in (12) and (13) respectively. But we will obtain these functions in terms of Narayana and Narayana-Lucas matrix sequences as a consequence of Theorems 1 and 2. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 1. For every integers n, the Binet's formulas for Narayana and Narayana-Lucas numbers are given as

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$
$$U_n = \alpha^n + \beta^n + \gamma^n.$$

Proof. From Theorem 2, we have

$$\begin{aligned}\mathcal{N}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n \\ &= \frac{\alpha\mathcal{N}_2 + \alpha(\alpha-1)\mathcal{N}_1 + \mathcal{N}_0}{\alpha(\alpha-\gamma)(\alpha-\gamma)}\alpha^n + \\ &\frac{\beta\mathcal{N}_2 + \beta(\beta-1)\mathcal{N}_1 + \mathcal{N}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n + \frac{\gamma\mathcal{N}_2 + \gamma(\gamma-1)\mathcal{N}_1 + \mathcal{N}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix}\alpha^2+1 & \alpha & \alpha^2 \\ \alpha^2 & 1 & \alpha \\ \alpha & \alpha(\alpha-1) & 1\end{pmatrix} \\ &+ \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix}\beta^2+1 & \beta & \beta^2 \\ \beta^2 & 1 & \beta \\ \beta & \beta(\beta-1) & 1\end{pmatrix} \\ &+ \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix}\gamma^2+1 & \gamma & \gamma^2 \\ \gamma^2 & 1 & \gamma \\ \gamma & \gamma(\gamma-1) & 1\end{pmatrix}.\end{aligned}$$

By Theorem 1, we known that

$$\mathcal{N}_n = \begin{pmatrix} N_{n+1} & N_n + N_{n-1} & N_n \\ N_n & N_{n-1}+N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-2}+N_{n-3} & N_{n-2}\end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}N_n &= \frac{\alpha^{n-1}\alpha^2}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}\beta^2}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}\gamma^2}{(\gamma-\beta)(\gamma-\alpha)} \\ &= \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\beta)(\gamma-\alpha)}.\end{aligned}$$

From Theorem 2, we obtain

$$\begin{aligned}\mathcal{U}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n \\ &= \frac{\alpha\mathcal{U}_2 + \alpha(\alpha-1)\mathcal{U}_1 + \mathcal{U}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n \\ &+ \frac{\beta\mathcal{U}_2 + \beta(\beta-1)\mathcal{U}_1 + \mathcal{U}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\ &+ \frac{\gamma\mathcal{U}_2 + \gamma(\gamma-1)\mathcal{U}_1 + \mathcal{U}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix}\alpha^2+3\alpha+1 & \alpha(3\alpha-2) & \alpha^2+3 \\ \alpha^2+3 & 3\alpha-2 & \alpha(3\alpha-2) \\ \alpha(3\alpha-2) & -2\alpha^2+2\alpha+3 & 3\alpha-2\end{pmatrix} \\ &+ \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix}\beta^2+3\beta+1 & \beta(3\beta-2) & \beta^2+3 \\ \beta^2+3 & 3\beta-2 & \beta(3\beta-2) \\ \beta(3\beta-2) & -2\beta^2+2\beta+3 & 3\beta-2\end{pmatrix} \\ &+ \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix}\gamma^2+3\gamma+1 & \gamma(3\gamma-2) & \gamma^2+3 \\ \gamma^2+3 & 3\gamma-2 & \gamma(3\gamma-2) \\ \gamma(3\gamma-2) & -2\gamma^2+2\gamma+3 & 3\gamma-2\end{pmatrix}.\end{aligned}$$

By Theorem 1, we known that

$$\mathcal{U}_n = \begin{pmatrix} U_{n+1} & U_n + U_{n-1} & U_n \\ U_n & U_{n-1}+U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2}+U_{n-3} & U_{n-2}\end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above last two equations, then we obtain

$$U_n = \frac{\alpha^{n-1}(\alpha^2+3)}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}(\beta^2+3)}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}(\gamma^2+3)}{(\gamma-\beta)(\gamma-\alpha)}.$$

Using the relations, $\alpha+\beta+\gamma=1, \alpha\beta\gamma=1$ and considering α,β and γ are the roots the equation $x^3-x^2-1=0$, we obtain

$$\begin{aligned}\frac{\alpha^2+3}{(\alpha-\gamma)(\alpha-\beta)} &= \frac{\alpha^2+3}{(\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma)} \\ &= \frac{\alpha}{\alpha}\frac{(\alpha^2+3)}{\alpha^2+\alpha(-\beta-\gamma)+\beta\gamma} = \frac{(\alpha^2+3)\alpha}{\alpha^3+\alpha^2(\alpha-1)+1} \\ &= \frac{(\alpha^2+3)\alpha}{2\alpha^3-\alpha^2+1} = \frac{(\alpha^2+3)\alpha}{2(\alpha^2+1)-\alpha^2+1} = \frac{(\alpha^2+3)\alpha}{\alpha^2+3} = \alpha, \\ \frac{\beta^2+3}{(\beta-\gamma)(\beta-\alpha)} &= \beta, \\ \frac{\gamma^2+3}{(\gamma-\beta)(\gamma-\alpha)} &= \gamma.\end{aligned}$$

So, we conclude that

$U_n = \alpha^n + \beta^n + \gamma^n$

as required.

Now, we present summation formulas for Narayana and Narayana-Lucas matrix sequences.

Theorem 3. For all m,j we have

$$\sum_{k=0}^{n-1}\mathcal{N}_{mk+j} = \frac{\mathcal{N}_{mn+m+j} + \mathcal{N}_{mn-m+j} + (1-U_m)\mathcal{N}_{mn+j} - N_{m+j} - N_{j-m} + (U_m-1)\mathcal{N}_j}{U_m + (1-U_{-m}) - 1} \tag{21}$$

and

$$\sum_{k=0}^{n-1}\mathcal{U}_{mk+j} = \frac{\mathcal{U}_{mn+m+j} + \mathcal{U}_{mn-m+j} + (1-U_m)\mathcal{U}_{mn+j} - \mathcal{U}_{m+j} - \mathcal{U}_{j-m} + (U_m-1)\mathcal{U}_j}{U_m + (1-U_{-m}) - 1}. \tag{22}$$

Proof. Note that

$$\begin{aligned}\sum_{i=0}^{n-1}\mathcal{N}_{mi+j} &= \sum_{i=0}^{n-1}(A_1\alpha^{mi+j} + B_1\beta^{mi+j} + C_1\gamma^{mi+j}) \\ &= A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + B_1\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + C_1\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right)\end{aligned}$$

and

$$\sum_{i=0}^{n-1}\mathcal{U}_{mi+j}=\sum_{i=0}^{n-1}\left(A_2\alpha^{mi+j}+B_2\beta^{mi+j}+C_2\gamma^{mi+j}\right)$$
$$=A_2\alpha^{j}\left(\frac{\alpha^{mn}-1}{\alpha^{m}-1}\right)+B_2\beta^{j}\left(\frac{\beta^{mn}-1}{\beta^{m}-1}\right)+C_2\gamma^{j}\left(\frac{\gamma^{mn}-1}{\gamma^{m}-1}\right).$$

Simplifying and rearranging the last equalities in the last two expression imply (21) and (22) as required.

As in Corollary 1, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

Corollary 2. For all m,j we have

$$\sum\nolimits_{k=0}^{n-1}N_{mk+j}$$
$$=\frac{N_{mn+m+j}+N_{mn-m+j}+(1+U_m)N_{mn+j}-N_{m+j}-N_{j-m}+(U_m-1)N_j}{U_m+(1-U_{-m})-1},$$
$$\sum\nolimits_{k=0}^{n-1}U_{mk+j}$$
$$=\frac{U_{mn+m+j}+U_{mn-m+j}+(1+U_m)U_{mn+j}-U_{m+j}-U_{j-m}+(U_m-1)U_j}{U_m+(1-U_{-m})-1}.$$

Note that using the above Corollary we obtain the followinng well known formulas (taking $m=1,j=0$):

$\sum\nolimits_{k=0}^{n-1}N_k=N_{n+1}+N_{n-1}-1$ and $\sum\nolimits_{k=0}^{n-1}U_k=U_{n+1}+U_{n-1}-1.$

We now give generating functions of $\mathcal{N}$ and $\mathcal{U}$.

Theorem 4. The generating function for the Narayana and Narayana-Lucas matrix sequences are given as

$$\sum_{n=0}^{\infty}\mathcal{N}_n x^n=\frac{1}{1-x-x^3}\begin{pmatrix}1 & x^2 & x\\ x & 1-x & x^2\\ x^2 & x-x^2 & 1-x\end{pmatrix}$$

and

$$\sum_{n=0}^{\infty}\mathcal{U}_n x^n=\frac{1}{1-x-x^3}\begin{pmatrix}3x^2+1 & 3x-2x^2 & 3-2x\\ 3-2x & 3x^2+2x-2 & 3x-2x^2\\ 3x-2x^2 & 2x^2-5x+3 & 3x^2+2x-2\end{pmatrix}$$

respectively.

Proof. We prove the Narayana case. Suppose that $g(x)=\sum_{n=0}^{\infty}\mathcal{N}_n x^n$ is the generating function fort he sequence $\{\mathcal{N}_n\}_{n\geq 0}$. Using the definition of the matrix sequence of Narayana numbers (15), and substracting $x\sum_{n=0}^{\infty}\mathcal{N}_n x^n$ and $x^3\sum_{n=0}^{\infty}\mathcal{N}_n x^n$ from $\sum_{n=0}^{\infty}\mathcal{N}_n x^n$ we obtain

$$(1-x-x^3)\sum_{n=0}^{\infty}\mathcal{N}_n x^n=\sum_{n=0}^{\infty}\mathcal{N}_n x^n-x\sum_{n=0}^{\infty}\mathcal{N}_n x^n-x^3\sum_{n=0}^{\infty}\mathcal{N}_n x^n$$
$$=\sum_{n=0}^{\infty}\mathcal{N}_n x^n-\sum_{n=0}^{\infty}\mathcal{N}_n x^{n+1}-\sum_{n=0}^{\infty}\mathcal{N}_n x^{n+3}$$
$$=\sum_{n=0}^{\infty}\mathcal{N}_n x^n-\sum_{n=1}^{\infty}\mathcal{N}_{n-1} x^n-\sum_{n=3}^{\infty}\mathcal{N}_{n-3} x^n$$
$$=(\mathcal{N}_0+\mathcal{N}_1x+\mathcal{N}_2x^2)-(\mathcal{N}_0x+\mathcal{N}_1x^2)$$
$$+\sum_{n=3}^{\infty}(\mathcal{N}_n+\mathcal{N}_{n-1}-\mathcal{N}_{n-3})x^n$$
$$=\mathcal{N}_0+\mathcal{N}_1x+\mathcal{N}_2x^2-\mathcal{N}_0x-\mathcal{N}_1x^2$$
$$=\mathcal{N}_0+(\mathcal{N}_1-\mathcal{N}_0)x+(\mathcal{N}_2-\mathcal{N}_1)x^2.$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}\mathcal{N}_n x^n=\frac{\mathcal{N}_0+(\mathcal{N}_1-\mathcal{N}_0)x+(\mathcal{N}_2-\mathcal{N}_1)x^2}{1-x-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{N}_n x^n$ in the Theorem. This completes the proof.

Narayana-Lucas case can be proved similarly.

The well known generating functions for Narayana and Narayana-Lucas numbers are as in (14). However, we will obtain these functions in terms of Narayana and Narayana-Lucas matrix sequences as as consequence of Theorem 4. To do this, we will again compare the 2nd row and 1st column entries with the matrices in Theorem 4. Thus we have the following corollary.

Corollary 3. The generating functions for the Narayana sequence $\{N_n\}_{n\geq 0}$ and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ are given as

$\sum\nolimits_{n=0}^{\infty}N_n x^n=\dfrac{x}{1-x-x^3}$ and $\sum\nolimits_{n=0}^{\infty}U_n x^n=\dfrac{3-2x}{1-x-x^3}$

respectively.

3.Relation Between Narayana and Narayana-Lucas Matrix Sequences

The following theorem shows that there always exist interrelation between Narayana and Narayana-Lucas matrix sequences.

Theorem 5. For the matrix sequences $\{\mathcal{N}_n\}$ and $\{\mathcal{U}_n\}$, we have the following identities.

(a): $\mathcal{U}_n=3\mathcal{N}_{n+4}-5\mathcal{N}_{n+3}+2\mathcal{N}_{n+2}.$

(b): $\mathcal{U}_n=-2\mathcal{N}_{n+3}+2\mathcal{N}_{n+2}+3\mathcal{N}_{n+1}.$

(c): $\mathcal{U}_n=3\mathcal{N}_{n+1}-2\mathcal{N}_n.$

(d): $\mathcal{U}_n=\mathcal{N}_n+3\mathcal{N}_{n-2}.$

(e): $31\mathcal{N}_n=-3\mathcal{U}_{n+4}+\mathcal{U}_{n+3}+11\mathcal{U}_{n+2}.$

(f): $31\mathcal{N}_n=-2\mathcal{U}_{n+3}+11\mathcal{U}_{n+2}-3\mathcal{U}_{n+1}.$

(g): $31\mathcal{N}_n=9\mathcal{U}_{n+2}-3\mathcal{U}_{n+1}-2\mathcal{U}_n.$

(h): $31\mathcal{N}_n=6\mathcal{U}_{n+1}-2\mathcal{U}_n+9\mathcal{U}_{n-1}.$

(i): $31\mathcal{N}_n=4\mathcal{U}_n+9\mathcal{U}_{n-1}+6\mathcal{U}_{n-2}.$

Proof. From (3)-(6) and (7)-(11), (a)-(i) follow.

Lemma 1. For all non-negative integers m and n, we have the following identities.

(a): $\mathcal{U}_0\mathcal{N}_n = \mathcal{N}_n\mathcal{U}_0 = \mathcal{U}_n$,

(b): $\mathcal{N}_0\mathcal{U}_n = \mathcal{U}_n\mathcal{N}_0 = \mathcal{U}_n$.

Proof. Identities can be established easily. Note that to show (a) we need to use the relations (3)-(6).

To prove the following Theorem we need the next Lemma.

Lemma 2. Let $A_1,B_1,C_1;A_2,B_2,C_2$ as in Theorem 2. Then the following relations hold:

$$A_1^2 = A_1, B_1^2 = B_1, C_1^2 = C_1,$$
$$A_1B_1 = B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0),$$
$$A_2B_2 = B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0).$$

Proof. Using $\alpha+\beta+\gamma=1, \alpha\beta+\alpha\gamma+\beta\gamma=0$ and $\alpha\beta\gamma=1$, required equalities can be established by matrix calculations.

Theorem 6. For all non-negative integers m and n, we have the following identities.

(a): $\mathcal{N}_m\mathcal{N}_n = \mathcal{N}_{m+n} = \mathcal{N}_n\mathcal{N}_m$.

(b): $\mathcal{N}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{N}_m = \mathcal{U}_{m+n}$.

(c): $\mathcal{U}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{U}_m = 9\mathcal{N}_{m+n+8} - 30\mathcal{N}_{m+n+7}$
$37\mathcal{N}_{m+n+6} - 20\mathcal{N}_{m+n+5} + 4\mathcal{N}_{m+n+4}$.

(d): $\mathcal{U}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{U}_m = 4\mathcal{N}_{m+n+6} - 8\mathcal{N}_{m+n+5}$
$-8\mathcal{N}_{m+n+4} + 12\mathcal{N}_{m+n+3} + 9\mathcal{N}_{m+n+2}$.

(e): $\mathcal{U}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{U}_m = 9\mathcal{N}_{m+n+2} - 12\mathcal{N}_{m+n+1} + 4\mathcal{N}_{m+n}$.

(f): $\mathcal{U}_m\mathcal{U}_n = \mathcal{U}_n\mathcal{U}_m = \mathcal{N}_{m+n} + 6\mathcal{N}_{m+n-2} + 9\mathcal{N}_{m+n-4}$.

Proof.

(a): Using Lemma 2. we obtain

$$\begin{aligned}\mathcal{N}_m\mathcal{N}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n)\\ &= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n\\ &+B_1A_1\alpha^n\beta^m + A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n\\ &+C_1B_1\beta^n\gamma^m = A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} = \mathcal{N}_{m+n}.\end{aligned}$$

(b): By Lemma 1, we have

$$\mathcal{N}_m\mathcal{U}_n = \mathcal{N}_m\mathcal{N}_n\mathcal{U}_0.$$

Now from (a) and again by Lemma 1. we obtain

$$\mathcal{N}_m\mathcal{U}_n = \mathcal{N}_{m+n}\mathcal{U}_0 = \mathcal{U}_{m+n}.$$

It can be shown similarly that $\mathcal{U}_n\mathcal{N}_m = \mathcal{U}_{m+n}$.

(c): Using (a) and Theorem 5 (a) we obtain

$$\begin{aligned}\mathcal{U}_m\mathcal{U}_n &= (3\mathcal{N}_{m+4} - 5\mathcal{N}_{m+3} + 2\mathcal{N}_{m+2})(3\mathcal{N}_{n+4} - 5\mathcal{N}_{n+3} + 2\mathcal{N}_{n+2})\\ &= 9\mathcal{N}_{m+4}\mathcal{N}_{n+4} - 15\mathcal{N}_{m+3}\mathcal{N}_{n+4} - 15\mathcal{N}_{m+4}\mathcal{N}_{n+3} + 25\mathcal{N}_{m+3}\mathcal{N}_{n+3}\\ &+6\mathcal{N}_{m+2}\mathcal{N}_{n+4} + 6\mathcal{N}_{m+4}\mathcal{N}_{n+2} - 10\mathcal{N}_{m+2}\mathcal{N}_{n+3} - 10\mathcal{N}_{m+3}\mathcal{N}_{n+2}\\ &+4\mathcal{N}_{m+2}\mathcal{N}_{n+2} = 9\mathcal{N}_{m+n+8} - 15\mathcal{N}_{m+n+7} - 15\mathcal{N}_{m+n+7} + 25\mathcal{N}_{m+n+6}\\ &+6\mathcal{N}_{m+n+6} + 6\mathcal{N}_{m+n+6} - 10\mathcal{N}_{m+n+5} - 10\mathcal{N}_{m+n+5} + 4\mathcal{N}_{m+n+4}\\ &= 9\mathcal{N}_{m+n+8} - 30\mathcal{N}_{m+n+7} + 37\mathcal{N}_{m+n+6} - 20\mathcal{N}_{m+n+5} + 4\mathcal{N}_{m+n+4}.\end{aligned}$$

It can be shown similarly that

$$\mathcal{U}_n\mathcal{U}_m = 9\mathcal{N}_{m+n+8} - 30\mathcal{N}_{m+n+7} + 37\mathcal{N}_{m+n+6} - 20\mathcal{N}_{m+n+5}$$
$$+4\mathcal{N}_{m+n+4}.$$

The remaining of identities can be proved by considering again (a) and Theorem 5. Comparing matrix entries and using Theorem 1 we have next result.

Corollary 4. For Narayana and Narayana-Lucas numbers, we have the following identities:

(a): $N_{m+n} = N_mN_{n+1} + N_{m-1}N_{n-1} + N_{m-2}N_n$.

(b): $U_{m+n} = U_mU_{n+1} + U_{m-1}U_{n-1} + U_{m-2}U_n$.

(c): $U_mU_{n+1} + U_{m-1}U_{n-1} + U_{m-2}U_n = 9N_{m+n+8} - 30N_{m+n+7}$
$+37N_{m+n+6} - 20N_{m+n+5} + 4N_{m+n+4}$.

(d): $U_mU_{n+1} + U_{m-1}U_{n-1} + U_{m-2}U_n = 4N_{m+n+6} - 8N_{m+n+5}$
$-8N_{m+n+4} + 12N_{m+n+3} + 9N_{m+n+2}$.

(e): $U_mU_{n+1} + U_{m-1}U_{n-1} + U_{m-2}U_n = 9N_{m+n+2} - 12N_{m+n+1}$
$+4N_{m+n}$.

(f): $U_mU_{n+1} + U_{m-1}U_{n-1} + U_{m-2}U_n = N_{m+n} + 6N_{m+n-2}$
$+9N_{m+n-4}$.

Proof.

(a): From Theorem 6. we know that $\mathcal{N}_m\mathcal{N}_n = \mathcal{N}_{m+n}$. Using Theorem 1, we can write this result as

$$\begin{pmatrix} N_{m+1} & N_{m-1} & N_m \\ N_m & N_{m-2} & N_{m-1} \\ N_{m-1} & N_{m-3} & N_{m-2} \end{pmatrix}\begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} N_{m+n+1} & N_{m+n-1} & N_{m+n} \\ N_{m+n} & N_{m+n-2} & N_{m+n-1} \\ N_{m+n-1} & N_{m+n-3} & N_{m+n-2} \end{pmatrix}.$$

Now, by multiplying the left-side matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identity in (a).

The remaining of identities can be proved by considering again Theorems 6 and 1.

The next two theorems provide us the convenience to obtain the powers of Narayana and Narayana-Lucas matrix sequences.

Theorem 7. For non-negative integers m,n and r with $n \geq r$, the following identities hold:

(a): $\mathcal{N}_n^m = \mathcal{N}_{mn}$,

(b): $\mathcal{N}_{n+1}^m = \mathcal{N}_1^m \mathcal{N}_{mn}$,

(c): $\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_n^2 = \mathcal{N}_2^n$.

Proof.

(a): We can write $\mathcal{N}_n^m$ as

$$\mathcal{N}_n^m = \mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n (m \; times).$$

Using Theorem 6. (a) iteratively, we obtain the required result:

$$\begin{aligned}
\mathcal{N}_n^m &= \underbrace{\mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n}_{m \; times} \\
&= \mathcal{N}_{2n} \underbrace{\mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n}_{m-1 \; times} \\
&= \mathcal{N}_{3n} \underbrace{\mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n}_{m-2 \; times} \\
&\vdots \\
&= \mathcal{N}_{(m-1)n} \mathcal{N}_n \\
&= \mathcal{N}_{mn}.
\end{aligned}$$

(b): As a similar approach in (a) we have

$$\begin{aligned}
\mathcal{N}_{n+1}^m &= \mathcal{N}_{n+1} \mathcal{N}_{n+1} ... \mathcal{N}_{n+1} = \mathcal{N}_m \mathcal{N}_{mn} \\
&= \mathcal{N}_1 \mathcal{N}_{m-1} \mathcal{N}mn.
\end{aligned}$$

Using Theorem 6 (a), we can write iteratively

$$\mathcal{N}_m = \mathcal{N}_1 \mathcal{N}_{m-1}, \mathcal{N}_{m-1} = \mathcal{N}_1 \mathcal{N}_{m-2}, ..., \mathcal{N}_2 = \mathcal{N}_1 \mathcal{N}_1.$$

Now it follows that

$$\mathcal{N}_{n+1}^m = \underbrace{\mathcal{N}_1 \mathcal{N}_1 ... \mathcal{N}_1}_{m \; times} \mathcal{N}_{mn} = \mathcal{N}_1^m \mathcal{N}_{mn}.$$

(c): Theorem 6 (a) gives

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \mathcal{N}_n \mathcal{N}_n = \mathcal{N}_n^2$$

and also

$$\mathcal{N}_{n-r}\mathcal{N}_{n+r} = \mathcal{N}_{2n} = \underbrace{\mathcal{N}_2 \mathcal{N}_2 ... \mathcal{N}_2}_{n \; times} = \mathcal{N}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{U}_n$.

Theorem 8. For non-negative integers m,n and r with $n \geq r$, the following identities hold:

(a): $\mathcal{U}_{n-r}\mathcal{U}_{n+r} = \mathcal{U}_n^2$,

(b): $\mathcal{U}_n^m = \mathcal{U}_0^m \mathcal{N}_{mn}$.

Proof.

(a): We use Binet's formula of Narayana and Narayana-Lucas matrix sequence which is given in Theorem 2. So

$$\begin{aligned}
&\mathcal{U}_{n-r}\mathcal{U}_{n+r} - \mathcal{U}_n^2 \\
&= (A_2\alpha^{n-r} + B_2\beta^{n-1} + C_2\gamma^{n-r})(A_2\alpha^{n+r} + B_2\beta^{n+1} + C_2\gamma^{n+r}) \\
&-(A_2\alpha^n + B_2\beta^n + C_2\gamma^n)^2 \\
&= A_2B_2\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + A_2C_2\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 \\
&+B_2C_2\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 = 0
\end{aligned}$$

since $A_2B_2 = A_2C_2 = C_2B_2$ (see Lemma 3). Now we get the result as required.

(b): By Theorem 8, we have

$$\mathcal{U}_0^m \mathcal{N}_{mn} = \underbrace{\mathcal{U}_0 \mathcal{U}_0 .... \mathcal{U}_0}_{m \; times} . \underbrace{\mathcal{N}_0 \mathcal{N}_0 .... \mathcal{N}_0}_{m \; times}.$$

When we apply Lemma 2 (a) iteratively, it follows that

$$\begin{aligned}
\mathcal{U}_0^m \mathcal{N}_{mn} &= (\mathcal{U}_0 \mathcal{N}_n)(\mathcal{U}_0 \mathcal{N}_n)....(\mathcal{U}_0 \mathcal{N}_n) \\
&= \mathcal{U}_n \mathcal{U}_n ... \mathcal{U}_n = \mathcal{U}_n^m.
\end{aligned}$$

This completes the proof.

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