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# Shehu Transform of Hilfer-Prabhakar Fractional Derivatives and Applications on some Cauchy Type Problems

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### Abstract

In this paper, we are interested on the Shehu transform of both Prabhakar and Hilfer–Prabhakar fractional derivative and its regularized version. These results are presented in terms of Mittag-Leffler type function and also utilized to obtain the solutions of some Cauchy type problems, such as Space-time Fractional Advection-Dispersion equation and Generalized fractional Free Electron Laser (FEL) equation, at which Hilfer-Prabhakar fractional derivative of fractional order and its regularized version are involved.

**Keywords:** Hilfer–Prabhakar derivatives, Prabhakar integrals, Mittag–Leffler functions, Shehu transform  
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### 1. Introduction

During the last few decades, fractional calculus have received considerable attention to solve different kind of problems in pure mathematics, applied mathematics and mathematical physics by modelling a wide range of phenomena such as: fluid mechanics, biology, chemistry, control theory, economics, electronics,

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finance, psychology and other area of science and engineering [1, 15, 16, 17, 18, 19, 22, 24]. In literature, there are many different definitions of fractional integrals and derivatives.

These definitions, although they do not always lead to identical results, they are equivalent for a wide range of functions. Among them, we have, in particular, Riemann-Liouville integral, Riemann-Liouville fractional derivative, Caputo fractional derivatives, the Prabhakar integral, the Hilfer-Prabhakar fractional derivative, ..., etc. We refer the reader to check e.g. [23, 11, 10, 8, 19, 14].

Many researchers use the Hilfer-Prabhakar fractional derivative operator for the purpose of modelling some physical aspects due to its specific properties, especially the combination with several integral transforms founded in the literature of fractional differentiations and integrations like Laplace, Fourier, Sumudu, Elzaki, ..., etc. In [10], the Laplace transformation of Hilfer-Prabhakar and its regularized version is considered, where the authors have also implemented their results in classical equations of mathematical physics such as heat, free electron laser equations and homogeneous Poisson process while in [21] W. Panchal et al. applied its Sumudu transform to some non-homogeneous Cauchy type problems. Indeed, V. Gill et al. in [12] derived the analytical solution of generalized space-time fractional advection-dispersion equation by coupling the Sumudu and Fourier transforms, associated with the Hilfer-Prabhakar fractional derivative. Recently, in [26] the authors use found the Elzaki transform of Hilfer-Prabhakar fractional derivative and its regularized version and gathering these results for solving free electron laser type integro-differential equation.

On the other hand, the Shehu transform is a new integral transform which is introduced at the first time in 2019 by Maitama and Zhao [25]. In fact, Shehu transform is a generalization of the Laplace and the Sumudu integral transforms and has some good features. In this context, the main objective of this paper is to find the Shehu transform of: Prabhakar fractional derivative, Hilfer-Prabhakar derivative and their regularized version. These results are presented in terms of Mittag-Leffler type function and employed to find the solutions of some Cauchy type problems such as space-time fractional advection-dispersion equation and generalized fractional free electron laser (FEL) equation, at which Hilfer-Prabhakar fractional derivative of fractional order and its regularized version are involved.

## 2. Preliminaries and notations

In this section, we study some important basic definition related to fractional calculus which are used in the sequel.

**Definition 2.1 (Riemann–Liouville integral [8, 19, 3]).** Let  $a, b \in \mathbb{R}$  such that  $a < b$  and  $f \in L^1(a, b)$ . If  $\alpha \geq 0$  then the left sided Riemann–Liouville fractional integral of order  $\alpha$  of  $f$  is defined by

$$I^\alpha f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(x, \tau) d\tau, \quad t > 0, \quad (1)$$

and

$$I^0 f(x, t) = f(x, t), \quad (\alpha = 0). \quad (2)$$

**Definition 2.2 (Riemann–Liouville derivative [8, 19, 3]).** Let  $a, b \in \mathbb{R}$  such that  $a < b$ ,  $m \in \mathbb{N}$  and  $f(x, \cdot) \in C^m(\mathbb{R}_+)$  for all  $x \in (a, b)$ . For all  $\alpha \geq 0$  such that  $m - 1 < \alpha \leq m$ , the Liouville fractional derivative of order  $\alpha > 0$  is defined as

$$D_{a+}^\alpha f(x, t) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dt^m} \left( \int_a^t \frac{f(\tau, x)}{(t - \tau)^{\alpha-m+1}} d\tau \right), \quad m = [\alpha] + 1. \quad (3)$$

**Definition 2.3 (Caputo derivative [8, 19, 3]).** Let  $f(x, \cdot) \in C^m(\mathbb{R}_+)$  for all  $x \in (a, b)$  and  $m - 1 < \alpha \leq m$ . The Caputo fractional derivative of order  $\alpha$  ( $\alpha > 0$ ) is

$${}^C D_{a+}^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - x)^{m-\alpha-1} f^{(m)}(x) dx, \quad m - 1 < \alpha \leq m. \quad (4)$$

**Theorem 2.1.** [14] For all  $f \in AC^m [a, b]$ ,  $m = \lceil \alpha \rceil$ ,  $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ , the Riemann–Liouville derivative of order  $\alpha$  of  $f$  exists almost everywhere and it can be written as

$$D_{a^+}^\alpha f(t) = {}^C D_{a^+}^\alpha f(t) + \sum_{k=0}^{m-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+). \tag{5}$$

**Definition 2.4 (Hilfer derivative [14]).** Let  $\beta \in (0, 1)$ ,  $\nu \in [0, 1]$ ,  $f \in L^1 [a, b]$ ,  $-\infty \leq a < t \leq \infty$ ,  $f * K_{(1-\nu)(1-\beta)} \in AC^1 [a, b]$ . The Hilfer derivative is expressed as

$$D_{a^+}^{\beta,\nu} f(t) = \left( I_{a^+}^{\nu(1-\beta)} \frac{d}{dt} \left( I_{a^+}^{(1-\nu)(1-\beta)} f \right) \right) (t), \tag{6}$$

where  $K_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ .

**Remark 2.1.** The Hilfer derivative (6) coincides with the Riemann–Liouville derivative (3) for  $\nu = 0$  and with the Caputo derivative (4) for  $\nu = 1$ .

**Definition 2.5 ([9]).** The one parameter Mittag-Leffler function (M-L function) is given by the following formula

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z, \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \tag{7}$$

The two parameter Mittag-Leffler function reads [27]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \tag{8}$$

such that  $E_{\alpha,1}(z) = E_\alpha(z)$ .

**Definition 2.6 ([23]).** The three parameter M-L function, also called Prabhakar function is given by

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^\infty \frac{\Gamma(\gamma+k)}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad z, \alpha, \beta, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \tag{9}$$

In applications it is usually used a further generalization of (9) which is given by

$$e_{\alpha,\beta,\omega}^\gamma(t; \omega) = t^{\beta-1} E_{\alpha,\beta}^\gamma(\omega t^\alpha), \tag{10}$$

where  $\omega \in \mathbb{C}$  is a parameter and  $t > 0$  the independent real variable.

**Definition 2.7 (Prabhakar integral [23]).** Let  $f \in L_{loc}^1([0, b])$ ,  $0 < t < b \leq \infty$ . The Prabhakar integral can be written in the form

$$\begin{aligned} \mathbf{E}_{\alpha,\beta,\omega,0^+}^\gamma f(t) &= \int_0^t (t-\tau)^{\beta-1} E_{\alpha,\beta}^\gamma[\omega(t-\tau)^\alpha] f(\tau) d\tau \\ &= \left( f * e_{\alpha,\beta,\omega}^\gamma \right) (t), \end{aligned} \tag{11}$$

where  $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ ,  $\operatorname{Re}(\alpha)$  and  $\operatorname{Re}(\beta) > 0$ .

**Definition 2.8 (Prabhakar derivative [23]).** Let  $f \in L_{loc}^1 [0, b]$ ,  $0 < t < b \leq \infty$ , and  $f * e_{\alpha,m-\beta,\omega}^{-\gamma} \in W^{m,1} [0, b]$ ,  $m = \lceil \beta \rceil$ . The Prabhakar derivative is defined by

$$\mathbf{D}_{\alpha,\beta,\omega,0^+}^\gamma f(t) = \frac{d^m}{dt^m} \mathbf{E}_{\alpha,m-\beta,\omega,0^+}^{-\gamma} f(t), \tag{12}$$

where  $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ , with  $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ .

The Riemann–Liouville derivative in (3) can be expressed in terms of Prabhakar integrals as

$$I_{0+}^{m-(\beta+\theta)} f(t) = \mathbf{E}_{\alpha, m-(\beta+\theta), \omega, 0+}^0 f(t), \quad (13)$$

we obtain that,

$$\begin{aligned} \mathbf{D}_{\alpha, \beta, \omega, 0+}^{\gamma} f(t) &= \frac{d^m}{dt^m} \mathbf{E}_{\alpha, m-\beta, \omega, 0+}^{-\gamma} f(t) \\ &= \frac{d^m}{dt^m} I_{0+}^{m-(\beta+\theta)} \mathbf{E}_{\alpha, \theta, \omega, 0+}^{-\gamma} f(t) \\ &= D_{0+}^{\beta+\theta} \mathbf{E}_{\alpha, \theta, \omega, 0+}^{-\gamma} f(t), \end{aligned} \quad (14)$$

where  $\theta \in \mathbb{C}$ ,  $\operatorname{Re}(\theta) > 0$  and  $D_{0+}^{\beta+\theta}$  the Riemann–Liouville derivative.

**Definition 2.9 (Regularized Prabhakar derivative [10]).** Let  $f \in AC[0, b]$ ,  $0 < t < b < \infty$ , and  $m = \lceil \beta \rceil$ . The regularized Prabhakar derivative is given

$${}^C \mathbf{D}_{\alpha, \beta, \omega, 0+}^{\gamma} f(t) = \mathbf{E}_{\alpha, m-\beta, \omega, 0+}^{-\gamma} \frac{d^m}{dx^m} f(t), \quad (15)$$

where  $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ ,  $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ .

**Definition 2.10 (Hilfer–Prabhakar derivative [10], [20]).** Let  $\beta \in (0, 1)$ ,  $\nu \in [0, 1]$ ,  $f \in L^1[a, b]$ ,  $0 < t < b \leq \infty$ ,  $f * e_{\alpha, (1-\nu)(1-\beta), \omega}^{-\gamma(1-\nu)}(\cdot) \in AC^1[a, b]$ . The Hilfer–derivative is

$$D_{\alpha, \omega, 0+}^{\gamma, \beta, \nu} f(t) = \left( \mathbf{E}_{\alpha, \nu(1-\beta), \omega, 0+}^{-\gamma\nu} \frac{d}{dt} \left( \mathbf{E}_{\alpha, (1-\nu)(1-\beta), \omega, 0+}^{-\gamma(1-\nu)} f \right) \right) (t), \quad (16)$$

where  $\gamma, \omega \in \mathbb{R}$ ,  $\alpha > 0$ , and where  $\mathbf{E}_{\alpha, 0, \omega, 0+}^0 f = f$ .

**Definition 2.11 (Regularized Version of Hilfer–Prabhakar Fractional Derivative [20]).** Let  $f \in AC[0, b]$ ,  $\beta \in (0, 1)$ ,  $\nu \in [0, 1]$ ,  $f \in L^1[a, b]$ ,  $0 < t < b < \infty$ . The regularized Prabhakar derivative of  $f(t)$  is

$$\begin{aligned} {}^C \mathbf{D}_{\alpha, \omega, 0+}^{\gamma, \beta, \nu} f(t) &= \left( \mathbf{E}_{\alpha, \nu(1-\beta), \omega, 0+}^{-\gamma\nu} \mathbf{E}_{\alpha, (1-\nu)(1-\beta), \omega, 0+}^{-\gamma(1-\nu)} \frac{d}{dt} f \right) (t) \\ &= \left( \mathbf{E}_{\alpha, 1-\beta, \omega, 0+}^{-\gamma} \frac{d}{dt} f \right) (t). \end{aligned} \quad (17)$$

Where  $\gamma, \omega \in \mathbb{R}$ ,  $\alpha > 0$ .

### 3. Fundamental Facts of the Shehu Transform

The Sumudu transform ([2], [4]) is obtained over the set of functions

$$A = \left\{ f(t) : \exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{t}{\eta_i}\right), \text{ if } t \in (-1)^i \times [0, \infty) \right\}, \quad (18)$$

by

$$\mathbb{S}[f(t)] = G(u) = \int_0^{\infty} f(ut) \exp(-t) dt, \quad u \in (-\eta_1, \eta_2). \quad (19)$$

Shehu transform of function  $f(t)$  is recently introduced by Shehu Maitama and Weidong Zhao [25] and it is a generalization of the Laplace and the Sumudu integral transforms.

The Shehu transform is obtained over the set  $A$  is defined as

$$\mathbb{H}[f(t)] = V(s, u) = \lim_{a \rightarrow \infty} \int_0^a \exp\left(-\frac{st}{u}\right) f(t) dt. \quad s > 0, \quad u > 0. \quad (20)$$

where  $s$  and  $u$  are the Shehu transform variables and  $a$  is a real constant.

Obviously, the Shehu transform is linear as the Laplace and Sumudu transformations. Inversion formula of (20), is given by

$$\mathbb{H}^{-1}[V(s, u)] = f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp\left(-\frac{st}{u}\right) f(t) dt. \quad \text{for } t \geq 0. \quad (21)$$

**Theorem 3.1.** [25] (*Derivative of Shehu transform*). Let  $f(t) \in A$ . If the function  $f^{(n)}(t)$  is the  $n^{\text{th}}$  derivative of the function  $f(t)$  with respect to  $t$ , then for  $n \geq 1$  its Shehu transform is defined by

$$\mathbb{H}[f^{(n)}(t)] = \left(\frac{s}{u}\right)^n V(s, u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} f^{(k)}(0). \quad (22)$$

Where  $V(s, u)$  denotes the Shehu transform of  $f(t)$ .

**Proposition 3.1.** [28] Shehu transform of  $n^{\text{th}}$  order partial derivative is defined as

$$\mathbb{H}\left[\frac{\partial^n u(x, t)}{\partial t^n}\right] = \left(\frac{s}{u}\right)^n V(x, s, u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} \frac{\partial^k u(x, 0)}{\partial t^k}. \quad (23)$$

Where  $V(x, s, u)$  denotes the shehu transform of the partial derivative of the function  $u(x, t)$ .

In the next theorem, we find relation between Sumudu and shehu transform.

**Theorem 3.2.** [6] (*Shehu-Sumudu duality*) Let  $f(t) \in A$  and  $G(u)$  be the Sumudu transform, the Shehu transform  $V(s, u)$  of  $f(t)$  is given by

$$V(s, u) = \frac{u}{s} G\left(\frac{u}{s}\right). \quad (24)$$

**Theorem 3.3.** [5] (*Convolution theorem for Shehu transform*) Let  $f(t)$  and  $g(t)$  be in  $A$ , having Sumudu transforms  $F(u)$  and  $G(u)$ , respectively, and Shehu transforms  $V(s, u)$  and  $W(s, u)$ , respectively. The Shehu transform of the convolution of  $f$  and  $g$  is given by

$$\mathbb{H}((f * g)(t)) = V(s, u) W(s, u). \quad (25)$$

where the convolution is defined as

$$(f * g)(t) = \int_0^\infty f(t) g(t - \tau) d\tau. \quad (26)$$

**Lemma 3.1.** [5] In the complex plane  $\mathbb{C}$ , for any  $\text{Re}(\alpha), \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0$  and  $\omega \in \mathbb{C}$ . Shehu transform of  $E_{\alpha, \beta}^\gamma(\omega t^\alpha)$  is given by

$$\mathbb{H}\left(t^{\beta-1} E_{\alpha, \beta}^\gamma(\omega t^\alpha)\right) = \left(\frac{u}{s}\right)^\beta \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^{-\gamma}. \quad (27)$$

### 4. Main Result

In this section, we find the Shehu transforms of: Prabhakar fractional derivative, regularized version of Prabhakar fractional derivative, Hilfer-Prabhakar fractional derivative and its regularized version. in the following, let  $f(t) \in A$  with Shehu transform  $V(s, u)$ .

**Lemma 4.1.** *The Shehu transform of Prabhakar derivative (12) is*

$$\mathbb{H}(\mathbf{D}_{\alpha,\beta,\omega,0+}^\gamma f(t))(s, u) = \left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma V(s, u) - \sum_{k=0}^{m-1} \left(\frac{s}{u}\right)^{m-(k+1)} \left[\mathbf{D}_{\alpha,k-m+\beta,\omega,0+}^\gamma f(t)\right]_{t=0+}. \tag{28}$$

*Proof.* Applying Shehu transforms of Prabhakar fractional derivative (12) with respect to variable  $t$  and using (22), (27) and convolution theorem for Shehu transform (25), we get

$$\begin{aligned} \mathbb{H}(\mathbf{D}_{\alpha,\beta,\omega,0+}^\gamma f(t))(s, u) &= \mathbb{H}\left[\frac{d^m}{dt^m} \mathbf{E}_{\alpha,m-\beta,\omega,0+}^{-\gamma} f(t)\right](s, u), \\ &= \left(\frac{s}{u}\right)^m \mathbb{H}\left[\left(\mathbf{e}_{\alpha,m-\beta,\omega,0+}^{-\gamma} * f\right)(t)\right](s, u) \\ &\quad - \sum_{k=0}^{m-1} \left(\frac{s}{u}\right)^{m-(k+1)} \left[\left(\frac{d}{dt}\right)^{(k)} \mathbf{E}_{\alpha,m-\beta,\omega,0+}^{-\gamma} f(t)\right]_{t=0+}, \\ &= \left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma V(s, u) \\ &\quad - \sum_{k=0}^{m-1} \left(\frac{s}{u}\right)^{m-(k+1)} \left[\left(\frac{d}{dt}\right)^{(k)} \mathbf{E}_{\alpha,k-(k-m+\beta),\omega,0+}^{-\gamma} f(t)\right]_{t=0+}, \end{aligned}$$

through some simplification, we obtain the required result (28). □

**Lemma 4.2.** *The Shehu transform of regularized version of Prabhakar fractional derivative (15) is*

$$\mathbb{H}({}^C\mathbf{D}_{\alpha,\beta,\omega,0+}^\gamma f(t))(s, u) = \left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega \left(\frac{u}{s}\right)^\alpha\right]^\gamma V(s, u) - \sum_{k=0}^{m-1} \left(\frac{u}{s}\right)^{k-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma f^{(k)}(0^+). \tag{29}$$

*Proof.* We follow the same method used in the previous proof, so, taking Shehu transforms of Prabhakar fractional derivative (15) with respect to variable  $t$  and using (22), (27) and (25), we get

$$\begin{aligned} \mathbb{H}({}^C\mathbf{D}_{\alpha,\beta,\omega,0+}^\gamma f(t))(s, u) &= \mathbb{H}\left[\mathbf{E}_{\alpha,m-\beta,\omega,0+}^{-\gamma} \frac{d^m}{dx^m} f(t)\right](s, u) \\ &= \mathbb{H}\left[\left(\mathbf{E}_{\alpha,m-\beta,\omega,0+}^{-\gamma} * \frac{d^m}{dx^m} f\right)(t)\right](s, u), \\ &= \left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma V(s, u) \\ &\quad - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{(k+1)-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma f^{(k)}(0). \end{aligned}$$

This is the desired result (29). □

**Lemma 4.3.** *The Shehu transform of Hilfer-Prabhakar fractional derivative (16) is expressed as*

$$\begin{aligned} \mathbb{H} \left( \mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} f(t) \right) (s, u) &= \left( \frac{u}{s} \right)^{-\beta} \left[ 1 - \omega \left( \frac{u}{s} \right)^\alpha \right]^\gamma V(s, u) \\ &\quad - \left( \frac{u}{s} \right)^{\nu(1-\beta)} \left[ 1 - \omega \left( \frac{u}{s} \right)^\alpha \right]^{\gamma\nu} \left[ \mathbf{E}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}^{-\gamma(1-\nu)} f(t) \right]_{t=0^+}. \end{aligned} \quad (30)$$

*Proof.* First, applying the Shehu transform of Hilfer-Prabhakar fractional derivative (16) with respect to variable  $t$ , then, using (11), (10), (27) and convolution theorem for Shehu transform (25), we can write

By other method, we can use the Sumudu transform of Hilfer-Prabhakar fractional derivative defined in([20],[21] )by

$$\begin{aligned} \mathbb{S} \left( D_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} f(t) \right) (u) &= u^{-\beta} [1 - \omega u^\alpha]^\gamma \mathbb{S}[f](u) \\ &\quad - u^{\nu(1-\beta)} [1 - \omega u^\alpha]^{\gamma\nu} \left[ \mathbf{E}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}^{-\gamma(1-\nu)} f(t) \right]_{t=0^+}, \end{aligned}$$

and by using Shehu-Sumudu duality theorem(3.2), we obtain the same result given by equation (30).  $\square$

**Lemma 4.4.** *The Shehu transforms of the regularized version of Hilfer-Prabhakar fractional derivative (17) of order  $\beta$  is*

$$\begin{aligned} \mathbb{H} \left( {}^C \mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} f(t) \right) (s, u) &= \left( \frac{u}{s} \right)^{-\beta} \left( 1 - \omega \left( \frac{u}{s} \right)^\alpha \right)^\gamma V(s, u) \\ &\quad - \left( \frac{u}{s} \right)^{1-\beta} \left( 1 - \omega \left( \frac{u}{s} \right)^\alpha \right)^\gamma f(0^+). \end{aligned} \quad (31)$$

*Proof.* Applying Shehu transforms of regularized version of Hilfer-Prabhakar fractional derivative (17) of order  $\beta$ , then using (11), (25), (10), (27) and (22). We have

$$\begin{aligned} \mathbb{H} \left( {}^C \mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} f(t) \right) (s, u) &= \mathbb{H} \left[ \left( \mathbf{E}_{\alpha,1-\beta,\omega,0^+}^{-\gamma} \frac{d}{dt} f \right) (t) \right] (s, u) \\ &= \mathbb{H} \left[ \left( \mathbf{e}_{\alpha,1-\beta,\omega,0^+}^{-\gamma} * \frac{d}{dt} f \right) (t) \right] (s, u), \\ &= \left( \frac{u}{s} \right)^{1-\beta} \left( 1 - \omega \left( \frac{u}{s} \right)^\alpha \right)^\gamma \left[ \left( \frac{s}{u} \right) V(s, u) - f(0^+) \right], \\ &= \left( \frac{u}{s} \right)^{-\beta} \left( 1 - \omega \left( \frac{u}{s} \right)^\alpha \right)^\gamma V(s, u) \\ &\quad - \left( \frac{u}{s} \right)^{1-\beta} \left( 1 - \omega \left( \frac{u}{s} \right)^\alpha \right)^\gamma f(0^+). \end{aligned}$$

Also, by a different method, we can get the same result given by equation (31), by using the Sumudu transform of regularized version of Hilfer-Prabhakar fractional derivative defined in( [20],[21] ) and by using Shehu-Sumudu duality theorem(3.2).  $\square$

### 5. Applications

In this section, we will provide some applications of Hilfer–Prabhakar derivatives using Shehu transform to find the solutions of Cauchy problems such as space-time fractional advection-dispersion equation and generalized fractional Free Electron Laser (FEL) equation [12, 10].

5.1. Generalized Space-time Fractional Advection-Dispersion Equation

Here we find, the solution of the generalized space-time advection-dispersion equation (32) under the initial condition (33) and the boundary condition(33).

**Theorem 5.1.** *The solution of Cauchy problem*

$$\mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} (u(x,t)) = -\eta D_x u(x,t) + \xi \Delta^{\frac{\lambda}{2}} (u(x,t)), \tag{32}$$

Subject to below constraints

$$\mathbf{E}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}^{-\gamma(1-\nu)} u(x,0^+) = g(x), \quad \omega, \gamma, x \in \mathbb{R}, \quad \alpha > 0, \tag{33}$$

$$\lim_{x \rightarrow \infty} u(x,t) = 0, \quad t \geq 0, \tag{34}$$

is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\beta)+n\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp^{-ikx} g(k) (i\eta k - \xi |k|^\lambda)^n E_{\alpha,\nu(1-\beta)+n\beta}^{\gamma(n-\nu)} (\omega t^\alpha) dk. \tag{35}$$

where  $\lambda \in (0, 2]$ ,  $x \in \mathbb{R}$ ,  $t \in \mathbb{R}^+$ ,  $\beta \in (0, 1)$ ,  $\nu \in [0, 1]$ .

$\Delta^{\frac{\lambda}{2}}$  is the fractional Laplace operator of order  $\lambda$  where it is defined in [7].

The positive constants  $\eta$  and  $\xi$  are the average fluid velocity and the dispersion coefficient respectively .

*Proof.* First, applying the Fourier transform of equation (32) with respect to the space variable  $x$ , we can write

$$\mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} (\bar{u}(k,t)) = \eta ik \bar{u}(k,t) - \xi |k|^\lambda \bar{u}(k,t), \tag{36}$$

where  $\bar{u}(k,t)$  represent Fourier transform of  $u(x,t)$  and the Fourier transform of  $\Delta^{\frac{\lambda}{2}}$  is given in [7], as

$$F \left\{ \Delta^{\frac{\lambda}{2}} (u(x,t)); k \right\} = -|k|^\lambda F \{u(x,t)\}, \quad \lambda \in (0, 2]. \tag{37}$$

Then, taking the Shehu transform on left sided of the above equation (36) with respect to the space variable  $t$  and by using (30), we obtain

$$\mathbb{H} \left\{ \mathbf{D}_{\alpha,\omega,0^+}^{\gamma,\beta,\nu} (\bar{u}(k,t)) \right\} = \left(\frac{u}{s}\right)^{-\beta} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^\gamma \bar{V}(k,s,u) - \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^{\gamma\nu} \left[ \mathbf{E}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}^{-\gamma(1-\nu)} \bar{u}(k,t) \right]_{t=0^+}, \tag{38}$$

where  $\bar{V}(k,s,u)$  represents Shehu transform of  $\bar{u}(k,t)$ .

Again, apply Shehu transform on right hand side of the equation (36) and using the initial condition (33), we get

$$\left(\frac{u}{s}\right)^{-\beta} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^\gamma \bar{V}(k,s,u) - \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^{\gamma\nu} g(k) = \eta ik \bar{V}(k,s,u) - \xi |k|^\lambda \bar{V}(k,s,u), \tag{39}$$

after some simplifications ,we can write

$$\bar{V}(k,s,u) = \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^{\gamma\nu} g(k) \frac{1}{1 - \frac{\xi |k|^\lambda - \eta ik}{\left(\frac{u}{s}\right)^{-\beta} \left[ 1 - \omega \left(\frac{u}{s}\right)^\alpha \right]^\gamma}}, \tag{40}$$

so, it gives

$$\bar{V}(k, s, u) = \sum_{n=0}^{\infty} \left(\frac{u}{s}\right)^{\nu(1-\beta)+\beta n} \left[1 - \omega \left(\frac{u}{s}\right)^{\alpha}\right]^{-\gamma(n-\nu)} \left(\xi |k|^{\lambda} - \eta ik\right)^n g(k). \tag{41}$$

Now, taking inverse Shehu transform of equation (41) using (27), we have

$$\bar{u}(k, t) = \sum_{n=0}^{\infty} \left(i\eta k - \xi |k|^{\lambda}\right)^n g(k) t^{\nu(1-\beta)+n\beta-1} E_{\alpha, \nu(1-\beta)+n\beta}^{\gamma(n-\nu)}(\omega t^{\alpha}). \tag{42}$$

Again, taking inverse Fourier transform of (42), get our required result (35). □

**Example 5.1.** If  $\eta = 0$ ,  $\xi = \frac{i\hbar}{2m}$  in above theorem 5.1, the solution of the resulting equation called one dimensional space-time Schrödinger equation of fractional order, for a free nature particle of mass  $m$  with  $\hbar$  Planck constant, is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\beta)+n\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp^{-ikx} g(k) \left(-\frac{i\hbar}{2m} |k|^{\lambda}\right)^n E_{\alpha, \nu(1-\beta)+n\beta}^{\gamma(n-\nu)}(\omega t^{\alpha}) dk, \tag{43}$$

Where  $\lambda, x, t, \beta, \nu$  and  $\Delta^{\frac{\lambda}{2}}$  are the same as we identified previously.

**Example 5.2.** To describe solute transport in aquifers, we take  $\eta = 1$ ,  $\xi = \frac{d}{v'L}$  in equation 32, and take  $g(x) = \exp(-x)$ ,  $0 < x < 1$  in the conditions (33).

The analytical expression of solute concentration of the resulting Cauchy type problem defined by equation (32) subject to constraints to (33) and (33), is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\beta)+n\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp^{-ikx} g(k) \left(ik - \mu' |k|^{\lambda}\right)^n E_{\alpha, \nu(1-\beta)+n\beta}^{\gamma(n-\nu)}(\omega t^{\alpha}) dk. \tag{44}$$

where  $\mu' = \frac{d}{v'L}$ ,  $L$  is the packing length,  $d$  is the dispersion coefficient and  $v'$  is the Darcy velocity.

### 5.2. Fractional Free Electron Laser (FEL) equation

Here we study the following fractional generalization of the FEL equation, involving Hilfer–Prabhakar derivatives.

**Theorem 5.2.** The solution of Cauchy problem

$$\mathbf{D}_{\alpha, \omega, 0+}^{\gamma, \beta, \nu} y(t) = \lambda \mathbf{E}_{\alpha, \beta, \omega, 0+}^{\delta} y(t) + f(t), \tag{45}$$

$$\left(\mathbf{E}_{\alpha, (1-\nu)(1-\beta), \omega, 0+}^{-\gamma(1-\nu)} f\right)_{t=0+} = K, \tag{46}$$

where  $f(x) \in L_1[0, \infty)$ ;  $\beta \in (0, 1)$ ,  $\nu \in [0, 1]$ ;  $\omega, \lambda \in \mathbb{C}$ ;  $t, \alpha > 0$ ,  $K, \gamma, \delta \geq 0$ , is given by

$$y(t) = K \sum_{n=0}^{\infty} \lambda^n t^{\nu(1-\beta)+\beta(2n+1)-1} E_{\alpha, \nu(1-\beta)+\beta(2n+1)}^{\gamma-\gamma\nu+n(\delta+\gamma)}(\omega t^{\alpha}) + \sum_{n=0}^{\infty} \mathbf{E}_{\alpha, \beta(2n+1), \omega, 0+}^{\gamma+n(\delta+\gamma)} f(t). \tag{47}$$

*Proof.* We denote by  $Y(s, u)$  and  $V(s, u)$  the Shehu transform of  $y(t)$  and  $f(t)$ , respectively. Applying Shehu transform of (45) and using (11), (25), (10) and (27), (46), we can write

$$\begin{aligned} \mathbb{H}\left(\mathbf{D}_{\alpha, \omega, 0+}^{\gamma, \beta, \nu} y(t)\right)(s, u) &= \mathbb{H}\left(\lambda \mathbf{E}_{\alpha, \beta, \omega, 0+}^{\delta} y(t) + f(t)\right)(s, u) \\ &= \lambda \mathbb{H}\left(\mathbf{E}_{\alpha, \beta, \omega, 0+}^{\delta} y(t)\right)(s, u) + V(s, u), \\ &= \lambda \mathbb{H}\left((\mathbf{e}_{\alpha, \beta, \omega, 0+}^{\delta} * y)(t)\right)(s, u) + V(s, u), \\ &= \lambda \mathbb{H}\left(t^{\beta-1} E_{\alpha, \beta}^{\delta}(\omega t^{\alpha})\right)(s, u) Y(s, u) + V(s, u), \\ &= \lambda \left(\frac{u}{s}\right)^{\beta} \left(1 - \omega \left(\frac{u}{s}\right)^{\alpha}\right)^{-\delta} Y(s, u) + V(s, u), \end{aligned}$$

and from lemma 4.3, we get

$$\begin{aligned} \left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^\gamma Y(s, u) - K \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^{\gamma\nu} \\ = \lambda \left(\frac{u}{s}\right)^\beta \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-\delta} Y(s, u) + V(s, u), \end{aligned} \tag{48}$$

so that

$$\begin{aligned} Y(s, u) &= \frac{V(s, u) + K \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^{\gamma\nu}}{\left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^\gamma} \frac{1}{\left[1 - \frac{\lambda\left(\frac{u}{s}\right)^\beta \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-\delta}}{\left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^\gamma}\right]}, \\ &= \left[\frac{V(s, u) + K \left(\frac{u}{s}\right)^{\nu(1-\beta)} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^{\gamma\nu}}{\left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^\gamma}\right] \sum_{n=0}^{\infty} \lambda^n \left(\frac{u}{s}\right)^{(2\beta)n} \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-(\delta+\gamma)n}, \\ &= V(s, u) \sum_{n=0}^{\infty} \lambda^n \left(\frac{u}{s}\right)^{\beta(2n+1)} \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-(\delta+\gamma)n-\gamma} \\ &\quad + K \sum_{n=0}^{\infty} \lambda^n \left(\frac{u}{s}\right)^{\beta(2n+1)+\nu(1-\beta)} \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-(\delta+\gamma)n+\gamma\nu-\gamma}. \end{aligned}$$

Last step is valid for  $\left|\frac{\lambda\left(\frac{u}{s}\right)^\beta \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-\delta}}{\left(\frac{u}{s}\right)^{-\beta} \left[1 - \omega\left(\frac{u}{s}\right)^\alpha\right]^\gamma}\right| \leq 1$ . The required solution (47) is obtained by applying the inverse of Shehu transform on both side of last equation,

$$\begin{aligned} y(t) &= \mathbb{H}^{-1} \left[ V(s, u) \sum_{n=0}^{\infty} \lambda^n \left(\frac{u}{s}\right)^{\beta(2n+1)} \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-(\delta+\gamma)n-\gamma} \right] \\ &\quad + \mathbb{H}^{-1} \left[ K \sum_{n=0}^{\infty} \lambda^n \left(\frac{u}{s}\right)^{\beta(2n+1)+\nu(1-\beta)} \left(1 - \omega\left(\frac{u}{s}\right)^\alpha\right)^{-(\delta+\gamma)n+\gamma\nu-\gamma} \right], \\ &= K \sum_{n=0}^{\infty} \lambda^n t^{\beta(2n+1)+\nu(1-\beta)} E_{\alpha, \beta(2n+1), \omega, 0^+}^{\gamma+n(\delta+\gamma)} + \sum_{n=0}^{\infty} E_{\alpha, \beta(2n+1), \omega, 0^+}^{\gamma+n(\delta+\gamma)} f(t), \end{aligned}$$

Which is required solution . □

**Theorem 5.3.** *The solution of Cauchy problem*

$${}^C \mathbf{D}_{\alpha, \omega, 0^+}^{\gamma, \beta, \nu} u(x, t) = K \frac{\partial^2}{\partial t^2} u(x, t), \quad t > 0, x \in \mathbb{R}, \tag{49}$$

$$[u(x, t)]_{t=0^+} = g(x), \tag{50}$$

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \tag{51}$$

where  $f(x) \in L_1[0, \infty)$ ;  $\beta \in (0, 1), \nu \in [0, 1]; \omega \in \mathbb{R}, K, \alpha > 0, \gamma \geq 0$ , is given by

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} E_{\alpha, \beta n+1}^{\gamma n} (\omega t^\alpha) (-K t^\gamma \eta^2)^n \exp^{-i\eta x} G(\eta) d\eta. \tag{52}$$

*Proof.* We denote by  $\bar{u}(\eta, t)$  the Fourier transform of  $u(x, t)$  with respect to the time variable  $x$  and  $\bar{V}(\eta, s, u)$  the Shehu transform of  $\bar{u}(\eta, t)$  with respect to the time variable  $t$ . Taking the Fourier transform of (49), we get

$${}^C D_{\alpha, \omega, 0^+}^{\gamma, \beta, \nu}(\bar{u}(x, t)) = -K\eta^2 \bar{u}(x, t), t > 0, x \in \mathbb{R}. \tag{53}$$

Then, by formula (31), we have

$$\left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma \bar{V}(\eta, s, u) - \left(\frac{u}{s}\right)^{1-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma G(\eta) = -K\eta^2 \bar{u}(x, t), \tag{54}$$

so that

$$\begin{aligned} \bar{V}(k, s, u) &= \left[ \frac{\left(\frac{s}{u}\right)}{G(\eta)} + \frac{K\eta^2}{\left(\frac{u}{s}\right)^{1-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma G(\eta)} \right]^{-1}, \\ &= \left(\frac{u}{s}\right) G(\eta) \frac{1}{\left(1 + \frac{K\eta^2}{\left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma}\right)}, \\ &= \sum_{n=0}^{\infty} \left(\frac{u}{s}\right) G(\eta) \left(\frac{-K\eta^2}{\left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma}\right)^n, \frac{K\eta^2}{\left(\frac{u}{s}\right)^{-\beta} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^\gamma} < 1, \\ &= \sum_{n=0}^{\infty} (-K)^n \eta^{2n} \left(\frac{u}{s}\right)^{1+\beta n} \left(1 - \omega \left(\frac{u}{s}\right)^\alpha\right)^{-\gamma n} G(\eta). \end{aligned}$$

Using first the Shehu transform, it produces

$$\bar{u}(k, t) = \sum_{n=0}^{\infty} (-K)^n \eta^{2n} G(\eta) E_{\alpha, \beta n+1}^{\gamma n}(\omega t^\alpha). \tag{55}$$

Finally, by inverting the Shehu transform to (55) we obtain the required solution (52). □

### 6. Conclusion

In this work, we present the Shehu transform of Hilfer-Prabhakar fractional derivative and its regularized version. We also present some its application of Cauchy type problems such as Space-time Fractional Convection-dispersion Equation and Generalized fractional Free Electron Laser (FEL) equation using the results of the third and the fourth section. The results shows that Shehu transform is very useful for solving fractional differential equations.

#### Conflict of interest

This work does not have any conflicts of interest.

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