Advances in the Theory of Nonlinear Analysis and its Applications **4** (2020) No. 4, 332–348.
https://doi.org/10.31197/atnaa.825294
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces

Abdelkrim Salim[a], Mouffak Benchohra[a], Jamal Eddine Lazreg[a], Johnny Henderson[b]

[a] *Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P.O. Box 89 Sidi Bel Abbes 22000, Algeria.*
[b] *Department of Mathematics, Baylor University, Waco, Texas, 76798-7328, USA.*

Abstract

In the present article, we prove some results concerning the existence of solutions for a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. An example is included to show the applicability of our results.

Keywords: Implicit fractional differential equations, generalized Hilfer fractional derivative, initial value problem, existence, measure of noncompactness, fixed point, non-instantaneous impulses, Ulam stability, Banach space.
2010 MSC: 34A08, 34A12, 34A37, 34G20.

1. Introduction

Fractional derivatives and fractional integrals generalize to noninteger order the derivative and the integral of a function. There are several kinds of fractional derivatives, such as, the Riemann–Liouville fractional derivative, the Grunwald–Letnikov fractional derivative, the Caputo derivative, the Marchaud fractional derivative, the generalized Hilfer derivative, etc. [4, 5, 6, 16, 17]. There are numerous books and articles

Email addresses: salimkarim24@yahoo.fr (Abdelkrim Salim), benchohra@yahoo.com (Mouffak Benchohra), lazregjamal@yahoo.fr (Jamal Eddine Lazreg), johnny_henderson@baylor.edu (Johnny Henderson)

Received August 14, 2020, Accepted: November 11, 2020, Online: November 13, 2020.

focused on linear and nonlinear problems for fractional differential equations involving different kinds of fractional derivatives, see, for example, [3, 7, 8, 10, 11, 13].

The class of problems for fractional differential equations with abrupt and instantaneous impulses is vastly studied, and different topics on the existence and qualitative properties of solutions are considered, [15, 18, 28]. In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of drugs into the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [1, 2, 9] the authors studied some new classes of abstract impulsive differential equations with not instantaneous impulses.

The stability of functional equations originated from a question by Ulam [27]. Hyers [21] gave a first affirmative partial answer to Ulam's question for Banach spaces. Hyers Theorem was generalized by Rassias [25] in 1978. Afterwards, many interesting results of the generalized Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations have been investigated by a number of mathematicians; one can see the monograph of Abbas *et al.* [8] and the paper by Rus [26] in which the Ulam-Hyers stability for operator equations is discussed.

Motivated by the works mentioned above, in this paper, we establish existence results for the initial value problem of a nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses,

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right);\ t \in I_k,\ k = 0,\dots,m, \tag{1}$$

$$u(t) = g_k(t, u(t));\ t \in \tilde{I}_k,\ k = 1,\dots,m, \tag{2}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+) = \phi_0, \tag{3}$$

where ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$ and ${}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}$ are, respectively, the generalized Hilfer-type fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and generalized fractional integral of order $1-\gamma, (\gamma = \alpha+\beta-\alpha\beta)$, $\rho > 0$, $\phi_0 \in E$, $I_k := (s_k, t_{k+1}]; k = 0,\dots,m$, $\tilde{I}_k := (t_k, s_k]; k = 1,\dots,m$, $a = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \dots \leq s_{m-1} < t_m \leq s_m < t_{m+1} = b < \infty$, $u(t_k^+) = \lim_{\epsilon\to 0^+} u(t_k+\epsilon)$ and $u(t_k^-) = \lim_{\epsilon\to 0^-} u(t_k+\epsilon)$ represent the right and left hand limits of $u(t)$ at $t = t_k$, $f : I_k \times E \times E \to E$ is a given function and $g_k : \tilde{I}_k \times E \to E$; $k = 1,\dots,m$, are given continuous functions such that $\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}g_k\right)(t, u(t))\,|_{t=s_k} = \phi_k \in E$, where $(E, \|\cdot\|)$ is a real Banach space.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about the generalized Hilfer fractional derivative and auxiliary results. In Section 3, two results for the problem (1)-(3) are presented which are based on the fixed point theorems of Mönch and Darbo associated with the technique of measure of noncompactness. In Section 4, we discuss the Ulam-Hyers-Rassias Stability for the problems. Finally, in the last section, we give an example to illustrate the applicability of our results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $0 < a < b, J = [a, b]$. Let $(E, \|\cdot\|)$ be a Banach space.

By C we denote the Banach space of all continuous functions from J into E with the norm

$$\|u\|_\infty = \sup\{\|u(t)\| : t \in J\}.$$

Consider the weighted Banach space

$$C_{\gamma,\rho}(I_k) = \left\{ u : I_k \to E : t \to \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t) \in C([s_k, t_{k+1}], E) \right\},$$

where $0 \le \gamma < 1$, $k = 0, \dots, m$, and

$$\begin{aligned} C^n_{\gamma,\rho}(I_k) &= \left\{ u \in C^{n-1}(I_k) : u^{(n)} \in C_{\gamma,\rho}(I_k) \right\}, n \in \mathbb{N}, \\ C^0_{\gamma,\rho}(I_k) &= C_{\gamma,\rho}(I_k). \end{aligned}$$

Also consider the Banach space

$$\begin{aligned} PC_{\gamma,\rho}(J) \;=\; & \Big\{ u : (a,b] \to E : u \in C_{\gamma,\rho}(\cup_{k=0}^m I_k) \cap C(\cup_{k=1}^m \tilde{I}_k, E) \text{ and there exist} \\ & u(t_k^-), u(t_k^+), u(s_k^-), \text{ and } u(s_k^+) \text{ with } u(t_k^-) = u(t_k) \Big\}, 0 \le \gamma < 1, \end{aligned}$$

and

$$\begin{aligned} PC^n_{\gamma,\rho}(J) &= \left\{ u \in PC^{n-1}(J) : u^{(n)} \in PC_{\gamma,\rho}(J) \right\}, n \in \mathbb{N}, \\ PC^0_{\gamma,\rho}(J) &= PC_{\gamma,\rho}(J), \end{aligned}$$

with the norm

$$\|u\|_{PC_{\gamma,\rho}} = \max\left\{ \max_{k=0,\dots,m} \left\{ \sup_{t\in I_k} \left\| \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} u(t) \right\| \right\}, \max_{k=1,\dots,m} \{ \sup_{t \in \tilde{I}_k} \|u(t)\| \} \right\}.$$

By $L^1(J)$, we denote the space of Bochner–integrable functions $f : J \longrightarrow E$ with the norm

$$\|f\|_1 = \int_a^b \|f(t)\| dt.$$

Definition 2.1. *[22] (Generalized fractional integral).*
Let $\alpha \in \mathbb{R}_+$ and $g \in L^1(J)$. The generalized fractional integral of order α is defined by

$$\left({}^\rho\mathcal{J}^\alpha_{a^+} g\right)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \;\; t > a, \rho > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \;\; \alpha > 0$.

Definition 2.2. *[22](Generalized fractional derivative).*
Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The generalized fractional derivative ${}^\rho\mathcal{D}^\alpha_{a^+}$ of order α is defined by

$$\begin{aligned} \left({}^\rho\mathcal{D}^\alpha_{a^+} g\right)(t) &= \delta^n_\rho ({}^\rho\mathcal{J}^{n-\alpha}_{a^+} g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds, \;\; t > a, \end{aligned}$$

where $n = [\alpha] + 1$ and $\delta^n_\rho = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Theorem 2.1. *[22] Let $\alpha > 0, \beta > 0, 1 \le p \le \infty, 0 < a < b < \infty$. Then, for $g \in L^1([s_k, t_{k+1}]), k = 0, \dots, m$, we have*

$$\left({}^\rho\mathcal{J}^\alpha_{s_k^+} \, {}^\rho\mathcal{J}^\beta_{s_k^+} g\right)(t) = \left({}^\rho\mathcal{J}^{\alpha+\beta}_{s_k^+} g\right)(t).$$

Lemma 2.1. *[22, 24] Let $\alpha > 0$, $0 \leq \gamma < 1$ and $k = 0, \ldots, m$. Then, ${}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}$ is bounded from $C_{\gamma,\rho}(I_k)$ into $C_{\gamma,\rho}(I_k)$.*

Lemma 2.2. *[24] Let $0 < a < b < \infty, \alpha > 0, 0 \leq \gamma < 1$, $u \in C_{\gamma,\rho}(I_k)$ and $k = 0, \ldots, m$. If $\alpha > 1 - \gamma$, then ${}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} u \in C([s_k, t_{k+1}], E)$ and*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} u\right)(s_k) = \lim_{t \to s_k^+} \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} u\right)(t) = 0.$$

Lemma 2.3. *[12] Let $t > s_k, k = 0, \ldots, m$. Then, for $\alpha \geq 0$ and $\beta > 0$, we have*

$$\left[{}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} \left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha+\beta-1},$$

$$\left[{}^{\rho}\mathcal{D}^{\alpha}_{s_k^+} \left(\frac{s^\rho - s_k^\rho}{\rho}\right)^{\alpha-1}\right](t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.4. *[24] Let $\alpha > 0, 0 \leq \gamma < 1, k = 0, \ldots, m$, and $g \in C_{\gamma,\rho}(I_k)$. Then,*

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{s_k^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} g\right)(t) = g(t), \quad \textit{for all} \quad t \in I_k, k = 0, \ldots, m.$$

Lemma 2.5. *[24] Let $0 < \alpha < 1, 0 \leq \gamma < 1, k = 0, \ldots, m$. If $g \in C_{\gamma,\rho}(I_k)$ and ${}^{\rho}\mathcal{J}^{1-\alpha}_{s_k^+} g \in C^1_{\gamma,\rho}(I_k)$, then for all $t \in I_k, k = 0, \ldots, m$,*

$$\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\alpha}_{s_k^+} g\right)(t) = g(t) - \frac{\left({}^{\rho}\mathcal{J}^{1-\alpha}_{s_k^+} g\right)(s_k)}{\Gamma(\alpha)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\alpha-1}.$$

Definition 2.3. *[24] Let order α and type β satisfy $n - 1 < \alpha < n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$, and $k = 0, \ldots, m$. The generalized Hilfer-type fractional derivative, with $\rho > 0$ of a function $g \in C_{\gamma,\rho}(I_k)$, is defined by*

$$\begin{aligned}\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} g\right)(t) &= \left({}^{\rho}\mathcal{J}^{\beta(n-\alpha)}_{s_k^+} \left(t^{\rho-1}\frac{d}{dt}\right)^n\ {}^{\rho}\mathcal{J}^{(1-\beta)(n-\alpha)}_{s_k^+} g\right)(t) \\ &= \left({}^{\rho}\mathcal{J}^{\beta(n-\alpha)}_{s_k^+} \delta^n_\rho\ {}^{\rho}\mathcal{J}^{(1-\beta)(n-\alpha)}_{s_k^+} g\right)(t).\end{aligned}$$

In this paper we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property 2.2. *[24] The operator ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$ can be written as*

$${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} = {}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{s_k^+} \delta_\rho\ {}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+} = {}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}, \quad \gamma = \alpha + \beta - \alpha\beta, k = 0, \ldots, m.$$

Property 2.3. *[24] The fractional derivative ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$), Hilfer–Hadamard ($\rho \to 0^+$), generalized ($\beta = 0$), Caputo–type ($\beta = 1$), Riemann–Liouville ($\beta = 0, \rho \to 1$), Hadamard ($\beta = 0, \rho \to 0^+$), Caputo ($\beta = 1, \rho \to 1$), Caputo–Hadamard ($\beta = 1, \rho \to 0^+$), Liouville ($\beta = 0, \rho \to 1, a = 0$) and Weyl ($\beta = 0, \rho \to 1, a = -\infty$).*

Consider the following parameters α, β, γ satisfying

$$\gamma = \alpha + \beta - \alpha\beta, \quad 0 < \alpha, \beta, \gamma < 1.$$

We define the spaces

$$C^{\alpha,\beta}_{\gamma,\rho}(I_k) = \left\{ u \in C_{\gamma,\rho}(I_k),\ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u \in C_{\gamma,\rho}(I_k) \right\},$$

and

$$C^{\gamma}_{\gamma,\rho}(I_k) = \left\{ u \in C_{\gamma,\rho}(I_k),\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u \in C_{\gamma,\rho}(I_k) \right\},$$

where $k = 0, \dots, m$.
Since ${}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u = {}^{\rho}\mathcal{J}^{\gamma(1-\alpha)}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u$, it follows from Lemma 2.1 that

$$C^{\gamma}_{\gamma,\rho}(I_k) \subset C^{\alpha,\beta}_{\gamma,\rho}(I_k) \subset C_{\gamma,\rho}(I_k).$$

Also,

$$PC^{\gamma}_{\gamma,\rho}(J) = \left\{ u : (a,b] \to E : u \in C^{\gamma}_{\gamma,\rho}(\cup_{k=0}^{m} I_k) \cap C(\cup_{k=1}^{m} \tilde{I}_k, E) \right\}.$$

Lemma 2.6. *[24] Let $0 < \alpha < 1, 0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$ and $k = 0, \dots, m$. If $u \in C^{\gamma}_{\gamma,\rho}(I_k)$, then*

$${}^{\rho}\mathcal{J}^{\gamma}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u = {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+} u,$$

and

$${}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} u = {}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{s_k^+} u.$$

Definition 2.4. *([14]) let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \longrightarrow [0, \infty)$ defined by*

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, diam(M_j) \leq \epsilon\},$$

where $M \in \Omega_X$. The map μ satisfies the following properties :

- $\mu(M) = 0 \Leftrightarrow \overline{M}$ *is compact (M is relatively compact).*
- $\mu(M) = \mu(\overline{M})$.
- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
- $\mu(M_1 + M_2) \leq \mu(B_1) + \mu(B_2)$.
- $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
- $\mu(convM) = \mu(M)$.

Lemma 2.7. *([19]) Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then*
(i) the function $t \to \mu(D(t))$ is continuous on J, and

$$\mu_{PC_{\gamma,\rho}} = \max\left\{ \max_{k=0,\dots,m} \left\{ \sup_{t\in I_k} \mu\left(\left(\frac{t^{\rho} - s_k^{\rho}}{\rho}\right)^{1-\gamma} u(t) \right) \right\}, \max_{k=1,\dots,m} \left\{ \sup_{t \in \tilde{I}_k} \mu\left(u(t)\right) \right\} \right\},$$

(ii) $\mu\left(\int_a^b u(s)ds : u \in D\right) \leq \int_a^b \mu(D(s))ds$, where

$$D(t) = \{u(t) : t \in D\},\ t \in J.$$

Lemma 2.8. *Let $f : I_k \times E \to E$ be a function such that $f(\cdot, u(\cdot), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(\cdot)) \in C(I_k, E), k = 0, \dots, m$, for any $u \in C_{\gamma,\rho}(I_k)$. Then $u \in C^{\gamma}_{\gamma,\rho}(I_k)$ is a solution of the differential equation, for $0 < \alpha < 1,\ 0 \leq \beta \leq 1$,*

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f(t, u(t), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(t)),\ \text{for each},\ t \in I_k,\ k = 0, \dots, m, \tag{4}$$

if and only if u satisfies the following Volterra integral equation,

$$\begin{aligned} u(t) =& \frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(s_k^+)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1} \\ &+\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}f(s, u(s), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(s))ds, \end{aligned} \tag{5}$$

where $\gamma = \alpha + \beta - \alpha\beta$.

Proof. Assume $u \in C^{\gamma}_{\gamma,\rho}(I_k)$ satisfies the equation (4) where $k = 0, \dots, m$. We prove that u is a solution to the equation (5). From the definition of the space $C^{\gamma}_{\gamma,\rho}(I_k)$ and by using Lemma 2.1 and Definition 2.2, we have

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(t) \in C_{\gamma,\rho}(I_k) \quad \text{and} \quad ({}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u)(t) = \left(\delta_{\rho}\ {}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(t) \in C_{\gamma,\rho}(I_k).$$

By the definition of the space $C^{n}_{\gamma,\rho}(I_k)$, we obtain

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(t) \in C^{1}_{\gamma,\rho}(I_k).$$

Hence, Lemma 2.5 implies that for all $t \in I_k, k = 0, \dots, m$,

$$\left({}^{\rho}\mathcal{J}^{\gamma}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u\right)(t) = u(t) - \frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(s_k)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1}.$$

Using Lemma 2.6 we have

$$\begin{aligned} \left({}^{\rho}\mathcal{J}^{\gamma}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u\right)(t) &= \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\ {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) \\ &= \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}f(s, u(s), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(s))\right)(t). \end{aligned}$$

Then,

$$u(t) = \frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_k^+}u\right)(s_k)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-s_k^{\rho}}{\rho}\right)^{\gamma-1} + \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}f(s, u(s), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(s))\right)(t),$$

where $t \in I_k, k = 0, \dots, m$, that is, u satisfies the equation (5).

Conversely, let $u \in C^{\gamma}_{\gamma,\rho}(I_k)$ satisfy the equation (5) where $k = 0, \dots, m$. We prove that u is a solution to the equation (4). Apply operator ${}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}$ on both sides of (5), where $k = 0, \dots, m$. Then, from Lemma 2.3 and Lemma 2.6 we obtain

$$({}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u)(t) = \left({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{s_k^+}f(s, u(s), {}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u(s))\right)(t). \tag{6}$$

Since $u \in C^{\gamma}_{\gamma,\rho}(I_k)$ and by definition of $C^{\gamma}_{\gamma,\rho}(I_k)$, we have ${}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u \in C_{\gamma,\rho}(I_k)$, then (6) implies that

$$({}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}u)(t) = \left(\delta_{\rho}\ {}^{\rho}\mathcal{J}^{1-\beta(1-\alpha)}_{s_k^+}f\right)(t) = \left({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{s_k^+}f\right)(t) \in C_{\gamma,\rho}(I_k). \tag{7}$$

As $f(\cdot, u(\cdot), {}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u(\cdot)) \in C(I_k, E)$ and from Lemma 2.1, it follows that

$$\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{s_k^+} f\right) \in C_{\gamma,\rho}(I_k), k = 0, \dots, m. \tag{8}$$

From (7), (8) and by the definition of the space $C^n_{\gamma,\rho}(I_k)$, we obtain

$$\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{s_k^+} f\right) \in C^1_{\gamma,\rho}(I_k), k = 0, \dots, m.$$

Applying operator ${}^\rho\mathcal{J}^{\beta(1-\alpha)}_{s_k^+}$ on both sides of (7) and using Lemma 2.5, Lemma 2.2 and Property 2.2, we have

$$\begin{aligned}\left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) &= {}^\rho\mathcal{J}^{\beta(1-\alpha)}_{s_k^+}\left({}^\rho\mathcal{D}^{\gamma}_{s_k^+}u\right)(t) \\ &= f(t, u(t), {}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u(t)) \\ &\quad - \frac{\left({}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{s_k^+} f\right)(s_k)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= f(t, u(t), {}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u(t)),\end{aligned}$$

that is, (4) holds. This completes the proof. □

Theorem 2.4. *(Mönch's fixed point Theorem ([23])). Let D be a closed, bounded and convex subset of a Banach space X such that $0 \in D$, and let T be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}T(V), \quad or \quad V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{9}$$

holds for every subset V of D, then T has a fixed point.

Theorem 2.5. *(Darbo's fixed point Theorem ([20])). Let D be a non-empty, closed, bounded and convex subset of a Banach space X, and let T be a continuous mapping of D into itself such that for any non-empty subset C of D,*

$$\mu(T(C)) \le k\mu(C), \tag{10}$$

where $0 \le k < 1$, and μ is the Kuratowski measure of noncompactness on X. Then T has a fixed point in D.

Now, we consider the Ulam stability for problem (1)−(3) that will be used in Section 4. Let $u \in PC_{\gamma,\rho}(J)$, $\epsilon > 0$, $\tau > 0$ and $\vartheta : (a, b] \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequality :

$$\begin{cases} \left\|\left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) - f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right)\right\| \le \epsilon\vartheta(t), t \in I_k, k = 0, \dots, m, \\ \\ \|u(t) - g_k(t, u(t))\| \le \epsilon\tau, t \in \tilde{I}_k, k = 1, \dots, m. \end{cases} \tag{11}$$

Definition 2.5. *Problem (1)−(3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ϑ, τ) if there exists a real number $a_{f,\vartheta} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC_{\gamma,\rho}(J)$ of inequality (11) there exists a solution $w \in PC_{\gamma,\rho}(J)$ of (1)−(3) with*

$$\|u(t) - w(t)\| \le \epsilon a_{f,\vartheta}(\vartheta(t) + \tau), \quad t \in J.$$

Remark 2.1. *A function $u \in PC_{\gamma,\rho}(J)$ is a solution of inequality (11) if and only if there exist $\sigma \in PC_{\gamma,\rho}(J)$ and a sequence σ_k, $k = 0, \dots, m$, such that*

1. $\|\sigma(t)\| \le \epsilon\vartheta(t)$, $t \in I_k$, $k = 0, \dots, m$, *and* $\|\sigma_k\| \le \epsilon\tau$, $t \in \tilde{I}_k$, $k = 1, \dots, m$,
2. $\left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)\right) + \sigma(t)$, $t \in I_k$, $k = 0, \dots, m$,
3. $u(t) = g_k(t, u(t)) + \sigma_k$, $t \in \tilde{I}_k$, $k = 1, \dots, m$.

3. Existence of Solutions

We consider the following linear fractional differential equation

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{s_k^+}u\right)(t)=\psi(t),\quad t\in I_k,\ k=0,\dots,m, \tag{12}$$

where $0<\alpha<1, 0\le\beta\le1, \rho>0$, with the conditions

$$u(t)=g_k(t,u(t)),\ t\in\tilde{I}_k,\ k=1,\dots,m, \tag{13}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a^+)=\phi_0, \tag{14}$$

where $\gamma=\alpha+\beta-\alpha\beta$ and $\phi_0\in E$, and let $\phi^*=\max\{\|\phi_k\|:k=0,\dots,m\}$. The following theorem shows that the problem (12)–(14) has a unique solution given by

$$u(t)=\begin{cases}\frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho-s_k^\rho}{\rho}\right)^{\gamma-1}+\left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+}\psi\right)(t), & t\in I_k,\ k=0,\dots,m,\\ g_k(t,u(t)), & t\in\tilde{I}_k,\ k=1,\dots,m.\end{cases} \tag{15}$$

Theorem 3.1. *Let $\gamma=\alpha+\beta-\alpha\beta$, where $0<\alpha<1$ and $0\le\beta\le1$. If $\psi:I_k\to E, k=0,\dots,m$, is a function such that $\psi(\cdot)\in C(I_k,E)$, then $u\in PC^{\gamma}_{\gamma,\rho}(J)$ satisfies the problem (12)–(14) if and only if it satisfies (15).*

Proof. Assume u satisfies (12)–(14). If $t\in I_0$, then

$$\left({}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}u\right)(t)=\psi(t),$$

Lemma 2.8 implies we have the solution can be written as

$$u(t)=\frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)(a)}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds.$$

If $t\in\tilde{I}_1$, then we have $u(t)=g_1(t,u(t))$.
If $t\in I_1$, then Lemma 2.8 implies

$$\begin{aligned}u(t)&=\frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_1^+}u\right)(s_1)}{\Gamma(\gamma)}\left(\frac{t^\rho-s_1^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{s_1}^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds\\&=\frac{\phi_1}{\Gamma(\gamma)}\left(\frac{t^\rho-s_1^\rho}{\rho}\right)^{\gamma-1}+\left({}^{\rho}\mathcal{J}^{\alpha}_{s_1^+}\psi\right)(t).\end{aligned}$$

If $t\in\tilde{I}_2$, then we have $u(t)=g_2(t,u(t))$.
If $t\in I_2$, then Lemma 2.8 implies

$$\begin{aligned}u(t)&=\frac{\left({}^{\rho}\mathcal{J}^{1-\gamma}_{s_2^+}u\right)(s_2)}{\Gamma(\gamma)}\left(\frac{t^\rho-s_2^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{s_2}^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds\\&=\frac{\phi_2}{\Gamma(\gamma)}\left(\frac{t^\rho-s_2^\rho}{\rho}\right)^{\gamma-1}+\left({}^{\rho}\mathcal{J}^{\alpha}_{s_2^+}\psi\right)(t).\end{aligned}$$

Repeating the process in this way, the solution $u(t)$ for $t \in (a,b]$ can be written as

$$u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}_{s_k^+}^{\alpha}\psi\right)(t), & t \in I_k,\ k = 0,\dots,m, \\ g_k(t,u(t)), & t \in \tilde{I}_k,\ k = 1,\dots,m. \end{cases}$$

Conversely, for $t \in I_0$, applying ${}^\rho\mathcal{J}_{a^+}^{1-\gamma}$ on both sides of (15) and using Lemma 2.3 and Theorem 2.1, we get

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(t) = \phi_0 + \left({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\right)(t). \tag{16}$$

Next, taking the limit as $t \to a^+$ of (16) and using Lemma 2.2, with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \phi_0, \tag{17}$$

which shows that the initial condition $\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)(a^+) = \phi_0$, is satisfied. Next, for $t \in I_k, k = 0,\dots,m$, apply operator ${}^\rho\mathcal{D}_{s_k^+}^{\gamma}$ on both sides of (15). Then, from Lemma 2.3 and Lemma 2.6 we obtain

$$({}^\rho\mathcal{D}_{s_k^+}^{\gamma}u)(t) = \left({}^\rho\mathcal{D}_{s_k^+}^{\beta(1-\alpha)}\psi\right)(t). \tag{18}$$

Since $u \in C_{\gamma,\rho}^{\gamma}(I_k)$ and by definition of $C_{\gamma,\rho}^{\gamma}(I_k)$, we have ${}^\rho\mathcal{D}_{s_k^+}^{\gamma}u \in C_{\gamma,\rho}(I_k)$, and then (18) implies that

$$({}^\rho\mathcal{D}_{s_k^+}^{\gamma}u)(t) = \left(\delta_\rho\ {}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right)(t) = \left({}^\rho\mathcal{D}_{s_k^+}^{\beta(1-\alpha)}\psi\right)(t) \in C_{\gamma,\rho}(I_k). \tag{19}$$

As $\psi(\cdot) \in C(I_k,E)$ and from Lemma 2.1, it follows that

$$\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}(I_k),\ k = 0,\dots,m. \tag{20}$$

From (19), (20) and by the definition of the space $C_{\gamma,\rho}^{n}(I_k)$, we obtain

$$\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right) \in C_{\gamma,\rho}^{1}(I_k),\ k = 0,\dots,m.$$

Applying operator ${}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)}$ on both sides of (18) and using Lemma 2.5, Lemma 2.2 and Property 2.2, we have

$$\begin{aligned} \left({}^\rho\mathcal{D}_{s_k^+}^{\alpha,\beta}u\right)(t) &= {}^\rho\mathcal{J}_{s_k^+}^{\beta(1-\alpha)}\left({}^\rho\mathcal{D}_{s_k^+}^{\gamma}u\right)(t) \\ = \psi(t) - \frac{\left({}^\rho\mathcal{J}_{s_k^+}^{1-\beta(1-\alpha)}\psi\right)(s_k)}{\Gamma(\beta(1-\alpha))}&\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \psi(t), \end{aligned}$$

that is, (12) holds.
Also, we have easily for $u \in C(\tilde{I}_k,E)$,

$$u(t) = g_k(t,u(t_k^-)),\ t \in \tilde{I}_k,\ k = 1,\dots,m.$$

This completes the proof. □

As a consequence of Theorem 3.1, we have the following result:

Lemma 3.1. *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $k = 0, \ldots, m$, let $f : I_k \times E \times E \to E$, be a function such that $f(\cdot, u(\cdot), w(\cdot)) \in C(I_k, E)$, for any $u, w \in PC_{\gamma,\rho}(J)$. If $u \in PC^{\gamma}_{\gamma,\rho}(J)$, then u satisfies the problem $(1)-(3)$ if and only if u is the fixed point of the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined by*

$$\Psi u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} h\right)(t), & t \in I_k, \ k = 0, \ldots, m, \\ \\ g_k(t, u(t)), & t \in \tilde{I}_k, \ k = 1, \ldots, m. \end{cases} \tag{21}$$

where $h \in C(I_k, E)$, $k = 0, \ldots, m$ is a function satisfying the functional equation

$$h(t) = f(t, u(t), h(t)).$$

Also, by Lemma 2.1, $\Psi u \in PC_{\gamma,\rho}(J)$.

The following hypotheses will be used in the sequel:

(*Ax1*) The function $t \mapsto f(t, u, w)$ is measurable on $I_k, k = 0, \ldots, m$, for each $u, w \in E$, and the functions $u \mapsto f(t, u, w)$ and $w \mapsto f(t, u, w)$ are continuous on E for a.e. $t \in I_k, k = 0, \ldots, m$, and

$$f(\cdot, u(\cdot), w(\cdot)) \in C^{\beta(1-\alpha)}_{\gamma,\rho}(I_k) \ \text{ for any } u, w \in PC_{\gamma,\rho}(J).$$

(*Ax2*) There exists a continuous function $p : [a, b] \longrightarrow [0, \infty)$ such that

$$\|f(t, u, w)\| \leq p(t), \text{ for a.e. } t \in I_k, k = 0, \ldots, m, \text{ and for each } u, w \in E.$$

(*Ax3*) For each bounded set $B \subset E$ and for each $t \in I_k, k = 0, \ldots, m$, we have

$$\mu(f(t, B, ({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} B))) \leq p(t)\mu(B),$$

where ${}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} B = \{{}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} w : w \in B\}$.

(*Ax4*) The functions $g_k \in C(\tilde{I}_k, E), k = 1, \ldots, m$, and there exists $l^* > 0$ such that

$$\|g_k(t, u)\| \leq l^*\|u\| \ \text{ for each } u \in E, k = 1, \ldots, m.$$

(*Ax5*) For each bounded set $B \subset E$ and for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\mu(g_k(t, B)) \leq l^*\mu(B), k = 1, \ldots, m.$$

Set $p^* = \sup\limits_{t \in [a,b]} p(t)$.

We are now in a position to state and prove our existence result for the problem $(1)-(3)$ based on Mönch's fixed point theorem.

Theorem 3.2. *Assume* (Ax1)–(Ax5) *hold. If*

$$L \ := \ max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha\right\} < 1, \tag{22}$$

then the problem (1)–(3) has at least one solution in $PC_{\gamma,\rho}(J)$.

Proof. Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in (21) and the ball $B_R := B(0,R) = \{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\}$, such that

$$R \geq \frac{\phi^*}{(1-l^*)\Gamma(\gamma)} + \frac{p^*}{(1-l^*)\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}.$$

For any $u \in B_R$, and each $t \in I_k$, $k = 0, \ldots, m$, we have

$$\begin{aligned}\|\Psi u(t)\| &\leq \frac{\|\phi_k\|}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+}\|h(s)\|\right)(t)\\ &\leq \frac{\phi^*}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + p^*\left({}^\rho\mathcal{J}^\alpha_{s_k^+}(1)\right)(t).\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}\left\|\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\Psi u(t)\right\| &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma+\alpha}\\ &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}.\end{aligned}$$

And for $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\|(\Psi u)(t)\| \leq l^*\|u(t)\| \leq l^* R.$$

Hence,

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq l^* R + \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \leq R.$$

This proves that Ψ transforms the ball B_R into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theorem 2.4. The rest of the proof will be given in several steps.

Step 1: $\Psi : B_R \to B_R$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k, k = 0, \ldots, m$, we have,

$$\left\|((\Psi u_n)(t) - (\Psi u)(t))\left(\tfrac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\right\| \leq \left(\tfrac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^\alpha_{s_k^+}\|h_n(s) - h(s)\|\right)(t),$$

where $h_n, h \in C(I_k, E); k = 0, \ldots, m$, such that

$$\begin{aligned}h_n(t) &= f(t, u_n(t), h_n(t)),\\ h(t) &= f(t, u(t), h(t)).\end{aligned}$$

For each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have,

$$\|((\Psi u_n)(t) - (\Psi u)(t))\| \leq \|(g_k(t, u_n(t)) - g_k(t, u(t)))\|.$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in (a, b]$, and since f and g_k are continuous, then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\Psi(B_R)$ is bounded and equicontinuous.
Since $\Psi(B_R) \subset B_R$ and B_R is bounded, then $\Psi(B_R)$ is bounded.

Next, let $\epsilon_1, \epsilon_2 \in I_k, k = 0, \dots, m$, $\epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\begin{aligned}
&\left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_2) \right\| \\
&\leq \left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathbb{J}^\alpha_{s_k^+} h(\tau)\right)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathbb{J}^\alpha_{s_k^+} h(\tau)\right)(\epsilon_2) \right\| \\
&\leq \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathbb{J}^\alpha_{\epsilon_1^+} \|h(\tau)\|\right)(\epsilon_2) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\epsilon_1} \left\| \tau^{\rho-1} H(\tau) h(\tau) \right\| d\tau,
\end{aligned}$$

where $H(\tau) = \left[\left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_1^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_2^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \right]$.
Then by Lemma 2.3, we have

$$\begin{aligned}
&\left\| \left(\frac{\epsilon_1^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(\epsilon_2) \right\| \\
&\leq \frac{p^*}{\Gamma(1+\alpha)} \left(\frac{\epsilon_2^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho}\right)^{\alpha} + p^* \int_{s_k}^{\epsilon_1} \left\| H(\tau) \frac{\tau^{\rho-1}}{\Gamma(\alpha)} \right\| \left(\frac{\tau^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} d\tau,
\end{aligned}$$

and for each $t \in \tilde{I}_k, k = 1, \dots, m$, we have

$$\|(\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2)\| \leq \|(g_k(\epsilon_1, u(\epsilon_1))) - (g_k(\epsilon_2, u(\epsilon_2)))\|.$$

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication (9) of Theorem 2.4 holds.
Now let D be an equicontinuous subset of B_R such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \longrightarrow d(t) = \mu(D(t))$ are continuous on J. By $(Ax3)$, $(Ax5)$ and the properties of the measure μ, for each $t \in I_k, k = 0, \dots, m$, we have

$$\begin{aligned}
\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} d(t) &\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \cup \{0\} \right) \\
&\leq \mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} (\Psi D)(t) \right) \\
&\leq \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}^\alpha_{s_k^+} p(s)\mu(D(s))\right)(t) \\
&\leq p^* \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho\mathcal{J}^\alpha_{s_k^+} d(s)\right)(t) \\
&\leq \left[\frac{p^* \Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha} \right] \|d\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

And for each $t \in \tilde{I}_k, k = 1, \dots, m$, we have

$$d(t) \leq \mu\left(g_k(t, D(t))\right) \leq l^* d(t).$$

Thus for each $t \in (a, b]$, we have

$$\|d\|_{PC_{\gamma,\rho}} \leq L \|d\|_{PC_{\gamma,\rho}}.$$

From (22), we get $\|d\|_{PC_{\gamma,\rho}} = 0$, that is, $d(t) = \mu(D(t)) = 0$, for each $t \in (a, b]$, and then $D(t)$ is relatively compact in E. In view of the Ascoli-Arzela Theorem, D is relatively compact in B_R. Applying now Theorem 2.4, we conclude that Ψ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$, which is solution of the problem (1)-(3).

Step 4: We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC^\gamma_{\gamma,\rho}(J)$.
Since u^* is the unique fixed point of operator Ψ in $PC_{\gamma,\rho}(J)$, then for each $t \in J$, we have

$$\Psi u^*(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} h\right)(t), & t \in I_k,\ k = 0, \dots, m, \\ \\ g_k(t, u^*(t)), & t \in \tilde{I}_k,\ k = 1, \dots, m. \end{cases}$$

where $h \in C(I_k, E); k = 0, \ldots, m$, such that

$$h(t) = f(t, u^*(t), h(t)).$$

For $t \in I_k; k = 0, \ldots, m$, applying ${}^{\rho}\mathcal{D}^{\gamma}_{s_k^+}$ to both sides and by Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned} {}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u^*(t) &= \left({}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} \, {}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} f(s, u^*(s), h(s)) \right)(t) \\ &= \left({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{s_k^+} f(s, u^*(s), h(s)) \right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, by $(Ax1)$, the right hand side is in $C_{\gamma,\rho}(I_k)$ and thus ${}^{\rho}\mathcal{D}^{\gamma}_{s_k^+} u^* \in C_{\gamma,\rho}(I_k)$ which implies that $u^* \in C^{\gamma}_{\gamma,\rho}(I_k)$. And since $g_k \in C(\tilde{I}_k, E); k = 1, \ldots, m$, then $u^* \in PC^{\gamma}_{\gamma,\rho}(J)$. As a consequence of Steps 1 to 4 together with Theorem 3.2, we can conclude that the problem $(1) - (3)$ has at least one solution in $PC_{\gamma,\rho}(J)$. □

Our second existence result for the problem (1)-(3) is based on Darbo's fixed point theorem.

Theorem 3.3. *Assume* (Ax1)–(Ax5) *hold. If*

$$L \ := \ \max\left\{ l^*, \tfrac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\tfrac{b^\rho - a^\rho}{\rho} \right)^{\alpha} \right\} < 1,$$

then the problem (1)−(3) has at least one solution in $PC_{\gamma,\rho}(J)$.

Proof. Consider the operator Ψ defined in (21). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuous and that $\Psi(B_R)$ is equicontinuous. We need to prove that the operator Ψ is an L-contraction.
Let $D \subset B_R$ and $t \in I_k, k = 0, \ldots, m$. Then we have

$$\begin{aligned} &\mu\left(\left(\tfrac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} (\Psi D)(t) \right) = \mu\left(\left(\tfrac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} (\Psi u)(t) : u \in D \right) \\ \leq \ &\left(\tfrac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left\{ \left({}^{\rho}\mathcal{J}^{\alpha}_{s_k^+} p^* \mu(u(s)) \right)(t), u \in D \right\}. \end{aligned}$$

By Lemma 2.3, we have for $t \in I_k, k = 0, \ldots, m$,

$$\mu\left(\left(\frac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} (\Psi D)(t) \right) \leq \left[\frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha} \right] \mu_{PC_{\gamma,\rho}}(D).$$

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have

$$\mu\left((\Psi D)(t) \right) \leq \mu\left(g_k(t, D(t)) \right) \leq l^* \mu\left(D(t) \right).$$

Hence, for each $t \in (a, b]$, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq L \mu_{PC_{\gamma,\rho}}(D).$$

So, by (22), the operator Ψ is an L-contraction. As consequence of Theorem 2.5 and using Step 4 of the last result, we deduce that Ψ has a fixed point which is a solution of the problem (1)-(3) □

4. Ulam-Hyers-Rassias (U-H-R) Stability

First, we are concerned with the Ulam-Hyers-Rassias (U-H-R) stability of our problem $(1)-(3)$.

Theorem 4.1. *Assume that in addition to (*Ax1*)–(*Ax5*) and (22), the following hypotheses hold.*

(Ax6) There exist a nondecreasing function $\vartheta : (a, b] \longrightarrow [0, \infty)$ and $\lambda_\vartheta > 0$ such that for each $t \in I_k, k = 0, \ldots, m$, we have

$$({}^\rho\mathcal{J}^\alpha_{s_k^+}\vartheta)(t) \leq \lambda_\vartheta \vartheta(t).$$

(Ax7) There exists a continuous function $\chi : \bigcup_{k=1}^{m} [s_k, t_{k+1}] \longrightarrow [0, \infty)$ such that for each $t \in I_k, k = 0, \ldots, m$, we have

$$p(t) \leq \chi(t)\vartheta(t).$$

Then problem (1)–(3) is U-H-R stable with respect to (ϑ, τ).

Proof. Consider the operator Ψ defined in (21). Let $u \in PC_{\gamma,\rho}(J)$ be a solution of inequality (11), and let us assume that w is the unique solution of the problem

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} w\right)(t) = f\left(t, w(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} w\right)(t)\right); \ t \in I_k, \ k = 0, \ldots, m, \\ w(t) = g_k(t, w(t_k^-)); \ t \in \tilde{I}_k, \ k = 1, \ldots, m, \\ \left({}^\rho\mathcal{J}^{1-\gamma}_{s_k^+} w\right)(s_k^+) = \left({}^\rho\mathcal{J}^{1-\gamma}_{s_k^+} u\right)(s_k^+) = \phi_k, \ k = 0, \ldots, m. \end{cases}$$

By Lemma 3.1, we obtain for each $t \in (a, b]$

$$w(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} h\right)(t) & if \ t \in I_k, k = 0, \ldots, m, \\ g_k(t, w(t)) \qquad if \ t \in \tilde{I}_k, k = 1, \ldots, m, \end{cases}$$

where $h \in C(I_k, E), k = 0, \ldots, m$, is a function satisfying the functional equation

$$h(t) = f(t, w(t), h(t)).$$

Since u is a solution of the inequality (11), by Remark 2.1, we have

$$\begin{cases} \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t) = f\left(t, u(t), \left({}^\rho\mathcal{D}^{\alpha,\beta}_{s_k^+} u\right)(t)\right) + \sigma(t), t \in I_k, k = 0, \ldots, m, \\ u(t) = g_k(t, u(t)) + \sigma_k, t \in \tilde{I}_k, k = 1, \ldots, m. \end{cases} \tag{23}$$

Clearly, the solution of (23) is given by

$$u(t) = \begin{cases} \frac{\phi_k}{\Gamma(\gamma)}\left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} g\right)(t) + \left({}^\rho\mathcal{J}^\alpha_{s_k^+} \sigma\right)(t), & t \in I_k, k = 1, \ldots, m, \\ g_k(t, u(t)) + \sigma_k, & t \in \tilde{I}_k, \ k = 1, \ldots, m, \end{cases}$$

where $g : I_k \to E, k = 0, \ldots, m$, is a function satisfying the functional equation

$$g(t) = f(t, u(t), g(t)).$$

Hence, for each $t \in I_k$, $k = 0, \ldots, m$, we have

$$\begin{aligned} \|u(t) - w(t)\| &\leq \left({}^\rho\mathcal{J}^\alpha_{a^+}\|g(s) - h(s)\|\right)(t) + \left({}^\rho\mathcal{J}^\alpha_{a^+}\|\sigma(s)\|\right) \\ &\leq \epsilon\lambda_\vartheta\vartheta(t) + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{2\chi(t)\vartheta(t)}{\Gamma(\gamma)}ds \\ &\leq \epsilon\lambda_\vartheta\vartheta(t) + 2\chi^*\left({}^\rho\mathcal{J}^\alpha_{a^+}\vartheta\right)(t) \\ &\leq (\epsilon + 2\chi^*)\lambda_\vartheta\vartheta(t) \\ &\leq (1 + \frac{2\chi^*}{\epsilon})\lambda_\vartheta\epsilon(\tau + \vartheta(t)), \end{aligned}$$

where

$$\chi^* = \max_{k=0,\dots,m}\left\{\sup_{t\in[s_k,t_{k+1}]}\chi(t)\right\}.$$

For each $t \in \tilde{I}_k$, $k = 1,\dots,m$, we have

$$\begin{aligned}\|u(t)-w(t)\| &\le \|g_k(t,u(t))-g_k(t,w(t))\|+\|\sigma_k\|\\ &\le l^*\|u(t)-w(t)\|+\epsilon\tau,\end{aligned}$$

and then by (22),

$$\|u(t)-w(t)\| \le \tfrac{\epsilon\tau}{1-l^*} \le \tfrac{\epsilon}{1-l^*}(\tau+\vartheta(t)).$$

Then for each $t \in (a,b]$, we have

$$\|u(t)-w(t)\| \le a_\vartheta \epsilon(\tau+\vartheta(t)),$$

where

$$a_\vartheta = \max\left\{(1+\frac{2\chi^*}{\epsilon})\lambda_\vartheta, \frac{1}{1-l^*}\right\}.$$

Hence, problem (1)–(3) is U-H-R stable with respect to (ϑ,τ). □

5. Example

Let

$$E = l^1 = \left\{v=(v_1,v_2,\dots,v_n,\dots), \sum_{n=1}^{\infty}|v_n| < \infty\right\}$$

be the Banach space with the norm

$$\|v\| = \sum_{n=1}^{\infty}|v_n|.$$

Consider the following initial value problem with non-instantaneous impulses

$$\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u\right)(t) = f\left(t,u(t),\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u\right)(t)\right), t\in(1,2]\cup(e,3], k\in\{0,1\} \tag{24}$$

$$u(t) = g(t,u(t)),\ t\in(2,e], \tag{25}$$

$$\left({}^1\mathcal{J}^{\frac{1}{2}}_{1^+}u\right)(1^+) = 0, \tag{26}$$

where

$$a = t_0 = s_0 = 1 < t_1 = 2 < s_1 = e < t_2 = 3 = b,$$

$$u = (u_1,u_2,\dots,u_n,\dots),$$

$$f = (f_1,f_2,\dots,f_n,\dots),$$

$${}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u = ({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u_1,\dots,{}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u_2,\dots,{}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u_n,\dots),$$

$$g = (g_1,g_2,\dots,g_n,\dots),$$

$$f_n(t,u_n(t),\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u_n\right)(t)) = \frac{(2t^3+5e^{-2})|u_n(t)|}{183e^{-t+3}(1+\|u(t)\|+\|\left({}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}u\right)(t)\|)},$$

with $t\in(1,2]\cup(e,3]$, $k\in\{0,1\}$, $n\in\mathbb{N}$, and

$$g_n(t,u_n(t)) = \frac{|u_n(t)|}{105e^{-t+5}+1},\ t\in(2,e], n\in\mathbb{N}.$$

We have
$$C^{\beta(1-\alpha)}_{\gamma,\rho}((1,2]) = C^{0}_{\frac{1}{2},1}((1,2]) = \left\{h : (1,2] \to E : (\sqrt{t-1})h \in C([1,2],E)\right\},$$
and
$$C^{\beta(1-\alpha)}_{\gamma,\rho}((e,3]) = C^{0}_{\frac{1}{2},1}((e,3]) = \left\{h : (e,3] \to E : (\sqrt{t-e})h \in C([e,3],E)\right\},$$
with $\gamma = \alpha = \frac{1}{2}$, $\rho = 1$, $\beta = 0$ and $k \in \{0,1\}$. Clearly, the continuous function $f \in C^{0}_{\frac{1}{2},1}((1,2]) \cup C^{0}_{\frac{1}{2},1}((e,3])$. Hence the condition $(Ax1)$ is satisfied.

For each $u, w \in E$ and $t \in (1,2] \cup (e,3]$,
$$\|f(t,u,w)\| \leq \frac{2t^3 + 5e^{-2}}{183e^{-t+3}}.$$
Hence condition $(Ax2)$ is satisfied with
$$p(t) = \frac{2t^3 + 5e^{-2}}{183e^{-t+3}},$$
and
$$p^* = \frac{54 + 5e^{-2}}{183}.$$
And for each $u \in E$ and $t \in (2,e]$ we have
$$\|g(t,u)\| \leq \frac{\|u\|}{105e^{5-e} + 1},$$
and so the condition $(Ax4)$ is satisfied with $l^* = \frac{1}{105e^{5-e}+1}$.
The condition (22) of Theorem 3.2 is satisfied, for
$$L := \max\left\{l^*, \frac{p^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right\} \approx 0.7489295248 < 1.$$
Let Ω be a bounded set in E where ${}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}\Omega = \left\{{}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}v : v \in \Omega\right\}, k \in \{0,1\}$. Then by the properties of the Kuratowski measure of noncompactness, for each $u \in \Omega$ and $t \in (1,2] \cup (e,3]$, we have
$$\mu\left(f(t,\Omega,{}^1\mathcal{D}^{\frac{1}{2},0}_{s_k^+}\Omega)\right) \leq p(t)\mu(\Omega),$$
and for each $t \in (2,e]$,
$$\mu\left(g(t,\Omega)\right) \leq l^*\mu(\Omega).$$
Hence conditions $(Ax3)$ and $(Ax5)$ are satisfied. Then the problem $(24)-(26)$ has at least one solution in $PC_{\frac{1}{2},1}([1,3])$.
Also, hypothesis $(Ax8)$ is satisfied with $\tau = 1$ and
$$\vartheta(t) = \begin{cases} \frac{1}{\sqrt{t-s_k}}, & t \in (1,2] \cup (e,3], \\ 1, & t \in (2,e], \end{cases}$$
and $\lambda_\vartheta = \sqrt{\pi}$. Indeed, for each $t \in (1,2]$, we get
$$({}^1\mathcal{J}^{\frac{1}{2}}_{1^+}\vartheta)(t) = \sqrt{\pi} \leq \frac{\sqrt{\pi}}{\sqrt{t-1}},$$
and for each $t \in (e,3]$, we get
$$({}^1\mathcal{J}^{\frac{1}{2}}_{e^+}\vartheta)(t) = \sqrt{\pi} \leq \frac{\sqrt{\pi}}{\sqrt{t-e}}.$$

Let the function $\chi : [1,2] \cup [e,3] \longrightarrow [0,\infty)$ be defined by

$$\chi(t) = \frac{(2t^3 + 5e^{-2})\sqrt{t - s_k}}{183e^{-t+3}}, k \in \{0,1\}.$$

Then, for each $t \in (1,2] \cup (e,3]$, we have

$$p(t) = \chi(t)\vartheta(t),$$

with $\chi^* = p^*$. Hence, the condition (*Ax9*) is satisfied. Consequently, Theorem 4.1 implies that the problem (24)–(26) is U-H-R stable.

References

[1] S. Abbas and M. Benchohra, Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses, *Appl. Math. Comput.* **257** (2015), 190-198.
[2] S. Abbas, M. Benchohra and M.A. Darwish, New stability results for partial fractional differential inclusions with not instantaneous impulses, *Frac. Calc. Appl. Anal.* **18** (1) (2015), 172-191.
[3] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Differential and Integral Equations: Existence and stability*, Walter de Gruyter, London, 2018.
[4] S. Abbas, M. Benchohra, J. E. Lazreg, A Alsaedi and Y. Zhou, Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type, *Adv. Difference Equ.* (2017) 14p.
[5] S. Abbas, M. Benchohra, J. E. Lazreg and G. N' Guérékata, Hilfer and Hadamard functional random fractional differential inclusions, *Cubo* **19** (2017), 17-38.
[6] S. Abbas, M. Benchohra, J. E. Lazreg and Y. Zhou, A survey on Hadamard and Hilfer fractional differential equations: analysis and stability, *Chaos Solitons Fractals* **102** (2017), 47-71.
[7] S. Abbas, M. Benchohra and G M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.
[8] S. Abbas, M. Benchohra and G M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2014.
[9] R.P. Agarwal, S. Hristova and D. O'Regan, *Non-Instantaneous Impulses in Differential Equations*, Springer, New York, 2017.
[10] B. Ahmad, A. Alsaedi, S.K. Ntouyas and J. Tariboon, *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*. Springer, Cham, 2017.
[11] B. Ahmad and S.K Ntouyas, Fractional differential inclusions with fractional separated boundary conditions, *Fract. Calc. Appl. Anal.* **15** (2012), 362-382.
[12] R. Almeida, A.B. Malinowska and T. Odzijewicz, Fractional differential equations with dependence on the Caputo–Katugampola derivative, *J. Comput. Nonlinear Dynam.* **11** (6) (2016), 1–11.
[13] D. Baleanu, Z.B. Güvenç and J.A.T. Machado *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.
[14] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*. Marcel Dekker, New York, 1980.
[15] M. Benchohra, J. Henderson and S. K. Ntouyas , *Impulsive Differential Equations and Inclusions*, vol. 2. Hindawi Publishing Corporation, New York, 2006.
[16] M. Benchohra and J. E. Lazreg, Existence and Ulam stability for Nonlinear implicit fractional differential equations with Hadamard derivative, *Stud. Univ. Babes-Bolyai Math.* **62** (2017), 27-38.
[17] M. Benchohra and J. E. Lazreg, On stability for nonlinear implicit fractional differential equations, *Le Matematiche (Catania)* **70** (2015), 49-61.
[18] P. Egbunonu and M. Guay, Identification of switched linear systems using subspace and integer programming techniques. *Nonlinear Anal. Hybrid Syst.* **1** (2007), 577-592.
[19] D.J. Guo, V. Lakshmikantham and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*. Kluwer Academic Publishers, Dordrecht, 1996.
[20] K. Goebel, *Concise course on Fixed Point Theorems*. Yokohama Publishers, Japan, 2002.
[21] D.H. Hyers. On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.* **27** (1941), 222-224.
[22] U. Katugampola, A new approach to a generalized fractional integral, *Appl. Math. Comput.* **218** (2011), 860-865.
[23] H. Mönch, BVP for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.
[24] D.S. Oliveira and E. Capelas de Oliveira, Hilfer–Katugampola fractional derivatives, *Comput. Appl. Math.* **37** (3)(2018), 3672-3690.
[25] T.M. Rassias. On the stability of the linear mappings in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
[26] I.A. Rus, Ulam stability of ordinary differential equations, *Stud. Univ. Babes-Bolyai, Math.* **LIV**(4), (2009), 125-133.
[27] S.M. Ulam. *A collection of mathematical problems*. Interscience Publishers, New York, 1968.
[28] F. Vaadrager and J. Van Schuppen (eds.), *Hybrid Systems, Computation and Control. Lecture Notes in Computer Sciences*, vol. 1569. Springer, New York, 1999.