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Research Article

Infimal generators and monotone sublinear operators

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ABSTRACT. We provide a version of Korovkin-type theorems for monotone sublinear operators in vector lattices and discuss the possibilities of further extensions and generalizations.

Keywords: Supremal generator, Korovkin-type theorem, sublinear operator.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

S. G. Gal and C. P. Niculescu recently proved a Korovkin-type theorem for monotone sublinear operators in spaces of continuous functions in N-dimensions [1, Theorem 2]. As the test functions they use the coordinate projections and the sum of their squares whose span is a classical supremal generator in $C(Q)$ with Q a compact subset of the N-dimensional Euclidean space $\mathbb{R}^N$. In this short note, we extend this theorem to order convergence.

2. THE MAIN RESULT

We will proceed with a vector sublattice X of a Dedekind complete vector lattice Y. Recall that a subset H of X is a supremal generator of X_+ with respect to Y provided that $x = \sup_Y\{h \in H : h \leq x\}$ for all $x \in X_+$, where as usual $X_+ = \{x \in X : x \geq 0\}$. Using the reverse order, we define an infimal generator of X_+ with respect to Y for convenience. We let $L_+(X, Y)$ stand for the set of positive linear operators from X to Y. Assume that Id_Y is the identity operator on Y. Then, the order interval $[0, \mathrm{Id}_Y]$ is the set of multipliers of Y.

Theorem 2.1. *Let H be a convex cone in X. Then, the following are equivalent:*

(1) H is an infimal generator of X_+ with respect to Y;

(2) there is a multiplier $\lambda \in [0, \mathrm{Id}_Y]$ such that $o{-}\limsup_{\alpha\in\mathrm{A}} P_\alpha(x) = \lambda x$ for all $x \in X_+$ and every net $(P_\alpha)_{\alpha\in\mathrm{A}}$ of monotone sublinear operators from X to Y such that $o{-}\limsup_\alpha P_{\alpha\in\mathrm{A}}(h) \leq h$ for all $h \in H$;

(3) H is cofinal and if $T \in L_+(X, Y)$ such that $Th \leq h$ for all $h \in H$; then T is a multiplier in Y.

Proof. (1) → (2): Clearly, $P(x) := o - \limsup_{\alpha\in\mathrm{A}} P_\alpha(x)$ exists for every $x \in X$. So, $P : X \to Y$ is a monotone sublinear operator and $P(x) = \sup\{Tx : T \in \partial(P)\}$, where $\partial(P)$ is the subdifferential of P.

Take $T \in \partial(P)$, $x \in X_+$, and $h \geq x, h \in H$. Since H is an infimal generator of X_+ with respect to Y and $Th \leq P(h) \leq h$; therefore, $Tx \leq x$ by the operator principle of preservation of

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inequalities [2, Theorem 2.1]. Consequently, $T \in [0, \mathrm{Id}_Y]$ and so $P(x) = \lambda x$ for all $x \in X_+$ since the pointwise supremum of multipliers is a multiplier itself.

(2)$\to$ (3): This is obvious.

(3) $\to$ (1): Let $q_H(x) := \inf\{h \in H : h \geq x\}$. Then, q_H is a monotone sublinear operator on X and $\partial(q_H) = \{T \in L_+(X, Y) : (\forall h \in H\,)Th \leq h\}$. By (3), q_H is a multiplier itself.

□

3. Comments and Extensions

In this section, we will discuss applications and extensions of Theorem 2.1 using the terminology and techniques of vector lattices, subdifferential calculus, and Boolean-valued analysis as presented in [3] and [4].

The ideas behind the main theorem make it possible to abstract the effects of monotone sublinear approximations along all lines thoroughly presented in [5]. For instance, if H is a finite-dimensional subspace of X, then X is a vector lattice of bounded elements. Moreover, H is coinitial to X (in fact, a supremal generator as well) and we have an analog of Theorem 2.1 for relatively uniform convergence to the embedding of X to Y. This yields uniform approximation by monotone sublinear operators in spaces of bounded continuous functions on compact subsets of finite-dimensional Euclidean space.

Many opportunities are open by Boolean valued analysis for Korovkin-type results in the realm of ordered modules over the ring of orthomorphisms of a Dedekind complete vector lattice Y. We can consider approximation of the embedding by monotone module-sublinear operators along the lines of Theorem 2.1. The rest of the matter is that the modular situation can be lifted to the Boolean valued universe $\mathbb{V}^{\mathbb{B}}$ over the base $\mathbb{B}$ of Y (i.e., the set of band projections in Y). Since by transfer Theorem 2.1 and its analogs are valid inside $\mathbb{V}^{\mathbb{B}}$, we can use their Boolean valued interpretations which yield, for instance, some Korovkin-type theorems for approximation of extensional continuous vector-valued functions on the so-called procompact spaces which are usually noncompact (for instance, the order intervals between two Lebesgue measurable functions).

We will not dwell on all these new model-theoretic opportunities since the possibilities of their application for the needs of the working mathematician seem dim these days.

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