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Novel Schemes for Cauchy-Riemann System of Equations with Cauchy Conditions

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Abstract

This communication deals with the analytical solutions of Cauchy problem for Cauchy-Riemann system of equations which is basically unstable according to Hadamard but its solution exists if its initial data is analytic. Here, we used the Vectorial Adomian Decomposition (VAD) method, Vectorial Variational Iteration (VVI) method, and Vectorial Modified Picard's Method (VMP) method to get the convergent series solution. These suggested schemes give analytical approximation in an infinite series form without using discretization. These methods are effectual and reliable which is demonstrated through six model problems having variety of source terms and analytic initial data.

Keywords: Cauchy-Riemann system of equations, Cauchy problem, Vectorial Adomian Decomposition method, Vectorial Variational Iteration method, Vectorial Modified Picard's Method.

1. Introduction

Many physical systems are used to predict the behavior of different phenomenon which are modeled in the form of partial differential equations. In order to understand the underlying physics of these phenomena, solutions of these expressions is a matter of great importance. Researchers have proposed several numerical and analytical methods to obtain the solutions of complex systems. For instance, Santra et al. [1] presented the oscillation theorem and also discussed the consistency analysis of second-order differential equations in

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their exploration. They proved several theorems for the validity of the solutions of this class of differential system. Also, they established the condition for the oscillation of solution. Ahmed et al. [2] studied the KdV equations via different well known approaches. They have analyzed and noticed the behavior of solution by different methods and presented the comparative analysis by computing the error tables. Also, they plotted the behavior of exact and approximate solutions. Moreover, they presented the absolute error analysis for the authenticity and applicability of utilized method. Turkyilmazoglu [3] presented the modification in variational iteration scheme by introducing the concept of accelerating convergence parameter. Oil diffusion model was studied by Ahmad et al. [4]. They presented the analytical solutions. They computed the residual errors and noticed that error diminish with the higher order approximation. Turkyilmazoglu [5] introduced the modification in Adomian scheme by using the convergence parameter. From several modeled examples, it is tested that the proposed modification increases the convergence rate as compared with the traditional approach. Several other important contributions can be found in [6]-[12].

In this article, we deal with the first-order system of linear partial differential equations [13]

$$\frac{\partial}{\partial y} \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}, x \in \mathbb{R}, y > 0, \quad (1)$$

with Cauchy condition

$$\begin{bmatrix} u(x, 0) \\ v(x, 0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, x \in \mathbb{R} \quad (2)$$

where $\phi(x)$ and $\psi(x)$ are analytic and taking y as a time variable for the desired solution vector

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

for real-valued functions $u(x, y)$ and $v(x, y)$. Collectively (1) is elliptic while individually both the partial differential equations [14] are hyperbolic. According to Hadamard [15], the Cauchy problem for the CRE is ill-posed. Tikhonov [16] and Bertero et al. [17] also discussed the ill-posed problems and their solution. Walter [18] proved the Cauchy-Kowalewsky theorem which guarantees existence and uniqueness of the Cauchy problem which are Hadamard unstable. The main source of motivation to our present article is Joseph et al. [19], according to them, “the problems which are Hadamard unstable cannot be solved unless the initial data are analytic”. Available literature witnessed that no study has been conducted to investigate the IVP for CR-equation in three dimensions. The next project will covers this type of problem in open literature.

2. Adomian’s decomposition method

George Adomian [20]-[22] introduced a powerful technique for solving functional equations of any kind and stochastic problems which is known as the Adomian’s Decomposition Method (ADM). This method provides an effective procedure for analytical solution of real physical problems. Its convergence is discussed by Abbaoui et al. [23], Abdelrazec et al. [24], and Turkyilmazoglu [25]. The ADM is used to tackle the problem directly without using linearization, perturbation, or any other restrictive assumptions.

2.1. Analysis of the Adomian’s Decomposition Method

Let us consider the equation

$$Fu = g(x, y), \quad (3)$$

where F is a differential operator involving both linear and non-linear terms. We write the Eq. (3) in operator form as

$$Lu + Ru + Nu = g. \quad (4)$$

where

L = Invertible Highest order linear derivative,

R = Contain the rest of the linear operators,

N = The non-linear terms if there are any,

g = The source term.

Solving Eq. (4) for Lu , we have

$$Lu = g - Ru - Nu. \quad (5)$$

The inverse of L is L^{-1} , which is considered as definite integral, i.e. if $L = \frac{d}{dx}$, then

$$L^{-1} = \int_0^x (\cdot) ds. \quad (6)$$

If L is second derivative, then L^{-1} can be defined as a two-fold integral. Applying L^{-1} on both sides of the Eq. (5) we have

$$u = f - L^{-1}(Ru) - L^{-1}(Nu) \quad (7)$$

where f consist of integration of the source term and Cauchy conditions. Finally, the solution u is represented as the infinite series

$$u = \sum_{n=0}^{\infty} u_n. \quad (8)$$

The terms u_0, u_1, u_2, \dots are obtained recursively through Eq. (11) and non-linear term Nu is disintegrated as follows

$$Nu = \sum_{n=0}^{\infty} A_n, \quad (9)$$

where A_n are Adomian polynomial in $u_i, i = 0, 1, 2, 3, \dots$ and generated by the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right)_{\lambda=0}, \quad n = 0, 1, 2, 3, \dots \quad (10)$$

Then calculating the components u_0, u_1, u_2, \dots from the following relation recursively

$$\left[\begin{array}{l} u_0 = f \\ u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), \quad k \geq 0, \end{array} \right] \quad (11)$$

and putting these terms in Eq. (8), we get the required Adomian's Decomposition solution. So once the term u_0 is defined then the remaining terms $u_k, k \geq 1$ are completely determined.

2.2. Analysis of Vectorial Adomian's Decomposition (VAD) Method for Cauchy-Riemann Equations

To demonstrate the VADM, we consider

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} + \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}, \quad (12)$$

with the given Cauchy conditions

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (13)$$

where invertible operator of highest order L_y is given by

$$L_y = d/dy, \text{ and } L_x = d/dx.$$

$f(x, y)$ and $g(x, y)$ are the forcing terms. The integral operator L_y^{-1} is defined by

$$L_y^{-1} = \int_0^y (\cdot) dy.$$

Applying L_y^{-1} on both sides of Eq. (12), and using the given conditions, we obtain

$$\begin{bmatrix} u_{n+1}(x, y) \\ v_{n+1}(x, y) \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} + L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} \right\}. \quad (14)$$

The function $f(x, y)$ and $g(x, y)$ consist of integration of the source term and all prescribed Cauchy conditions.

$$\left\{ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}, \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, n \geq 0. \right. \quad (15)$$

So the series solution is determined as

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} u_n(x, y) \\ v_n(x, y) \end{bmatrix}. \quad (16)$$

2.3. Implementation of VAD Method

The proposed method is applied on the Cauchy problems for the Cauchy-Riemann systems of equations.

2.3.1. Model Problem-A

Consider the Cauchy-Riemann system of equation

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \quad (17)$$

with Cauchy data

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh(x) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (18)$$

and the exact solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sin(y) \cosh(x) \\ \cos(y) \sinh(x) \end{bmatrix}. \quad (19)$$

Applying VAD procedure on Eqs. (17)-(18) which gives the recurrence relation

$$\left\{ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh(x) \end{bmatrix}, \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, n \geq 0, \right. \quad (20)$$

and Eq. (20) yields the following vectors

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh(x) \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \cosh(x) y \\ 0 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sinh(x) \frac{y^2}{2!} \end{bmatrix}, \dots \quad (21)$$

Hence the solution in closed form is

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \sum_{n=0}^{\infty} \begin{bmatrix} u_n(x, y) \\ v_n(x, y) \end{bmatrix} = \begin{bmatrix} \left\{ y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right\} \cosh(x) \\ \left\{ 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right\} \sinh(x) \end{bmatrix}, \\ \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} &= \begin{bmatrix} \sin(y) \cosh(x) \\ \cos(y) \sinh(x) \end{bmatrix}, \end{aligned} \quad (22)$$

which is the same as exact solution.

Comparison of exact and approximated solution via VADM is given by the following graph

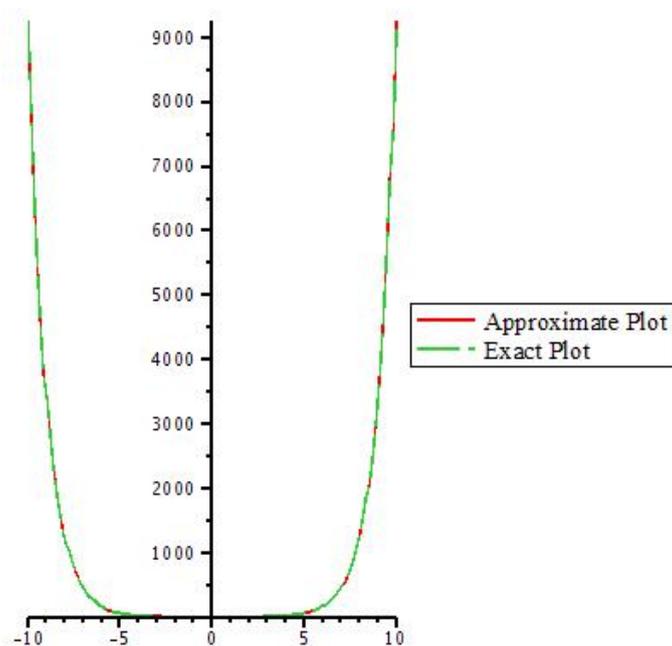


Figure 1: Graph of exact solution and 10th iteration of u by taking $y=1$.

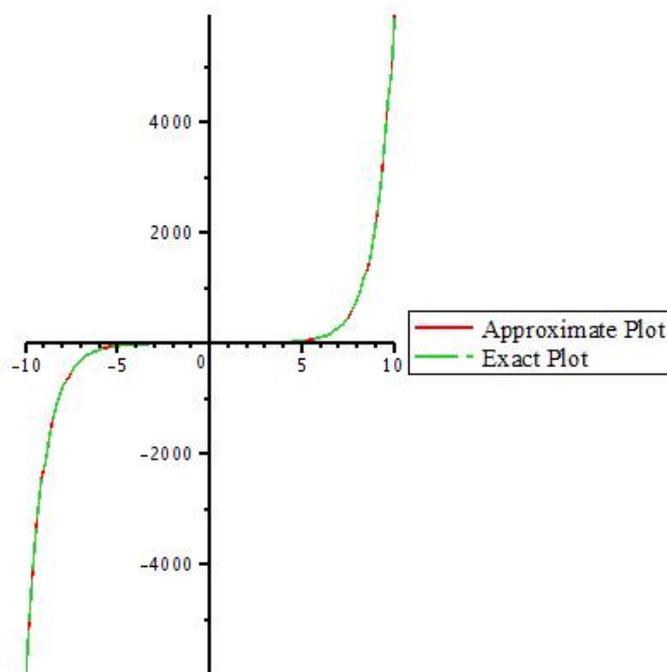


Figure 2: Graph of exact solution and 10th iteration of v by taking $y=1$.

2.3.2. Model Problem-B

Consider the Eq. (17) subject to the initial data

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \tag{23}$$

with the exact solution

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e^{-y} \sin(x) \\ e^{-y} \cos(x) \end{bmatrix} \tag{24}$$

The recurrence relation is given as

$$\left\{ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \right. \tag{25}$$

$$\left. \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, n \geq 0. \right.$$

Thus we obtained the following vectors

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -\sin(x) y \\ -\cos(x) y \end{bmatrix},$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sin(x) \frac{y^2}{2!} \\ \cos(x) \frac{y^2}{2!} \end{bmatrix}, \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \sin(x) \frac{y^3}{3!} \\ \cos(x) \frac{y^3}{3!} \end{bmatrix} \dots \tag{26}$$

So we have

$$\begin{bmatrix} u \\ v \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} u_n(x, y) \\ v_n(x, y) \end{bmatrix} = \begin{bmatrix} \left\{ 1 - y + \frac{y^2}{2!} - \dots \right\} \sin(x) \\ \left\{ 1 - y + \frac{y^2}{2!} - \dots \right\} \cos(x) \end{bmatrix},$$

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} e^{-y} \sin(x) \\ e^{-y} \cos(x) \end{bmatrix} \quad (27)$$

which is the same as exact solution.

Comparison of exact and approximated solution is given by the following graph

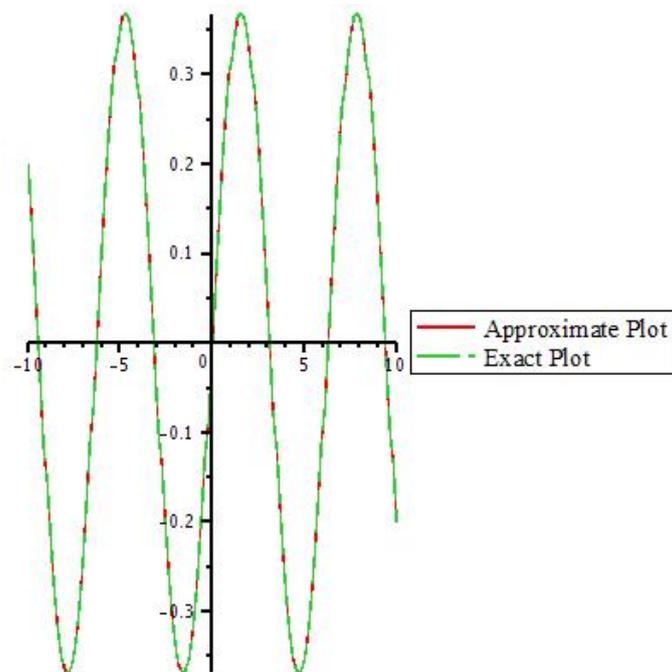


Figure 3: Graph of exact solution and 10th iteration of u by taking $y=1$.

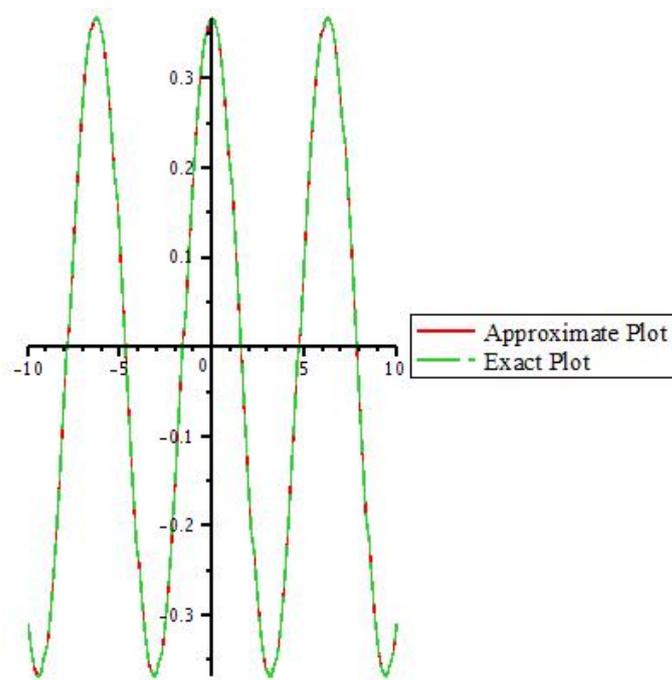


Figure 4: Graph of exact solution and 10th iteration of v by taking $y=1$.

2.3.3. Model Problem-C

Suppose the inhomogeneous Cauchy-Riemann system

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} + \begin{bmatrix} (a^2 - 1) e^{-y} \sin(ax) \\ 0 \end{bmatrix}, x \in \mathbb{R}, y > 0, \tag{28}$$

with the Cauchy conditions

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \sin(ax) \\ \operatorname{acos}(ax) \end{bmatrix}, x \in \mathbb{R}. \tag{29}$$

With the exact solution

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e^{-y} \sin(ax) \\ ae^{-y} \cos(ax) \end{bmatrix} \tag{30}$$

The recurrence relation is

$$\left\{ \begin{array}{l} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \sin(ax) (a^2 - a^2 e^{-y} + e^{-y}) \\ \operatorname{acos}(ax) \end{bmatrix}, \\ \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, n \geq 0. \end{array} \right. \tag{31}$$

Thus we obtained the following vectors

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -a^2 \sin(ax) y \\ a^3 \cos(ax) - \operatorname{acos}(ax) \\ -a^3 \cos(ax) y - a^3 \cos(ax) e^{-y} + e^{-y} \operatorname{acos}(ax) \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} a^4 \sin(ax) - a^2 \sin(ax) - a^4 \sin(ax) y + \\ a^2 \sin(ax) y + a^4 \sin(ax) \frac{y^2}{2} \\ -e^{-y} a^4 \sin(ax) + e^{-y} a^2 \sin(ax) \\ \frac{1}{2} a^3 \cos(ax) y^2 \end{bmatrix} \tag{32}$$

and so on. Compare the result by making graph of exact and approximate solution by

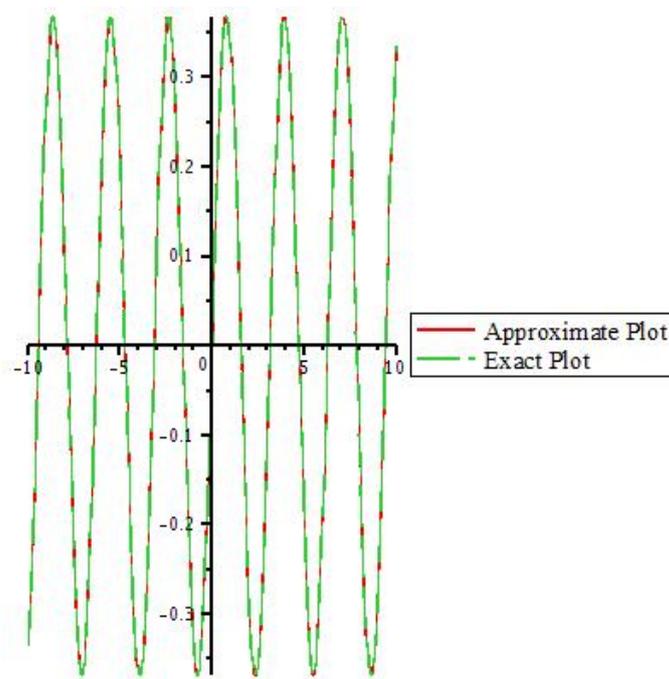


Figure 5: Graph of exact solution and 10th iteration of u by taking y=1 and a=2.

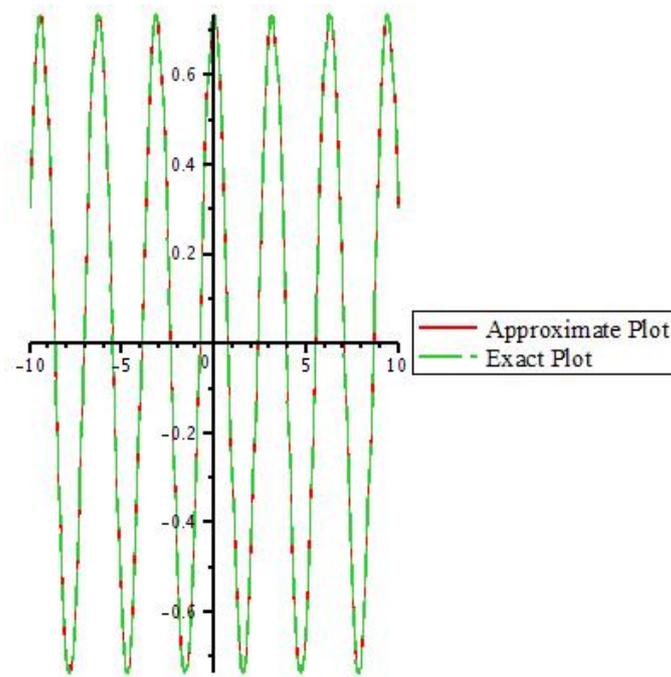


Figure 6: Graph of exact solution and 10th iteration of v by taking $y=1$ and $a=2$.

2.3.4. Model Problem-D

Consider the following inhomogeneous Cauchy-Riemann system

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} + \begin{bmatrix} (a+b) \sin(ax) \cos(by) \\ (a-b) \cos(ax) \sin(by) \end{bmatrix}, \quad (33)$$

$x \in \mathbb{R}, y > 0,$

with the Cauchy conditions

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (34)$$

whose exact solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sin(ax) \sin(by) \\ \cos(ax) \cos(by) \end{bmatrix} \quad (35)$$

The recurrence relation is given by

$$\begin{cases} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \frac{(a+b) \sin(ax) \sin(by)}{b} \\ \cos(ax) - \frac{(a-b) \cos(ax) (-1 + \cos(by))}{b} \end{bmatrix}, \\ \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, \quad n \geq 0. \end{cases} \quad (36)$$

Thus we obtained the following vectors

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{\sin(ax) a(-aby + a \sin(by)) - b \sin(by)}{b^2} \\ \frac{(a+b) \cos(ax) a(-1 + \cos(by))}{b^2} \end{bmatrix} \quad (37)$$

and so on. Compare the result by making graph of exact and approximate solution by

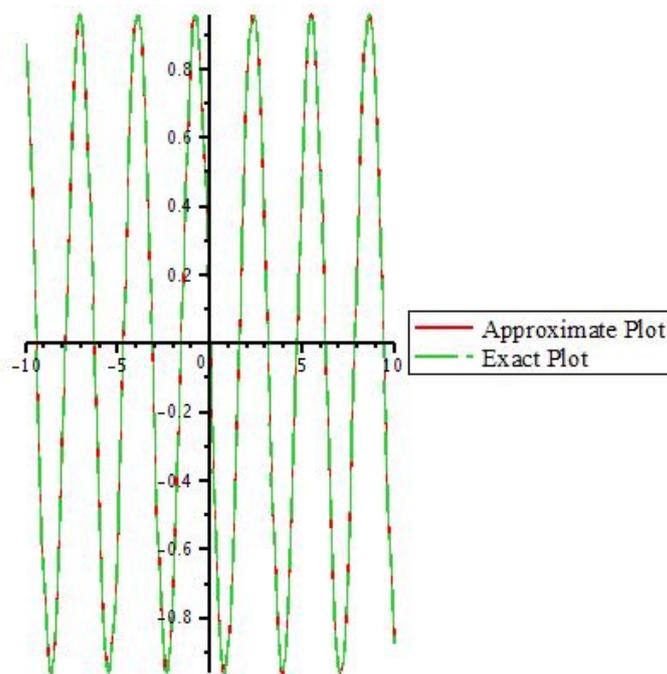


Figure 7: Graph of exact solution and 10th iteration of u by taking $y=1$, $a=2$, and $b=5$.

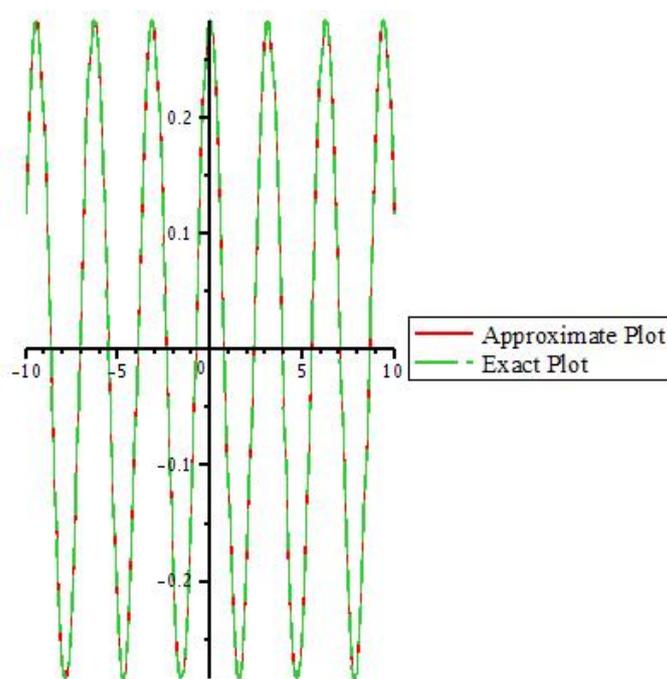


Figure 8: Graph of exact solution and 10th iteration of v by taking $y=1$, $a=2$, and $b=5$.

2.3.5. Model Problem-E

Consider the following inhomogeneous Cauchy-Riemann system of equations

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} + \begin{bmatrix} 4y^3 \sin(4x) - 2(1-x) \sin(4y) \\ 4x(2-x) \cos(4y) + 4y^4 \cos(4x) \end{bmatrix}, \quad (38)$$

$x \in \mathbb{R}, y > 0,$

with the homogeneous Cauchy conditions

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}, \tag{39}$$

having exact solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} y^4 \sin(4x) \\ x(2-x)\sin(4y) \end{bmatrix}. \tag{40}$$

The recurrence relation is given by

$$\left\{ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}x + y^4 \sin(4x) + \frac{1}{2} \cos(4y) - \frac{1}{2} \cos(4y) x \\ 2x \sin(4y) - x^2 \sin(4y) + \frac{4}{5} y^5 \cos(4x) \end{bmatrix}, \right. \tag{41}$$

$$\left. \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, \quad n \geq 0. \right.$$

Thus we obtained the following vectors

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}x - \frac{1}{2} \cos(4y) + \frac{1}{2}x \cos(4y) - \frac{8}{15} y^6 \sin(4x) \\ -\frac{1}{2}y - \frac{4}{5} y^5 \cos(4x) + \frac{1}{8} \sin(4y) \end{bmatrix},$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{8}{15} y^6 \sin(4x) \\ \frac{1}{2}y - \frac{1}{8} \sin(4y) + \frac{32}{105} y^7 \cos(4x) \end{bmatrix}, \tag{42}$$

Compare the result by making graph of exact and approximate solution by

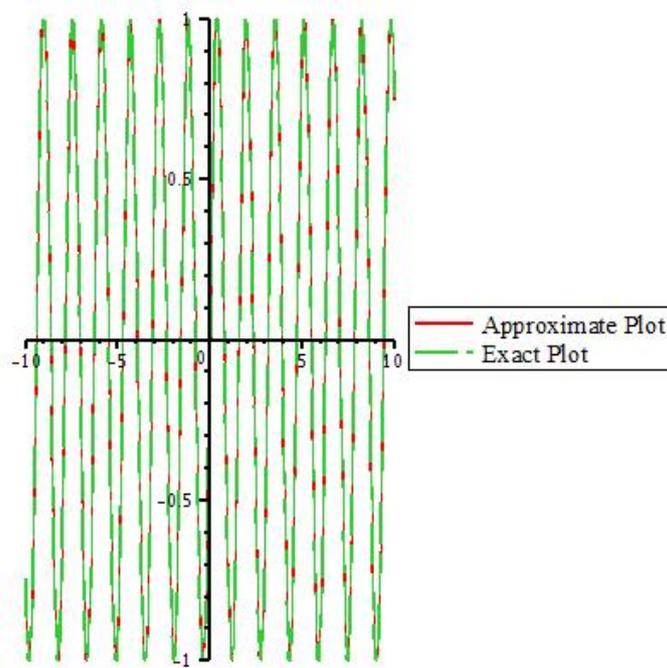


Figure 9: Graph of exact solution and 10th iteration of u by taking $y=1$.

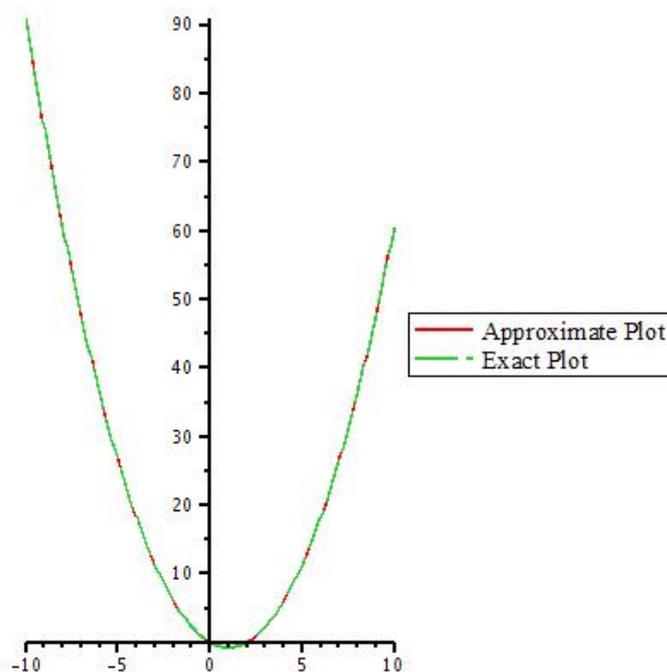


Figure 10: Graph of exact solution and 10th iteration of v by taking $y=1$.

2.3.6. *Model Problem-F*

Consider the inhomogeneous CREs

$$L_y \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} + \begin{bmatrix} 4y^3 \sin(x) \\ 4y^3 + y^4 \cos(x) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \tag{43}$$

subject to the homogeneous Cauchy conditions

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.001 \end{bmatrix}, \quad x \in \mathbb{R}, \tag{44}$$

the exact solution is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} y^4 \sin(x) + 0.001 \\ y^4 + 0.001 \end{bmatrix}. \tag{45}$$

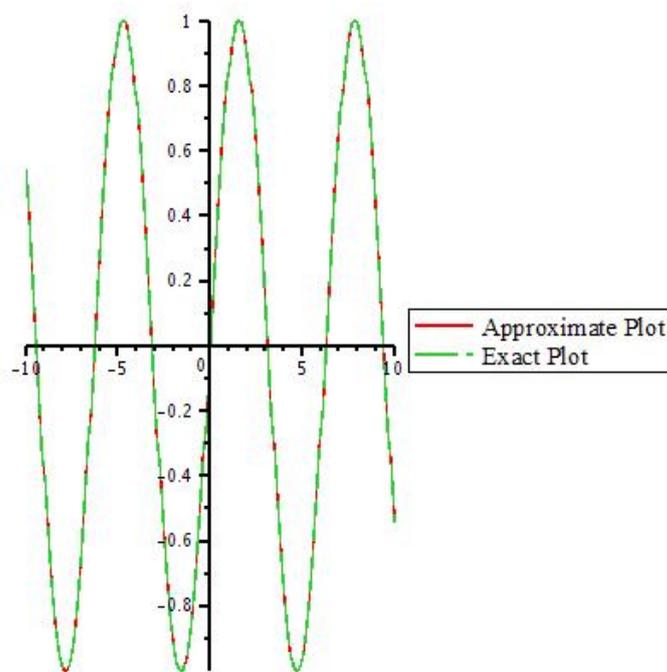
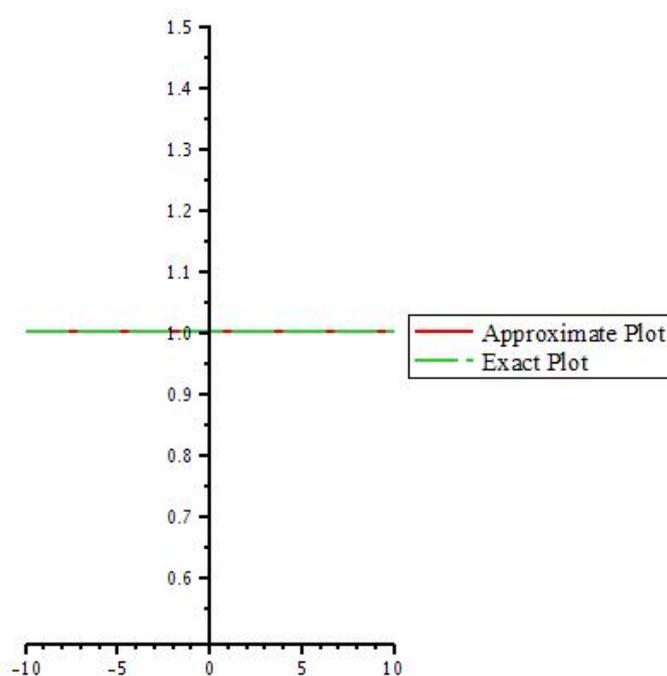
The recurrence relation (12) for (41)-(42) is given by

$$\left\{ \begin{array}{l} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0.001 + y^4 \sin(x) \\ 0.001 + y^4 + \frac{1}{5}y^5 \cos(x) \end{bmatrix}, \\ \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = L_y^{-1} \left\{ L_x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} \right\}, \quad n \geq 0. \end{array} \right. \tag{46}$$

Thus we obtained the following vectors

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{30} \sin(x)y^6 \\ -\frac{1}{5} \cos(x)y^5 \end{bmatrix}, \quad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{30} \sin(x)y^6 \\ \frac{1}{210} \cos(x)y^7 \end{bmatrix}, \quad \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{1680} y^8 \sin(x) \\ -\frac{1}{210} \cos(x)y^7 \end{bmatrix}, \dots \tag{47}$$

Compare the result by making graph of exact and approximate solution by

Figure 11: Graph of exact solution and 10th iteration of u by taking $y=1$.Figure 12: Graph of exact solution and 10th iteration of v by taking $y=1$.

3. Vectorial Variational Iteration (VVI) Method

For the solution of system of differential equations many analytical and numerical methods have been developed. We consider the Eq. (4) and construct the correction functional for VIM as

$$u_{n+1} = u_n + \int_0^t \lambda (Lu_n + Ru_n + N\tilde{u}_n - g) ds, \quad (48)$$

where the Lagrange multiplier λ can be identified via variational theory, u_n is the approximate value, and \tilde{u}_n denotes the restricted variation, i.e. $\delta\tilde{u}_n = 0$. Consequently,

$$u = \lim_{n \rightarrow \infty} u_n \quad (49)$$

3.1. Analysis of Vectorial Variation Iteration Method (VVIM) for Cauchy-Riemann SYSTEM OF equations

Consider the elliptic system of two first-order linear equations in two independent variables given in equation whose correction functional is

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_{n+1} &= \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n + \int_0^y \lambda(s) L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n \\ &\quad - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n - \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} ds, \end{aligned} \quad (50)$$

where the Lagrange multiplier λ can be identified optimally via variational theory. The Lagrange multiplier in the case of Cauchy-Riemann equations is -1 because this is 1st -order linear system of equations. Consequently, the solution is given by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n. \quad (51)$$

3.2. Implementation of VVI Method

Some modal problems are presented here for the suitability and compatibility of the Vectorial Variational Iteration Method (VVIM).

3.2.1. Model Problem-A

Consider the Eqs. (17-18), whose correction functional will be

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} 0 \\ \sinh(x) \end{bmatrix}, \\ \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_{n+1} &= \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n - \int_0^y \begin{bmatrix} L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n \end{bmatrix} ds, \end{aligned} \quad (52)$$

which yields the following vectors

$$\begin{bmatrix} u \\ v \end{bmatrix}_1 = \begin{bmatrix} \cosh(x) y \\ \sin(x) \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix}_2 = \begin{bmatrix} \cosh(x) y \\ \sinh(x) - \sinh(x) \frac{y^2}{2} \end{bmatrix}, \dots \quad (53)$$

By using the Eq. (51) we have

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u \\ v \end{bmatrix}_n = \begin{bmatrix} (y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots) \cosh(x) \\ (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots) \sinh(x) \end{bmatrix},$$

In closed form, we have

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sin(y) \cosh(x) \\ \cos(y) \sinh(x) \end{bmatrix}.$$

The graphical comparison of exact and 10^{th} iterates of approximate solution with $y=1$ is given as

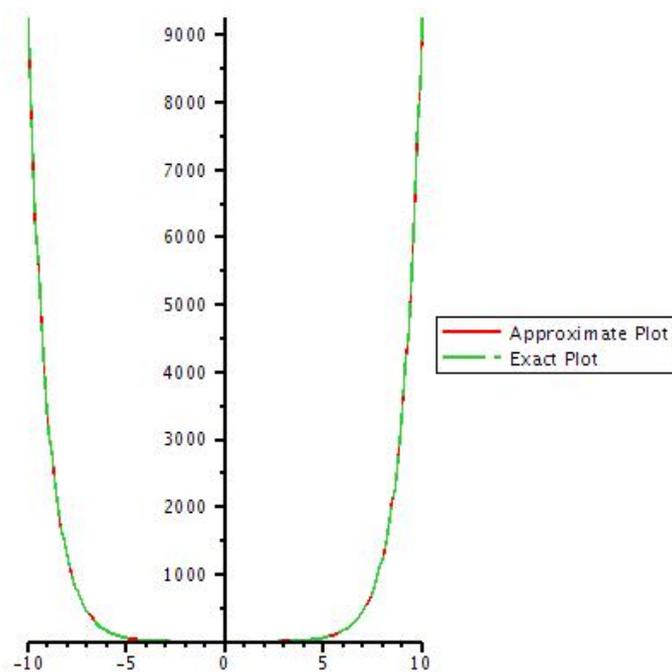


Figure 13: Graph of exact solution and 10th iteration of u by taking $y=1$.

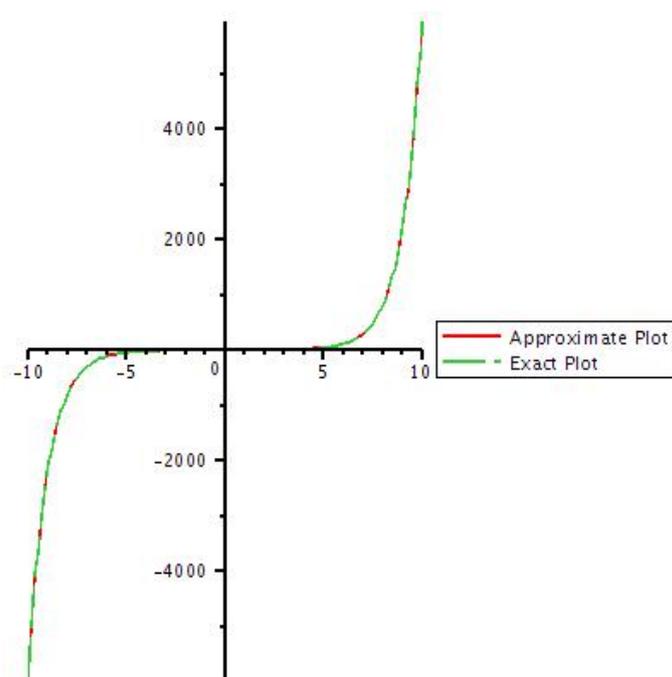


Figure 14: Graph of exact solution and 10th iteration of v by taking $y=1$.

3.2.2. Model Problem-B

Consider the Eqs. (17) and (23) whose correction functional is

$$\begin{bmatrix} u \\ v \end{bmatrix}_0 = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix},$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_{n+1} = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n - \int_0^y \left[L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n \right] ds. \quad (54)$$

which yields the following vectors

$$\begin{bmatrix} u \\ v \end{bmatrix}_1 = \begin{bmatrix} \sin(x) - \sin(x)y \\ \cos(x) - \cos(x)y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}_2 = \begin{bmatrix} \sin(x) - \sin(x)y + \sin(x)\frac{y^2}{2!} \\ \cos(x) - \cos(x)y + \cos(x)\frac{y^2}{2!} \end{bmatrix}, \dots \quad (55)$$

By using the (51) we have

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u \\ v \end{bmatrix}_n = \begin{bmatrix} \left(1 - y + \frac{y^2}{2!} - \dots\right) \sin(x) \\ \left(1 - y + \frac{y^2}{2!} - \dots\right) \cos(x) \end{bmatrix} = \begin{bmatrix} e^{-y} \sin(x) \\ e^{-y} \cos(x) \end{bmatrix}.$$

The graphical comparison of exact and 10^{th} iterates of approximate solution with $y = 1$ is given as

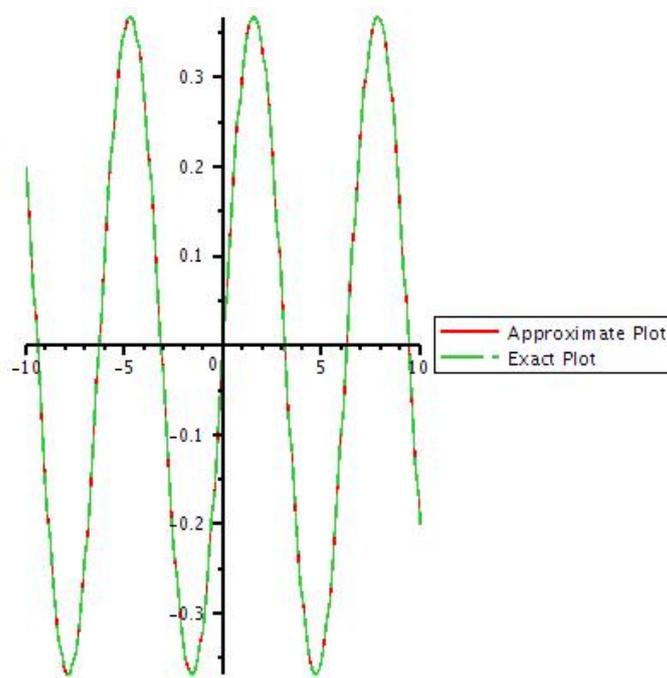


Figure 15: Graph of exact solution and 10th iteration of u by taking $y=1$.

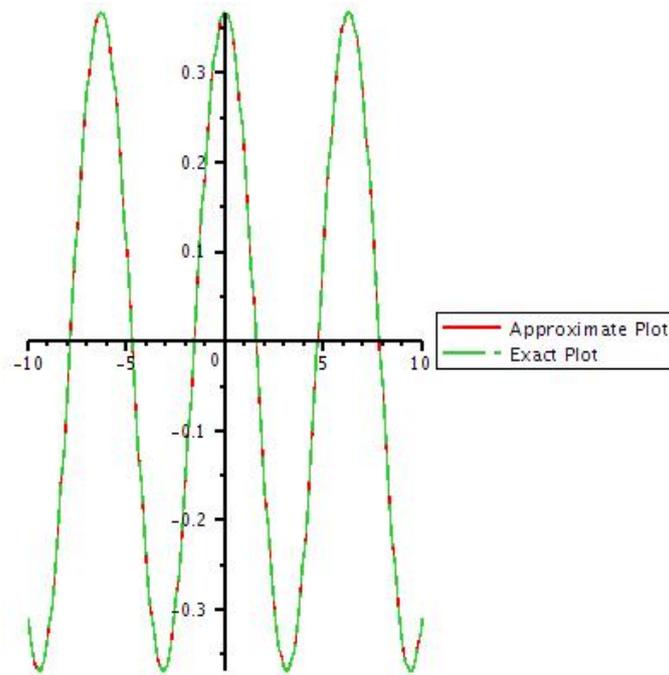


Figure 16: Graph of exact solution and 10th iteration of v by taking $y=1$.

3.2.3. Model Problem-C

Consider the CREs as discussed in Eqs. (28)-(29), and use the VVIM so that the correction functional becomes

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} \sin(ax) \\ \cos(ax) \end{bmatrix}, \\ \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_{n+1} &= \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n - \int_0^y \begin{bmatrix} L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n - \begin{bmatrix} (a^2 - 1)e^{-y} \sin(ax) \\ 0 \end{bmatrix} \end{bmatrix} ds. \end{aligned} \tag{56}$$

The correction functional Eq. (56) yield the following vectors

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} a^2 \sin(ax) \{1 - y - e^{-y}\} + e^{-y} \sin(ax) \\ \cos(ax) \{1 - y\} \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_2 &= \begin{bmatrix} a^2 \sin(ax) \left\{1 - y - e^{-y} + \frac{y^2}{2!}\right\} + e^{-y} \sin(ax) \\ a^3 \cos(ax) \left\{1 - y - e^{-y} + \frac{y^2}{2!}\right\} + e^{-y} \cos(ax) \end{bmatrix}, \dots \end{aligned}$$

The graphical comparison of exact and 10th iterates of approximate solution for $a = 2$ and $y = 1$ is depicted as

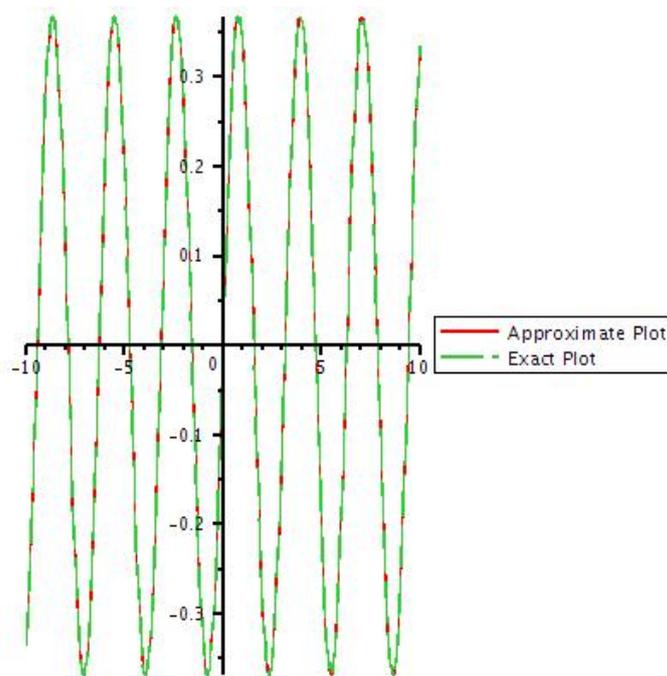


Figure 17: Graph of exact solution and 10th iteration of u by taking $y=1$ and $a=2$.

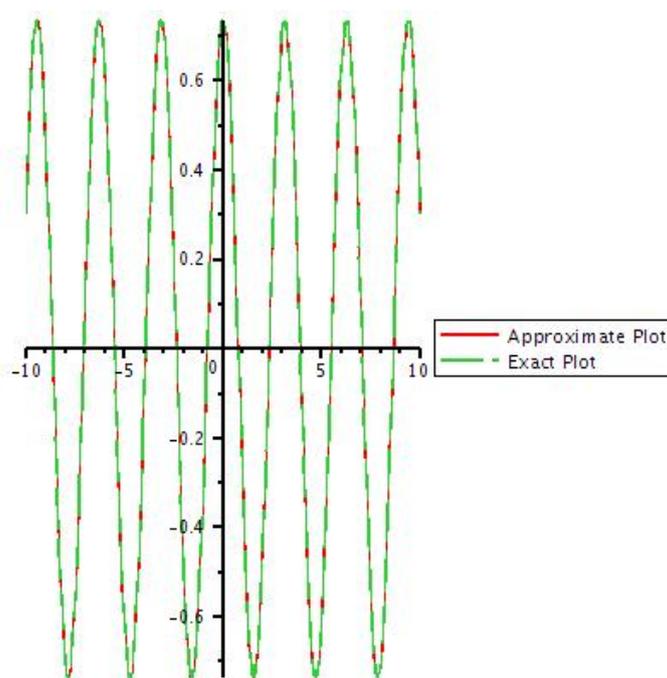


Figure 18: Graph of exact solution and 10th iteration of v by taking $y=1$ and $a=2$.

3.2.4. Model Problem-D

Consider the inhomogeneous CRE with source terms and the initial data as trigonometric functions. The correction functional is

$$\begin{bmatrix} u \\ v \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_{n+1} &= \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}_n - \int_0^y \begin{bmatrix} L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n - \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n - \begin{bmatrix} (b+a)\sin(ax)\cos(by) \\ (a-b)\cos(ax)\sin(by) \end{bmatrix} \end{bmatrix} ds. \end{aligned} \quad (57)$$

The correction functional Eq. (57) yields the following vectors

$$\begin{bmatrix} u \\ v \end{bmatrix}_1 = \begin{bmatrix} \frac{\sin(ax)\{-aby+asin(by) +bsin(by)\}}{b} \\ \cos(ax) - \frac{(a-b)\cos(ax)\{-1+\cos(by)\}}{b} \end{bmatrix}, \dots \quad (58)$$

The comparison of exact and 10th iterates of approximate solution when $a = 2$, $b = 5$ and $y = 1$ is expressed as

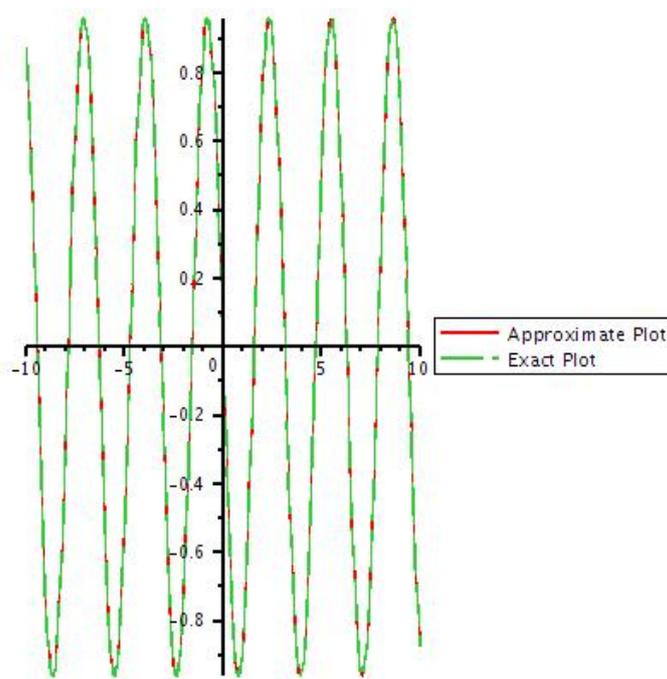


Figure 19: Graph of exact solution and 10th iteration of u by taking $y=1$, $a=2$, and $b=5$.

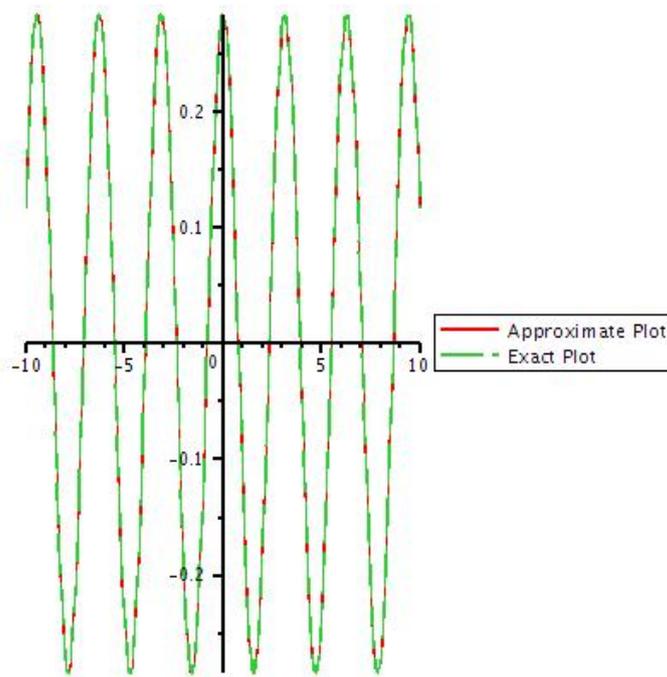


Figure 20: Graph of exact solution and 10th iteration of v by taking $y=1$, $a=2$, and $b=5$.

3.2.5. Model Problem-E

If consider the inhomogeneous system Eqs. (38)-(39), then the correction functional is

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_{n+1} &= \begin{bmatrix} u \\ v \end{bmatrix}_n - \int_0^y \left[L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n \right. \\ &\quad \left. - \begin{bmatrix} 4y^3 \sin(4x) & -2(1-x) \sin(4y) \\ 4x(2-x) \cos(4y) & +4y^4 \cos(4x) \end{bmatrix} \right] ds. \end{aligned} \tag{59}$$

The correction functional Eq. (59) yields the following vectors

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}x + y^4 \sin(4x) + \frac{1}{2} \cos(4y) - \frac{1}{2}x \cos(4y) \\ 2x \sin(4x) - x^2 \sin(4y) + \frac{4}{5}y^5 \cos(4x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_2 &= \begin{bmatrix} y^4 \sin(4x) - \frac{8}{15}y^6 \sin(4x) \\ 2x \sin(4y) - x^2 \sin(4y) - \frac{1}{2}y + \frac{1}{8} \sin(4y) \end{bmatrix}, \dots \end{aligned} \tag{60}$$

The graphical comparison of exact and 10th iterates of approximate solution with $y = 1$ is given as

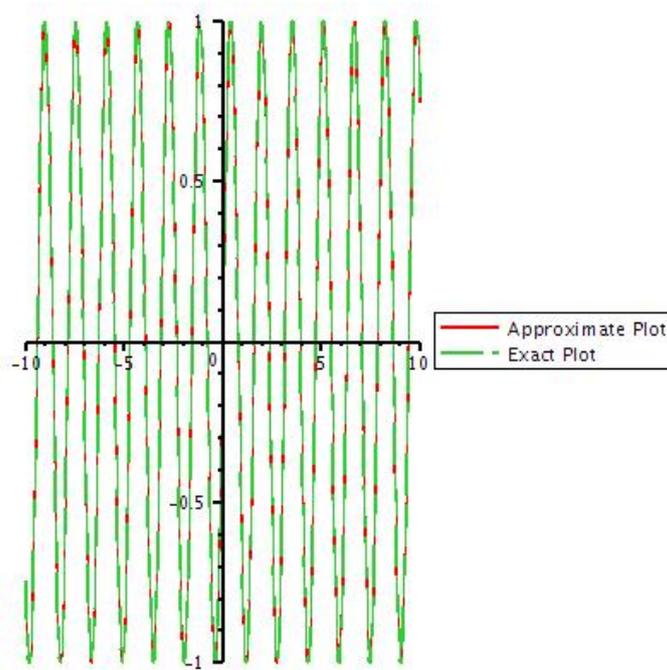


Figure 21: Graph of exact solution and 10th iteration of u by taking $y=1$.

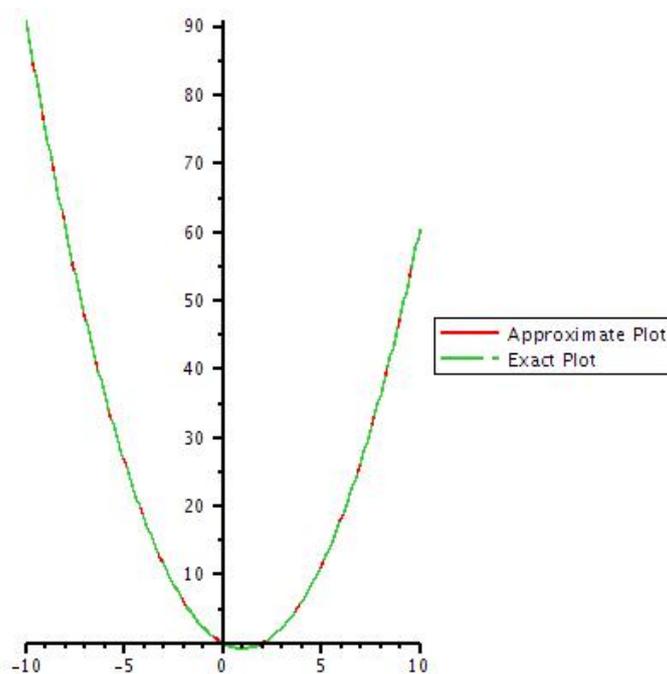


Figure 22: Graph of exact solution and 10th iteration of u by taking $y=1$.

3.2.6. Model Problem-F

Let us take the inhomogeneous Cauchy-Riemann system with their corresponding correction functional

$$\begin{bmatrix} u \\ v \end{bmatrix}_0 = \begin{bmatrix} 0.001 \\ 0.001 \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_{n+1} &= \begin{bmatrix} u \\ v \end{bmatrix}_n - \int_0^y \left[L_s \begin{bmatrix} u(x, s) \\ v(x, s) \end{bmatrix}_n - \right. \\ &\left. \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n - \begin{bmatrix} 4y^3 \sin(x) \\ 4y^3 + y^4 \cos(x) \end{bmatrix} \right] ds. \end{aligned} \tag{61}$$

The correction functional Eq. (61) yields the following vectors

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} 0.001 + y^4 \sin(x) \\ 0.001 + y^4 + \frac{y^5}{5} \cos(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_2 &= \begin{bmatrix} 0.001 + y^4 \sin(x) - \frac{1}{30} \sin(x)y^6 \\ 0.001 + y^4 \end{bmatrix}, \dots \end{aligned} \tag{62}$$

The graphical comparison of exact and 10th iterates of approximate solution with $y = 1$ is given as

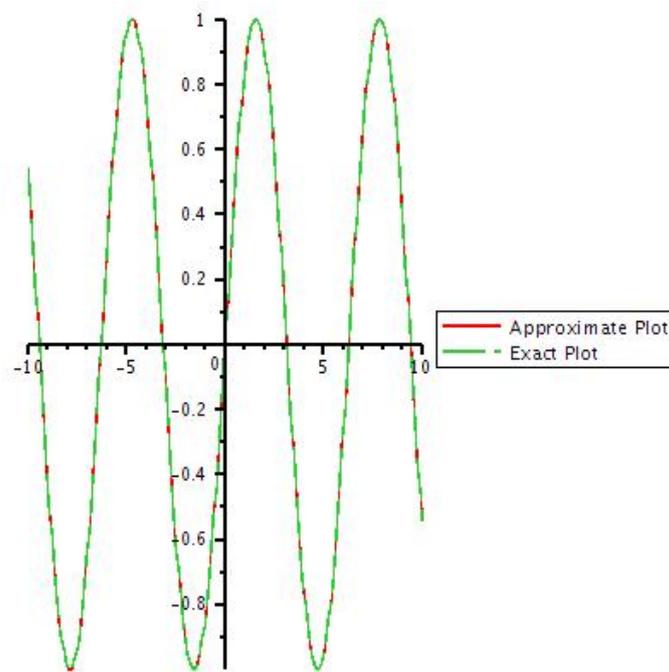


Figure 23: Graph of exact solution and 10th iteration of u by taking $y=1$.

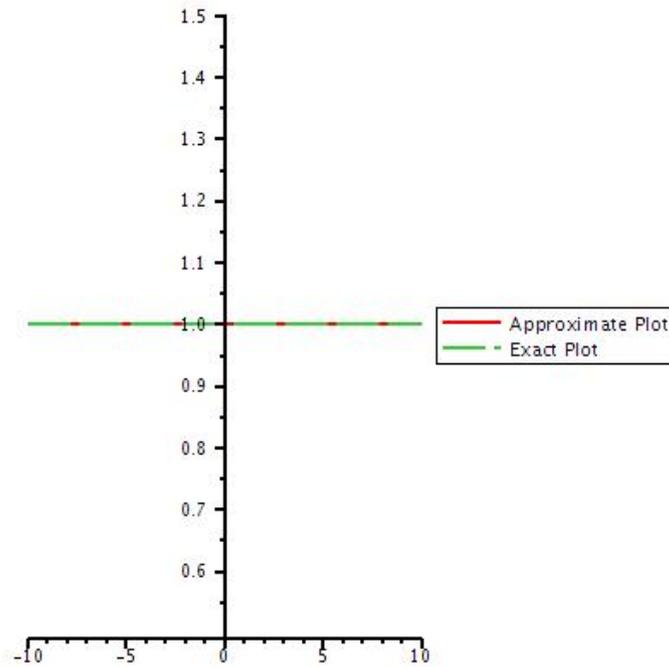


Figure 24: Graph of exact solution and 10th iteration of u by taking $y=1$.

4. Vectorial Modified Picard (VMP) Method

The Picard’s method in the PDE form, one can consider

$$\left\{ \left[\begin{array}{l} u_t = P(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots) \\ u(., 0) = q(.) \end{array} \right] \right\}, \tag{63}$$

P and q are n variable polynomial. In the Parker and Sochacki work the computational method is as follow

$$\left\{ \left[\begin{array}{l} \phi_0(t) = q(.) \\ \phi_{n+1}(., t) = q(.) + \int_0^t P(\phi_n(., s)) ds \end{array} \right] \right\}, n = 0, 1, 2, \dots \tag{64}$$

“They also showed that the n th Picard iterates is the Maclaurin polynomial of degree n for $y(t)$ if $\phi^n(t)$ is truncated to degree n at each step. This form of Picard’s method is called the Modified Picard Method (MPM).”

4.1. Analysis of the Vectorial Modified Picard’s Method (VMPM) for Cauchy-Riemann Equations

We consider the Eq. (4) and applying the VMPM

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_{n+1} &= \begin{bmatrix} u \\ v \end{bmatrix}_0 + \int_0^y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L_x \begin{bmatrix} v(x, s) \\ u(x, s) \end{bmatrix}_n + \begin{bmatrix} f(x, s) \\ g(x, s) \end{bmatrix} ds, \end{aligned} \tag{65}$$

where n is the order of the approximation. The solution is, therefore, given by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u_n \\ v_n \end{bmatrix}. \tag{66}$$

4.2. Implementation of the Method

To check that the proposed method is effective for the initial data we solve six model problems.

4.2.1. Model Problem-A

Consider the Eqs. (17)-(18) and then using the iterative relation (65), we have

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} 0 \\ \sinh(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} y \cosh(x) \\ \sinh(x) \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}_2 = \begin{bmatrix} y \cosh(x) \\ \sinh(x) - \frac{1}{2}y^2 \sinh(x) \end{bmatrix}, \dots \end{aligned} \quad (67)$$

Now by using Eq. (66), we obtain the solution in the closed form as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} (y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots) \cosh(x) \\ (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots) \sinh(x) \end{bmatrix} = \begin{bmatrix} \sin(y) \cosh(x) \\ \cos(y) \sinh(x) \end{bmatrix}.$$

This is the same as exact solution.

The graphical comparison of exact and 10th iterates of approximate solution with $y = 1$ is given as

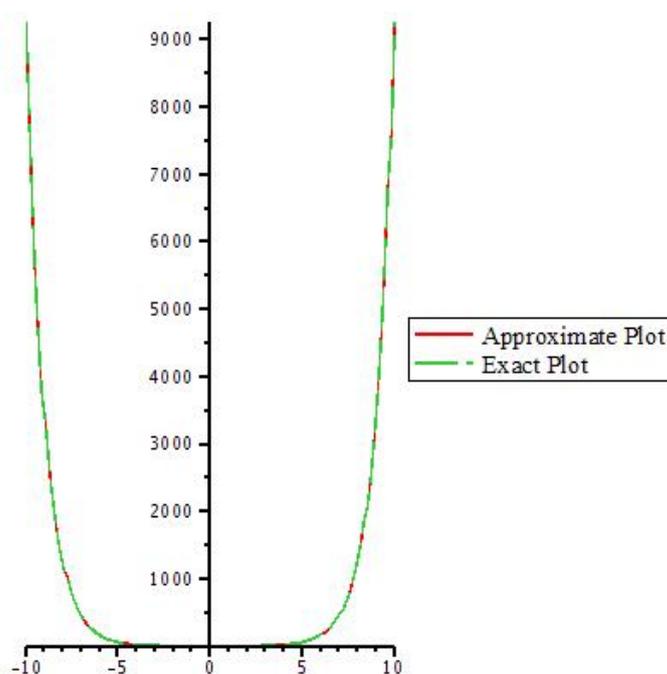


Figure 25: Graph of exact solution and 10th iteration of u by taking $y=1$.

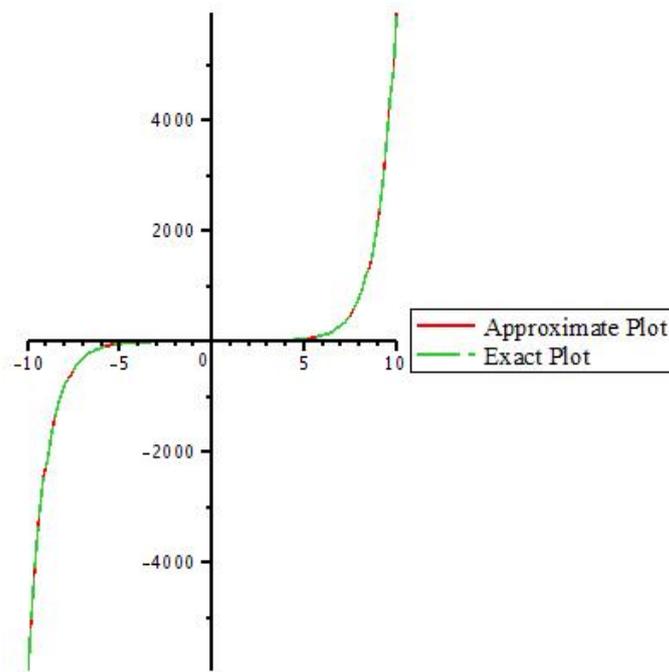


Figure 26: Graph of exact solution and 10th iteration of v by taking $y=1$.

4.2.2. Model Problem-B

Consider the problem given in Eqs. (17) and (23) and handling it with the iterative relation (65), we have

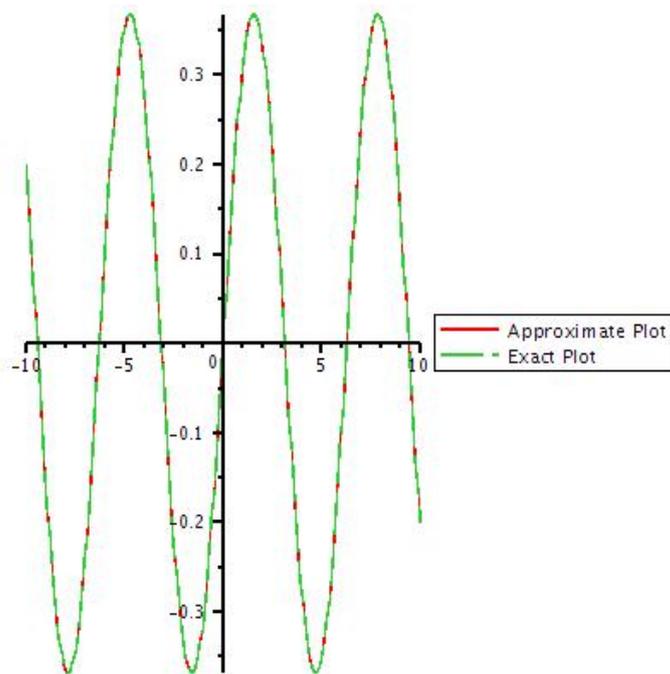
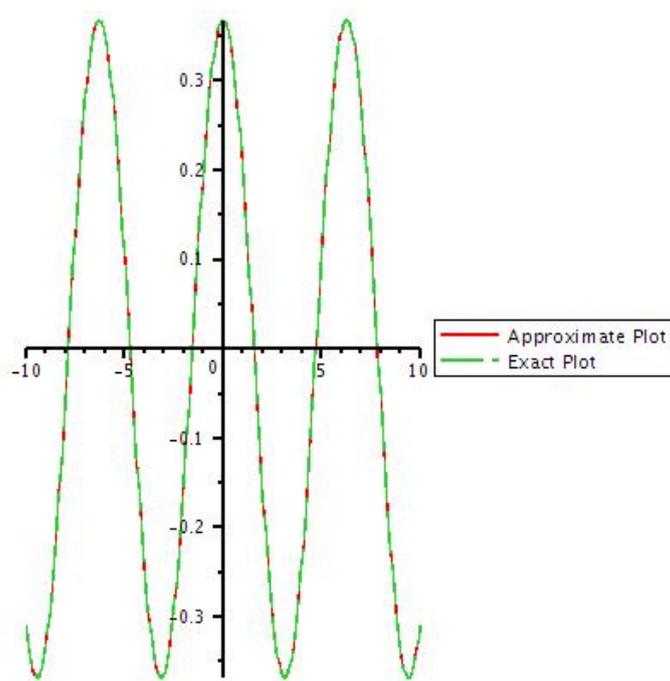
$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} \sin(x) - y\sin(x) \\ \cos(x) - y\cos(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_2 &= \begin{bmatrix} \sin(x) - y\sin(x) + \frac{1}{2}y^2\sin(x) \\ \cos(x) - y\cos(x) + \frac{1}{2}y^2\cos(x) \end{bmatrix}, \dots \end{aligned} \tag{68}$$

Now by using equation (??), we obtain the solution in the closed form as

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} u_n(x, y) \\ v_n(x, y) \end{bmatrix} = \begin{bmatrix} \left(1 - y + \frac{y^2}{2!} - \dots\right) \cosh(x) \\ \left(1 - y + \frac{y^2}{2!} - \dots\right) \sinh(x) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(x) \\ e^{-y}\cos(x) \end{bmatrix}.$$

This is the same as exact solution.

The graphical comparison of exact and 10th iterates of approximate

Figure 27: Graph of exact solution and 10th iteration of u by taking $y=1$.Figure 28: Graph of exact solution and 10th iteration of v by taking $y=1$.

4.2.3. Model Problem-C

Consider Eqs. (28)-(29) Now we use iterative relation (65) to solve this problem and have

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} \sin(ax) \\ \cos(ax) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} a^2 \sin(ax) - a^2 y \sin(ax) - a^2 e^{-y} \sin(ax) + e^{-y} \sin(ax) \\ \cos(ax) - ay \cos(ax) \end{bmatrix}, \end{aligned}$$

and so on The graphical comparison of exact and 10^{th} iterates of approximate solution with $a = 2$, and $y = 1$ is given as

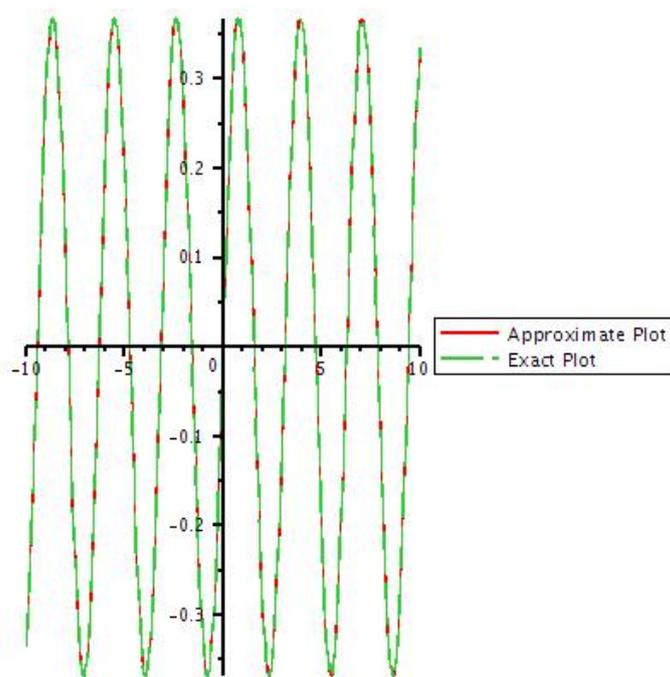


Figure 29: Graph of exact solution and 10th iteration of u by taking $y=1$ and $a=2$.

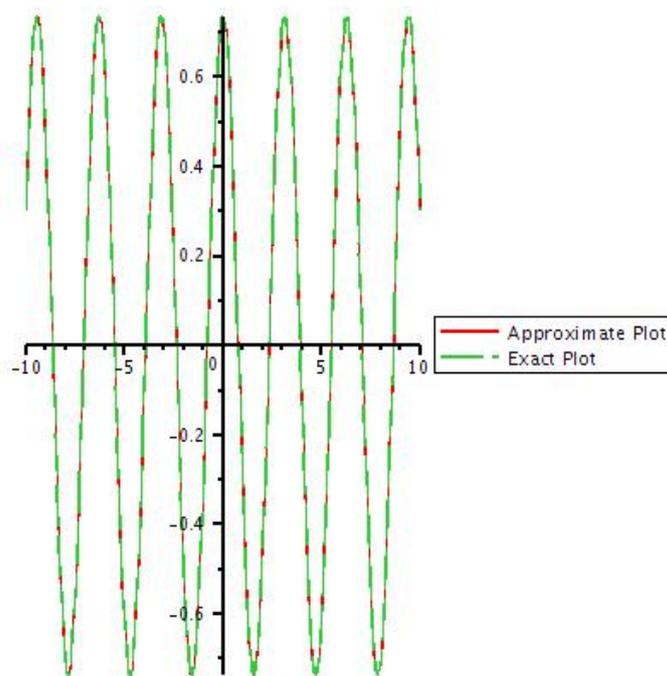


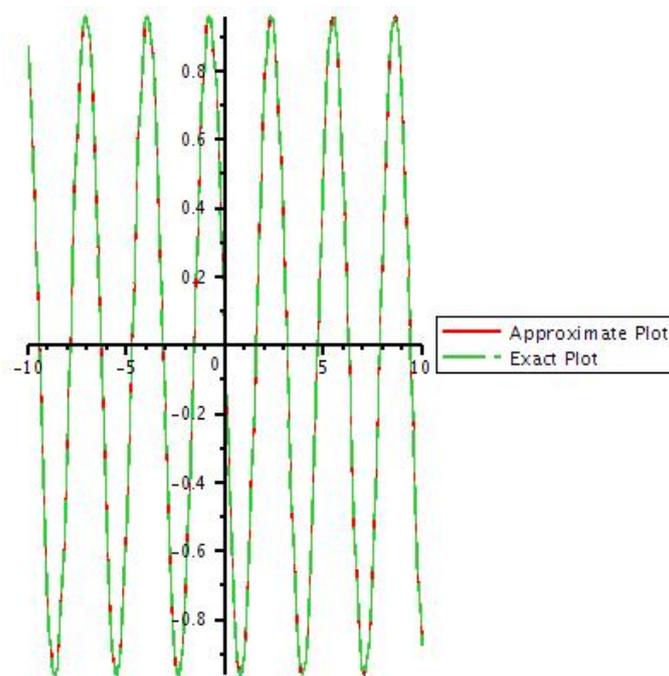
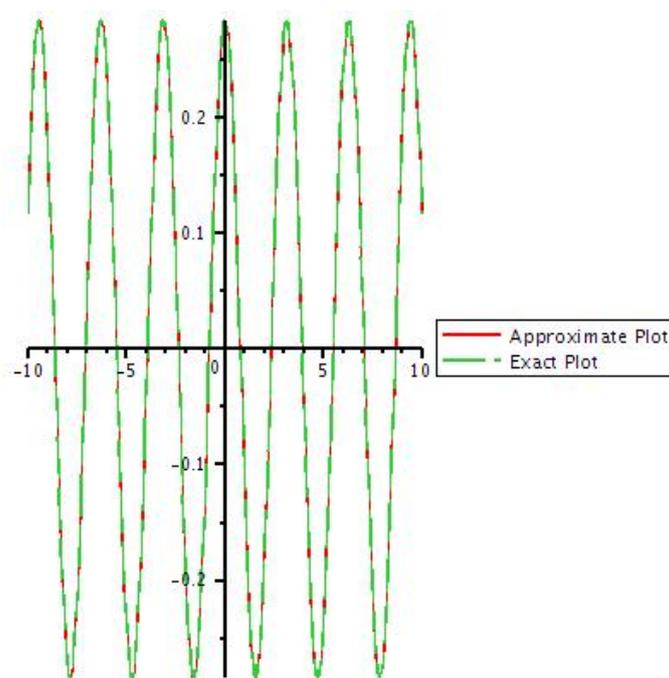
Figure 30: Graph of exact solution and 10th iteration of v by taking $y=1$ and $a=2$.

4.2.4. Model Problem-D

Let us suppose Eqs. (33)-(34) then on using the iterative relation (65) to solve this problem and have

$$\begin{bmatrix} u \\ v \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix}_1 = \begin{bmatrix} \frac{\sin(ax) (-aby + a\sin(by) - b\sin(by))}{b} \\ \cos(ax) - \frac{(a-b)\cos(ax) (-1 + \cos(by))}{b} \end{bmatrix},$$

and so on. The graphical comparison of exact and 10th iterates of approximate solution with $y = 1$, $a = 2$, $b = 5$ is given as

Figure 31: Graph of exact solution and 10th iteration of u by taking $y=1$, $a=2$, and $b=5$.Figure 32: Graph of exact solution and 10th iteration of v by taking $y=1$, $a=2$, and $b=5$.

4.2.5. Model Problem-E

Consider Eqs. (38)-(39) then by using relation (65), we have the following iterates

$$\begin{bmatrix} u \\ v \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix}_1 = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}x + y^4 \sin(4x) + \frac{1}{2} \cos(4y) - \frac{1}{2}x \cos(4y) \\ 2x \sin(4y) - x^2 \sin(4y) + \frac{4}{5}y^5 \cos(4x) \end{bmatrix}$$

The graphical comparison of exact and 10th iterates of approximate solution with $y = 1$ is given as

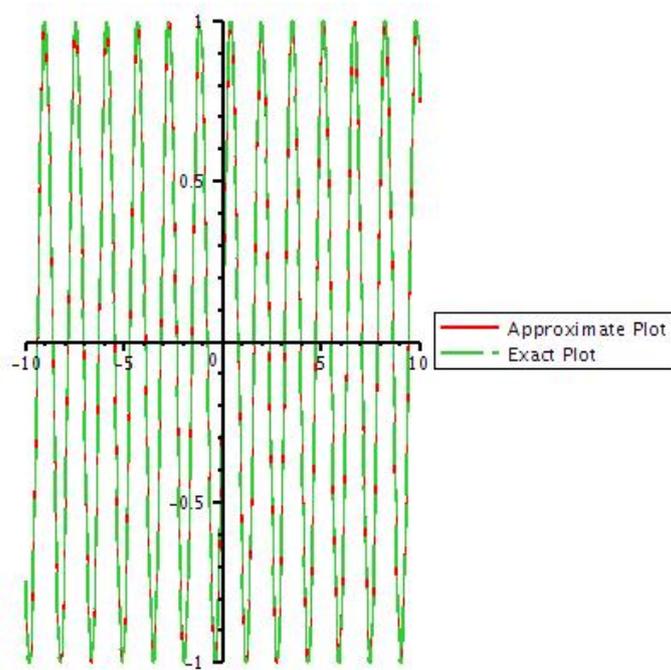


Figure 33: Graph of exact solution and 10th iteration of u by taking $y=1$.

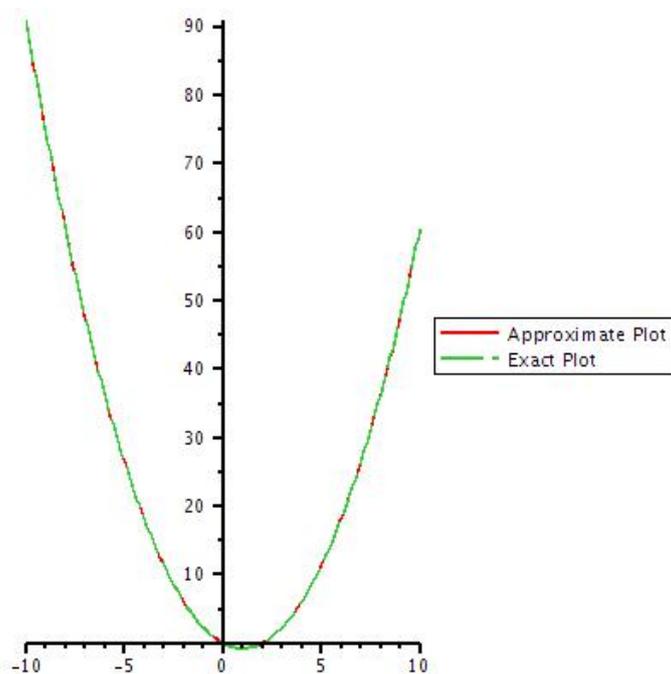


Figure 34: Graph of exact solution and 10th iteration of v by taking $y=1$.

4.2.6. Model Problem-F

Consider Eqs. (43)-(44) and using relation (65), we have

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix}_0 &= \begin{bmatrix} 0.001 \\ 0.001 \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_1 &= \begin{bmatrix} 0.001 + y^4 \sin(x) \\ 0.001 + y^4 + \frac{y^5}{5} \cos(x) \end{bmatrix}, \\ \begin{bmatrix} u \\ v \end{bmatrix}_2 &= \begin{bmatrix} 0.001 - \frac{1}{30}y^6 \sin(x) + y^4 \sin(x) \\ 0.001 + y^4 \end{bmatrix} \dots \end{aligned}$$

The graphical comparison of exact and 10^{th} iterates of approximate solution with $a = 2$, $b = 5$ and $y = 1$ is given as

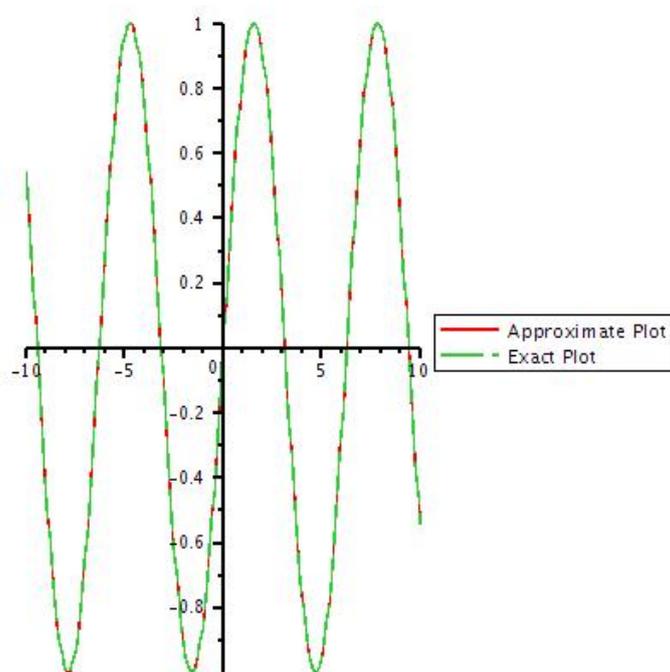


Figure 35: Graph of exact solution and 10th iteration of u by taking $y=1$.

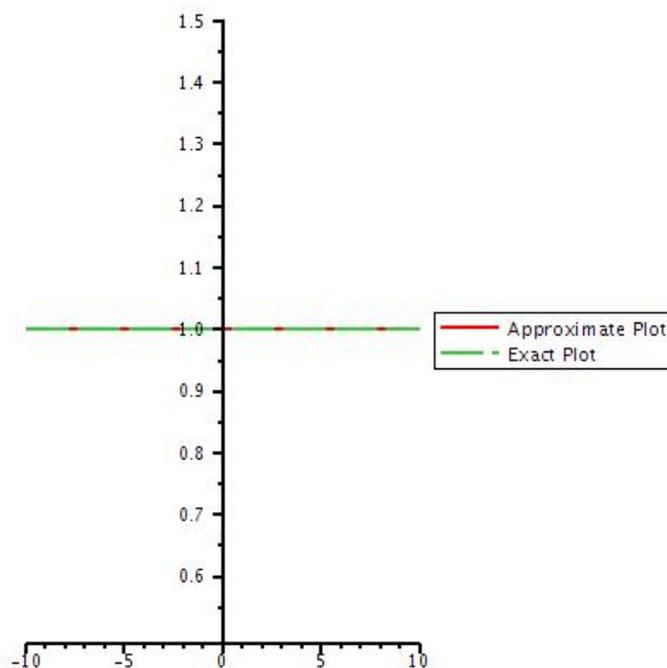


Figure 36: Graph of exact solution and 10th iteration of u by taking $y=1$.

5. Conclusion

Vectorial Adomian Decomposition Method, Vectorial Variational Iteration Method and Vectorial Modified Picard's Method has been used on the Cauchy problem for Cauchy-Riemann system of equations to obtain the analytic expressions. These utilized methods give the analytic approximation in the form of rapidly convergent series which can be converted into exact form in most of the cases. Efficiency of these schemes has been checked by variety of model problems which confirms the reliability of the utilized schemes. The obtained results are signified graphically.

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